QUANTIZATION OF A GAUGE THEORY WITH SECOND-CLASS CONSTRAINTS

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Hamiltonian formulation of the non-gravitational part of the unified Yang-Mills and Gravitational theory is derived. A correct construction of a relativistic functional integral is obtained in the presence of second class constraints.

I. Introduction

The attempt of formulating quantum gravity, renormalizable, unitary and causal at the same time, has brought to the attention all alternative theories to General Relativity. By allowing connections and metrics to become dynamical variables in both Einstein and Yang-Mills theories, unified action was found in rather simple form [1]. When written in terms of the component fields the unified action splits into a purely gravitational part and an internal part. The non-gravitational part turns out to be a gauge theory of the SL (N, C) group proposed by Cahill [2].

II. Hamiltonian form and the functional integral

The Lagrangian density of the theory is given by:

$$\mathcal{L} = -\frac{1}{4g^2}\left[G_{\mu\nu}\, G^{\mu\nu} + W_{\mu\nu}\, W^{\mu\nu}\right] + \frac{M^2}{2} B_\mu\, \bar{B}^\mu \tag{1}$$

where

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] + [B_\mu, B_\nu] \\ W_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + [A_\mu, B_\nu] - [A_\nu, B_\mu] \end{aligned} \tag{2}$$

In the usual way one obtains for the canonical Hamiltonian

$$\mathcal{H}_c = -\frac{1}{2} g^2 (\pi^i_a \pi^a_i + p^i_a p^a_i) + \pi^i_a (\partial_i A^a_o + \varepsilon^{abc} A^b_o A^c_i -$$

$$- \varepsilon^{abc} B^b_o B^c_i) + P^i_a (\partial_i B^a_o + \varepsilon^{abc} A^b_o B^c_i - \varepsilon^{abc} A^b_i B^c_o) + \qquad (3)$$

$$+ \frac{1}{4g^2} (G_{ij} G^{ij} + W_{ij} W^{ij}) - \frac{1}{2} M^2 B_o B^o - \frac{1}{2} M^2 B_i B^i$$

Following the Dirac systematic method [3] for the theory with constraints, we find that the primary constraints are

$$\pi^o_a = 0, \qquad P^o_a = o \qquad (4)$$

while the secondary ones are

$$C^a = \partial_i \pi^i_a + \varepsilon^{abc} \pi^i_b A^c_i + \varepsilon^{abc} P^i_b B^c_i$$

$$D^a = \partial_i P^i_a - \varepsilon^{abc} P^i_c A^b_i - \varepsilon^{abc} \pi^i_b B^c_i + M^2 B^a_o \qquad (5)$$

After working out all Poisson brackets for the constraints (4) and (5) we have found [4] that C^*_a and π^o_a are the first class constraints while D_a and P^o_a are the second class ones.

Now we can proceed in formulating the quantum theory. Besides the trivial solutions for π^o_a and P^o_a , by solving the constraints $D_a = 0$ we can eliminate B^a_o from the Hamiltonian.

In solving the constraints $C_a = 0$ we may proceed in the usual way of decomposing the remaining fields into transverse and longitudinal parts. Next, one has to fix a gauge for every first class constraints. For $\pi^o_a = 0$ it is $A^a_o = 0$, and for C_a as in the Yang-Mills case $\mathcal{X}^a \equiv \partial_i A^{ia} = 0$. Then to replace the Poisson brackets with the Dirac ones.

Now we are able to write the set of physical variables (P^*, q^*) as

$$(P^*, q^*) = (\pi^{iT}_a, P^{iT}_a, P^{iL}_a, A^{aT}_i, B^{aL}_i) \qquad (6)$$

In constructing the functional integral, the first thought is to write it in terms of physical variables. But this is not a good idea because the physical Hamiltonian is a highly nonlinear function of corresponding variables (P^*, q^*)

$$< \text{out} \,|S|\, \text{in}> = \int dP^* dq^* e^{i \int d^4x \,[P^* \dot{q}^* - h^* (q^*, P^*)]} \qquad (7)$$

This is why we usually derive the functional integral over the entire phase space. In doing so one must reintroduce unphysical degrees A_i^{aL}, π_a^{iL} and B_o^a back to (7)

$$
\langle out | S | in \rangle = \int dP^* \, dq^* \, dA_i^{aL} \, d\pi_a^{iL} \, dB_o^a \, \delta(A_i^{aL}) \, \delta(\pi_a^{iL} - \bar{\pi}_a^{iL}) \cdot
$$

$$
\cdot \, \delta(B_o^a - \bar{B}_o^a) \exp i \int d^4x \, [p^* \dot{q}^* + \pi_a^{iL} \dot{A}_i^{aL} - \mathcal{H}(P^*, q^*, A_i^{aL}, \tag{8}
$$

$$
\pi_a^{iL}, B_o^a)] .
$$

By changing variables in the δ functionals $\pi_a^{iL} \to C_a$, $A_i^{aL} \to \chi^a$ and $B_o^a \to D^a$, and by simple inspection of our constraints and the gauge conditions one finds

$$
\langle out | S | in \rangle = \int dA_i^a \, d\pi_a^i \, dB_\mu^a \, dP_a^i \, \delta(\chi^a) \det | [\chi^a, C_b] |
$$

$$
\delta(C_a) \, \delta(D_a) \exp i \int d^4x \, [\pi_a^i \dot{A}_i^a + P_a^i \dot{B}_i^a - \mathcal{H}(A_i^a, B_\mu^a, \pi_a^i, P_a^i)] \tag{9}
$$

We now exponentiate $\delta(C_a)$ as $dA_o^a \, e^{iA_o^a C_a}$ and finally obtain

$$
\langle out | S | in \rangle = \int \exp i \int d^4x \, [\pi_a^i \dot{A}_i^a + P_a^i \dot{B}_i^a - \mathcal{H}_c] \cdot
$$

$$
\cdot \, dA_\mu^a \, dB_\mu^a \, d\pi_a^i \, dP_a^i \, \delta(\chi^a) \det | \{ C^a, \chi^b \} | \, \delta(D_a) \tag{10}
$$

The authors in ref. [5] claim that the presence of the second class constraints in the functional integral through the δ - function prevents us from geting a manifestly relativistic integral. But, as it was shown by Senjanović [6] the presence of second class constraints does not spoil manifest relativistic invariance, their presence only means that we cannot use directly the Faddeev-Poppov method. Let us show how to obtain a manifestly relativistic functional integral.

One can rewrite the integral (10) as

$$
\langle out | S | in \rangle = \int \exp i \int d^4x \, | \pi_a^i \dot{A}_i^a + P_a^i \dot{B}_i^a - \mathcal{H}_c + \lambda^a D_a | \cdot
$$

$$\cdot\, dA^a_\mu\, dB^a_\mu\, d\pi^i_a\, dP^i_a\, \delta(\chi^a)\, \det|\{C^a, \chi^b\}|\, d\lambda^a \qquad (11)$$

Using (3) and (5), after changing variables $B^a_o \to B^a_o - \lambda^a$ into (11) and performing Gaussian integration over λ^a, P^i_a and π^i_a, we obtain [4]

$$\langle out | S | in \rangle = \int \exp i \int d^4x \mathcal{L}(x)\, dA^a_\mu\, dB^a_\mu\, \delta(\chi^a)\, \det|\{C^a, \chi^b\}| \qquad (12)$$

As it could be seen from (12) the proposed theory is relativistically invariant in spite of the presence of the second class constraints.

References

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