

STATISTICS AND COLLECTIVE FIELDS IN ONE AND TWO DIMENSIONS

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1. Introduction

During the last few years considerable attention has been devoted to the statistics of different objects, such as particles, anyons and monopoles¹⁻³⁾. In one dimension, the interaction simulates statistics and varying coupling constants can be smoothly interpolated between bosons and fermions⁴⁾. It has been shown that in two dimensions, the statistics of "anyons" can be intermediate. It has been argued that in three dimensions (there are only two well-known statistics allowed), the fermion system can be treated by extracting an antisymmetric pre-factor from the wave function⁵⁾. To treat non-relativistic quantum mechanical systems of N identical objects, Jevicki and Sakita⁶⁾ have shown that introduction of collective fields is an efficient method for obtaining a large- N limit of such systems. In such a limit, statistics enters in the form of an effective potential. In sec. II we shall treat "anyons" in one dimension and in sec. III we shall treat them in two dimensions. We have stated a Thomas-Fermi functional for anyons.

2. Anyons in one dimension

Calogero⁴⁾ has shown that in one dimension a singular interaction simulates boson or fermion statistics. This statistical interaction is vanishing for bosons and fermions, but owing to Klauder phenomena⁷⁾, we cannot disregard this term. If we start from the N -particle Hamiltonian

$$\left(-\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \lambda(\lambda-1) \sum_{i < j} \frac{1}{(x_i - x_j)^2} + \sum_{i < j} V(x_i - x_j)\right) \psi = E\psi, \quad (1.1)$$

the solution of the problem can be factorized as

$$\psi(x_1, x_2, \dots, x_N) = \Delta^\lambda(x_1, x_2, \dots, x_N) \phi(x_1, x_2, \dots, x_N), \quad (1.2)$$

where

$$\Delta(x_1, x_2, \dots, x_N) = \prod_{i < j} (x_i - x_j) \quad (1.3)$$

is an antisymmetric factor which takes care of the singular interaction. $\phi(x_1, x_2, \dots, x_N)$ is a totally symmetric function. After performing a similarity transformation

$$H \rightarrow \tilde{H} = \Delta^{-\lambda} H \Delta^{\lambda}, \quad (1.4)$$

we obtain an equation for the symmetric function ϕ :

$$\left(-\frac{1}{2} \Delta^{-2\lambda} \sum_i \frac{\partial}{\partial x_i} \Delta^{2\lambda} \frac{\partial}{\partial x_i} + \sum_{i < j} V(x_i - x_j) \right) \phi = E \phi. \quad (1.5)$$

Owing to the symmetry of the kinetic energy term, we can now proceed by using the standard methods for reformulation of the problem in terms of collective fields:

$$\rho(x) = \sum_{i=1}^N \delta(x - x_i) \quad (1.6)$$

and, in the limit of large N ($N \rightarrow \infty$), after hermitization we obtain

$$\begin{aligned} \tilde{H} = & \frac{1}{2} \int dx dy \pi(x) \Omega(x, y) \pi(y) + \frac{(\lambda-1)^2}{8} \int \frac{\rho^-(x)^2}{\rho(x)} dx \\ & + \frac{\pi^2}{6} \lambda^2 \int \rho^3(x) dx + \frac{\lambda(\lambda-1)}{2} \int dx dy \frac{\rho^-(x) \rho(y)}{x-y} \\ & + \int \rho(x) V(x-y) \rho(y) dx dy, \end{aligned} \quad (1.7)$$

where $\Omega(x, y) = \partial_x \partial_y (\delta(x-y) \rho(x))$. For $\lambda = 0$, we obtain a purely bosonic von Weizsäcker Hamiltonian; for $\lambda = 1$ ⁸⁾, we obtain a pure Thomas-Fermi Hamiltonian. For $0 < \lambda < 1$, we obtain intermediate statistics.

3. Anyons in two dimensions

As shown by Wilczek, in two space dimensions there exists a remarkable two-body interaction which changes a many-body system from a Bose to a Fermi gas as the interaction constant is varied. Yong-Shi Wu has shown that this vector-potential interaction can be gauged away by a singular gauge transformation to a multivalued gauge factor. This gauge

factor can be written using complex co-ordinates $z_i = x_i + iy_i$. The wave function for anyons can be factorized³⁾ as

$$\psi(z_i) = \prod_{i < j}^N (z_i - z_j)^\lambda \phi(z_1, \dots, z_N), \quad (3.1)$$

where $\phi(z_1, \dots, z_N)$ is a symmetric function which satisfies the Schrödinger equation

$$\hat{H} = -\frac{1}{2} \sum \nabla_i^2 - 2\lambda \sum_{i \neq j} \frac{1}{z_i - z_j} \frac{\partial}{\partial z_i} + V.$$

Now we can reformulate the Hamiltonian in terms of collective fields, introducing the density

$$\rho(\vec{r}) = \sum_{i=1}^N \delta(\vec{r} - \vec{r}_i) \quad (3.2)$$

and, in the large-N limit, we obtain

$$\begin{aligned} \hat{H}_{\text{coll}} = & -\frac{1}{2} \int ds ds' \nabla \nabla' (\delta(\vec{r} - \vec{r}') \rho(\vec{r})) \frac{\delta^2}{\delta \rho(\vec{r}) \delta \rho(\vec{r}')} \quad (3.3) \\ & + \int ds \left[\frac{\lambda-2}{4} \Delta \rho(\vec{r}) + \lambda \nabla(\rho(\vec{r})) \int \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} \rho(\vec{r}') ds' \right] \frac{\delta}{\delta \rho(\vec{r})} \\ & + i\lambda \int ds \nabla(\rho(\vec{r})) \hat{n} \times \int \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} \rho(\vec{r}') ds' \frac{\delta}{\delta \rho(\vec{r})} + \int V(\vec{r}) \rho(\vec{r}) ds, \end{aligned}$$

where the unit vector \hat{n} is perpendicular to the x-y plane.

The real part of the linear derivative term in (3.3) can be eliminated by hermitization. The effective potential is

$$\begin{aligned} \mathcal{H}_{\text{eff}}^{\sim} = & \frac{(\lambda-2)^2}{32} \int \frac{(\nabla \rho)^2}{\rho} ds - \frac{\lambda(\lambda-2)}{2} \pi \int \rho^2(\vec{r}) ds + V(\vec{r}) \\ & + \frac{\lambda}{2} \int ds \rho(\vec{r}) \left(\int \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} \rho(\vec{r}') ds' \right)^2 \quad (3.4) \end{aligned}$$

We have assumed spherical symmetry for the ground state described by ρ . In this case, the imaginary term in the Hamiltonian vanishes identically. After minimization we obtain the equation for $\rho(\vec{r})$:

$$\begin{aligned}
& - \frac{(\lambda-2)^2}{32} \frac{(\vec{\nabla}\rho)^2}{\rho} - \frac{(\lambda-2)^2}{16} \nabla \left(\frac{\vec{\nabla}\rho}{\rho} \right) - \lambda(\lambda-2)\pi\rho + V(\vec{r}) - \mu \\
& + \frac{\lambda^2}{2} \left(\int \frac{\vec{r}' - \vec{r}''}{|\vec{r}' - \vec{r}''|^2} \rho(\vec{r}') \right)^2 + \lambda^2 \int ds' \rho(\vec{r}') \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^2} \\
& \times \int ds'' \frac{\vec{r}' - \vec{r}''}{|\vec{r}' - \vec{r}''|^2} \rho(\vec{r}'') , \tag{3.5}
\end{aligned}$$

where μ is a Lagrange multiplier which takes care of the condition $\int \rho(\vec{r}) ds = N$.

We can exactly solve eq. (3.5) for the case of the harmonic potential $V = \frac{\omega^2}{2} r^2$. The solution is a constant density

$$\rho(\vec{r}) = \frac{1}{\lambda\pi} ,$$

$$\mu = \lambda\omega N - (\lambda-2)\omega ,$$

and

$$E = N(N-1) \frac{\omega\lambda}{2} + \omega N .$$

In the limit $\lambda \rightarrow 1$, we do not obtain a result for the ground state of two-dimensional fermions. The exact result for fermions is lower than the ground state of anyons. This is clear because the pre-factor for fermions must be different from the factor (3.1). For three fermions, the antisymmetric pre-factor is

$$\vec{r}_1 \times \vec{r}_2 + \vec{r}_2 \times \vec{r}_3 + \vec{r}_3 \times \vec{r}_1 ,$$

which is quadratic in \vec{r} !

The functional for anyons (3.4) does not contain the same factor as in the Thomas-Fermi functional for fermions and there is in addition a diamagnetic term. Let us determine the order of various terms in the effective functional by rescaling⁹⁾ $x = a(N)\tilde{x}$. The orders are N/a^2 , N^2/a^2 , Na^λ and N^3/a^2 , respectively. The leading behaviour comes from the cubic term in $\rho(\vec{r})$. Minimizing the leading terms, we obtain

$$\begin{aligned}
V(\vec{r}) - \mu + \frac{\lambda^2}{2} \left(\int \frac{\vec{r}' - \vec{r}''}{|\vec{r}' - \vec{r}''|^2} \rho(\vec{r}') ds' \right)^2 \\
+ \lambda^2 \int ds' \rho(\vec{r}') \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^2} \int ds'' \frac{\vec{r}' - \vec{r}''}{|\vec{r}' - \vec{r}''|^2} \rho(\vec{r}'') = 0 . \tag{3.6}
\end{aligned}$$

From here we obtain for the spherical symmetric regular potential

$$\rho(r) = \frac{1}{2\pi\lambda} \frac{1}{r} \frac{\partial}{\partial r} \sqrt{r^3 \frac{\partial V}{\partial r}}. \quad (3.7)$$

Because of the condition $2\pi \int \rho(r) r dr = N$, we conclude that $\rho(r)$ must be defined on a compact support. Then from (3.7) we obtain for the radius of the domain

$$R^3 \frac{\partial V}{\partial r} \Big|_{r=R} = N^2 \lambda^2$$

and the energy in the large-N approximation is

$$E = \frac{1}{3} V(R)N + \frac{\lambda^2 N^2}{6R^2} + \frac{2}{3\lambda} \int_0^R \left[r \frac{dV}{dr} \right]^{3/2} dr.$$

As an example let us treat the case $V = gr^4$. Then $\rho(r) = \frac{3\sqrt{g}}{\pi\lambda} r$, which is a rather peculiar distribution, because it is empty in the middle of the disc.

It will be interesting to see whether the method of collective fields can be used to treat the quantized Hall effect, for which the relevance of intermediate statistics was emphasized¹⁰⁾.

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