# CANONICAL STRUCTURE OF POINCARÉ GAUGE-INVARIANT THEORY OF GRAVITY

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## 1. Introduction

The classical Poincaré gauge-invariant theory of gravity[1] (PGITG) is a natural extension of Einstein's general theory of relativity which admits torsion propagation and offers an atractive possibility to avoid gravitational singularities[2]. From the (perturbative) quantum-field-theoretic point of view, PGITG cannot be simultaneously renormalizable and unitary[3], and should be regarded as an effective theory.

In this work, we investigate the Hamiltonian formulation of the theory using Dirac's method for constrained systems[4]. Here, we will confine ourselves to kinematical aspects of PGITG, i.e. to general structure of the Hamiltonian and the Poisson brackets between the first class constraints[5]. Although the second class constraints are not completely analysed in the literature, a general setting can be found in Ref. 6.

1. Brief review of PGITG. The basic gravitational fields in the theory are the tetrads $b^k{}_\mu$ and the spin connections $A^{ij}{}_\mu$, whereas the corresponding field strenghts are the torsion $T^k{}_{\mu\nu}$ and the curvature $R^{ij}{}_{\mu\nu}$

$$
\begin{aligned}
T^k{}_{\mu\nu} &\equiv 2\nabla_{[\nu} b^k{}_{\mu]} = \nabla_\nu b^k{}_\mu - (\mu \leftrightarrow \nu) \\
&\equiv 2\,(b^k{}_{\mu,\nu} + A^k{}_{l[\nu}\, b^l{}_{\mu]}) \qquad (1)\\
R^{ij}{}_{\mu\nu} &= 2\,(A^{ij}{}_{[\mu,\nu]} + A^i{}_{n[\nu}\, A^{nj}{}_{\mu]}) .
\end{aligned}
$$

The Latin indices are local Lorentz (anholonomic) indices, whereas the Greek indices are coordinate (holonomic) indices. Both types of indices can be transferred into each other by using the tetrads and the inverse tetrad fields $h_k{}^\mu$

$$h_k{}^\mu\, b^k{}_\nu = \delta^\mu{}_\nu . \qquad (2)$$

For example: $T^{k}{}_{lm} = h_{l}{}^{\mu} h_{m}{}^{\nu} T^{k}{}_{\mu\nu}$. Note that contravariant metric is defined by:

$$g_{\mu\nu} = b^{l}{}_{\mu} b^{k}{}_{\nu} \eta_{kl}, \qquad \sqrt{-g} = \det b^{k}{}_{\mu} \equiv b \tag{3}$$

where $\eta_{ij} = \mathrm{diag}\,(1, -1, -1, -1)$. Covariant derivative of an arbitrary matter field u is defined by

$$D_{k} u \equiv h_{k}{}^{\mu} \nabla_{\mu} u \equiv h_{k}{}^{\mu} (\partial_{\mu} + \tfrac{1}{2} A^{ij}{}_{\mu} S_{ij})\, u \tag{4}$$

where $S_{ij}$ are the Lorentz group generators in the u-representation.

The most general PGITG action is given by:

$$S = \int d^4 x\, b\, \mathcal{L}(u,\, D_k u\, ,\, T^{k}{}_{lm},\, R^{ij}{}_{kl})$$

where $\mathcal{L}$ is a scalar function. This action can be further restricted assuming minimal coupling: $\mathcal{L} = \mathcal{L}^{M}(u,\, D_k u) + \mathcal{L}^{G}(T^{k}{}_{lm},\, R^{ij}{}_{kl})$ and at most second order derivatives in equations of motion. The latter assumption leads to a Lagrangian which is at most quadratic in the torsion and curvature and depends on nine arbitrary parameters[7]. This and other possible physical requirements (positiveness of the energy, standard gravitational tests, etc.) are not important for our further exposition.

2. Hamiltonian and the first class constraints. Let us denote by $\pi, \pi_{k}{}^{\mu}$ and $\pi_{ij}{}^{\mu}$ the momenta which correspond to u, $b^{k}{}_{\mu}$ and $A^{ij}{}_{\mu}$, respectively. Since the torsion and the curvature are antisymmetric in the derivatives $b^{k}{}_{\mu,\nu}$ and $A^{ij}{}_{\mu,\nu}$, one easily obtains the following primary constraints:

$$\pi_{k}{}^{0} \approx 0 \qquad\qquad \pi_{ij}{}^{0} \approx 0 \tag{5}$$

If some of the above mentioned nine parameters in the Lagrangian take on certain critical values, further primary constraints appear, and they diminish the number of propagating fields. Without loss of generality we assume that there are no extra primary constraints.

For a Hamiltonian approach in the curved space-time it is very convenient to use the ADM basis $(\vec{n}, \vec{e}_{\bar{\alpha}})$, $\alpha = 1, 2, 3$ instead of the local Lorentz $(\vec{u}_k)$ or the general coordinate basis $(\vec{e}_\mu)$. Here, $\vec{n}$ is the unit normal to the hypersurface $x^0 = \text{const}$ (which is spanned by $\vec{e}_\alpha$). Any vector, say $\vec{D}$, can be decomposed as

$$\vec{D} = D_\perp \vec{n} + D^{\bar{\alpha}} \vec{e}_\alpha \equiv (n^\mu D_\mu)\vec{n} + ({}^3g^{\alpha\beta} D_\beta)\vec{e}_\alpha$$
$$n^\mu = h_k{}^\mu n^k, \quad n_k \equiv h_k{}^0 / \sqrt{g^{00}} \tag{6}$$

where ${}^3g^{\alpha\beta}$ is inverse to $g_{\alpha\beta}$ ; from now on, we assume $\alpha, \beta, \gamma, \delta = 1, 2, 3$. Since $\vec{e}_0$ determines a direction of the time evolution of the system, it is very useful to introduce lapse and shift functions $N$ and $N^\alpha$, respectively:

$$(\vec{e}_0)_\perp \quad N = 1 / \sqrt{g^{00}} \equiv b^k{}_0 n_k$$
$$(\vec{e}_0)^{\bar{\alpha}} \equiv N^\alpha = - g^{0\alpha} / g^{00} \equiv {}^3g^{\alpha\beta} b_{k\beta} b^k{}_0 \tag{7}$$

which are linear functions of $b^k{}_0$ (note that $n^k$, given by (6), is independent of $b^k{}_0$).

Using the above decomposition, the canonical Hamiltonian density ($\mathcal{H}_{can} \equiv \pi^A \dot{Q}_A - b\mathcal{L}$) can be written in the form

$$\mathcal{H}_{can} = N\mathcal{H}_\perp + N^\alpha\mathcal{H}_\alpha - \frac{1}{2} A^{ij}{}_0 \mathcal{H}_{ij} + D^\alpha{}_{,\alpha} \tag{8}$$

where $D^\alpha \equiv b^k{}_0 \pi_k{}^\alpha + \frac{1}{2} A^{ij}{}_0 \pi_{ij}{}^\alpha$, and

$$\mathcal{H}_{ij} = \pi S_{ij} u + 2\pi_{[i}{}^\alpha b_{j]\alpha} + \nabla_\alpha \pi_{ij}{}^\alpha,$$
$$\mathcal{H}_\alpha = T^0{}_\alpha - b^k{}_\alpha \nabla_\beta \pi_k{}^\beta ; \quad \mathcal{H}_\perp = T^0{}_\perp - n^k \nabla_\beta \pi_k{}^\beta , \tag{9}$$
$$T^0{}_\mu \equiv \pi\nabla_\mu u + \pi_k{}^\alpha T^k{}_{\alpha\mu} + \frac{1}{2}\pi_{kl}{}^\alpha R^{kl}{}_{\alpha\mu} - \delta^0_\mu b\mathcal{L},$$

Since $T^0{}_\mu$ is the covariant canonical energy-momentum tensor (EMT) and $\mathcal{H}_{ij}$ is the canonical spin tensor $(\sigma^0{}_{ij})$, the equation (8) is nothing but a consequence of certain identities in PGITG [9].

An important feature of the above decomposition is that $\mathcal{H}_{ij}$, $\mathcal{H}_\alpha$ and $\mathcal{H}_\perp$ are simultaneously independent of velocities and $b^k{}_o{}^\alpha$ [5] (since $T^o{}_o$ is Legendre's transformation of L with respect to $\nabla_\perp u$, $T^k{}_{\alpha\perp}$ and $R^{ij}{}_{\alpha\perp}$). Therefore, consistency conditions of the primary constraints (5) result in the following secondary constraints

$$\mathcal{H}_{ij} \approx 0 \ , \quad \mathcal{H}_\alpha \approx 0 \ , \quad \mathcal{H}_\perp \approx 0 \tag{10}$$

which means that the Hamiltonian $H = \int \mathcal{H} \, d^3 x$ vanishies weakly (as a consequence of the general coordinate invariance of the PGITG).

3. Poisson brackets (PB) between constraints. Another important feature of the above decomposition (8) is that the PB between constraints have a "universal form" [10],

$$\{\mathcal{H}_{ij}, \mathcal{H}'_{kl}\} = \frac{1}{2} f_{ij}{}^{mn}{}_{kl} \mathcal{H}_{mn} \, \delta(\vec{x} - \vec{x}') \tag{11}$$

$$\{\mathcal{H}_{ij}, \mathcal{H}'_\alpha\} = 0 \ , \quad \{\mathcal{H}_{ij}, \mathcal{H}_\perp'\} = 0 \tag{12}$$

$$\{\mathcal{H}_\alpha, \mathcal{H}'_\beta\} = (\mathcal{H}'_\alpha \partial_\beta + \mathcal{H}_\beta \partial_\alpha + \frac{1}{2} R^{ij}{}_{\alpha\beta} \mathcal{H}_{ij}) \, \delta(\vec{x} - \vec{x}') \tag{13}$$

$$\{\mathcal{H}_\alpha, \mathcal{H}_\perp'\} = (\mathcal{H}_\perp \partial_\alpha + \frac{1}{2} R^{ij}{}_{\alpha\perp} \mathcal{H}_{ij}) \, \delta(\vec{x} - \vec{x}') \tag{14}$$

$$\{\mathcal{H}_\perp, \mathcal{H}_\perp'\} = - ({}^3g^{\alpha\beta} \mathcal{H}_\alpha + {}^3g'^{\alpha\beta} \mathcal{H}'_\alpha) \, \partial_\beta \, \delta(\vec{x} - \vec{x}') \tag{15}$$

which is of great importance if one attempts to quantize the theory. Since the proof of the above PB requires a rather hard algebra, let us just make few comments on them.

Equation (11) shows that $\mathcal{H}_{ij}$ satisfies the Lorentz group algebra, whereas (12) is a consequence of the fact that $\mathcal{H}_\alpha$ and $\mathcal{H}_\perp$ are Lorentz invariant constraints. Brackets (13) can be directly verified since $\mathcal{H}_\alpha$ are explicitely known functions of fields and momenta. Equation (14) can be proved by exploiting chain rule of PB's: $\{\mathcal{H}_\alpha, \mathcal{H}'_\perp\} = \{\mathcal{H}_\alpha, \xi'^A\} \frac{\partial \mathcal{H}_\perp'}{\partial \xi'^A}$, where $\xi^A$

denotes the arguments of (implicitely known) lapse Hamiltonian:
$\mathcal{H}_\perp(\xi^A) = \mathcal{H}_\perp(u, \nabla_\alpha u, \pi/J, T^k{}_{\alpha\beta}, \pi_k{}^\alpha/J, R^{ij}{}_{\alpha\beta}, \pi_{ij}{}^\alpha/J, b^k{}_\alpha)$
where $J = b/N$.

The most important PB are those between $\mathcal{H}_\perp$ and $\mathcal{H}_\perp'$, since $\mathcal{H}_\perp$ governs the dynamical evolytion of the theory (it is the only constraint which depends on the choice of the Lagrangian). Using again the chain rule

$$\{\mathcal{H}_\perp, \mathcal{H}_\perp'\} = \frac{\partial \mathcal{H}_\perp}{\partial \xi^A}\{\xi^A, \xi'^B\}\frac{\partial \mathcal{H}_\perp'}{\partial \xi'^B} - (x \leftrightarrow x')$$

and keeping only terms proportional to the derivatives of the $\delta$-function, one obtains the equation (15) (with the help of already mentioned identity, which relates canonical EMT, $T^\mu{}_\nu$ and $\sigma^\mu{}_{ij}$).

If there are another second class constraints in the theory, one should replace PB's by new, Dirac's brackets. That is the reason why terms quadratic in the constraints may appear on the right hand sides of the above PB [11]

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