# A NEW APPROACH TO THE STATISTICAL THEORY OF NUCLEAR REACTIONS*

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A statistical theory of nuclear reactions is constructed by proposing a trial distribution for the $S$-matrix, whose parameters, determined through a variational principle, depend only upon the optical $\overline{S}$-matrix. These are the quantities that are physically relevant when equilibrium has been reached, and they are thus introduced from the outset. Results are excellent for the average and variance of the cross section in the region of strong absorption. The extension to weaker absorption is considered in a 2-channel problem.

In most statistical theories of nuclear reactions (see Ref. 1-4 and references contained therein), the $S$-matrix is written in terms of microscopic quantities (like the poles and residues of the $S$ or of the $K$-matrix, or the matrix elements of the underlying Hamiltonian), for which a statistical law is assumed and used to calculate the fluctuation cross section $\sigma_{ab}^{fl} \sim \overline{|S_{ab}^{fl}|^2}$ (here $S = \overline{S} + S^{fl}$). This, added to the direct (optical) term, gives the average cross section $\overline{\sigma}_{ab}$. If the compound system has reached equilibrium, $\sigma^{fl}$ can be expressed entirely in terms of the macroscopic quantities $\overline{S}_{ab}$, i.e. the optical $S$-matrix elements, as in the familiar Hauser-Feshbach[5] theory, in which $\sigma_{ab}^{fl}$ is calculated in terms of the transmission factors, computed from the optical model for each channel.

Thus, at least in the case of $\sigma_{ab}^{fl}$, the microscopic quantities play the role of a scaffolding, which is eliminated at the end in favor of the macroscopic quantities $\overline{S}_{ab}$. It is natural to ask[7] whether the scaffolding can be eliminated from the very beginning, by proposing a trial statistical law directly for the $S$-matrix elements, the input to the problem being the exact expectation values $\overline{S}_{ab}$. In other words, we shall attempt to construct an ensemble of $S$-matrices[7], such that the physically relevant quantities can be obtained as ensemble averages.

A convenient measure[6,7] for unitary and symmetric matrices $S$ is provided by Dyson's measure[8] $\mu(dS)$, defined uniquely by the property of remaining invariant under the transformation $S = USU^T$ ($U$ being any unitary matrix), which in turn preserves the property of unitarity and symmetry. In a simple 1-channel problem ($S = exp(i\theta)$), $\mu(dS) = d\theta$.

In the case of complete absorption, $\overline{S} = 0$, we shall make the assumption[7] that the frequency of occurrence of $S$ in a subspace of the space of unitary and symmetric matrices, is proportional to Dyson's measure for that subspace. The results obtained in this limit for the average[7,11] and variance[14] of the elastic and inelastic cross sections agree with those obtained in Refs. 1-4.

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For the general case $\overline{S} \neq 0$, one can propose a trial probability density for $S$, containing $k$ functions $f_i(S)$ and $k$ parameters; the latter can be varied so as to give the "best" trial probability density, in the sense of a variational principle[9]; one basically finds the best upper bound to the entropy of the exact distribution, and when this is reached, the expectation values of the $f_i(S)$ associated with the exact and trial probability densities coincide. If we reached the exact entropy, then the trial and exact distributions would coincide. In our case the functions $f_i(S)$ are chosen[7] as the real and imaginary parts of the various matrix elements $S_{ab}$, so that we fix the expectation value $\overline{S}$, i.e. the optical $S$-matrix. The trial distribution is then

$$p(dS) = \frac{exp\ (-ReTr\beta S) \cdot \mu(dS)}{\int exp(-ReTr\beta S) \cdot \mu(dS)}\ , \qquad (1)$$

where the matrix $\beta$ has to be fixed so that $\overline{S}$ has the required value.

When $\overline{S} \neq 0$ but small, one can expand the exponential in (1) in a power series and keep only the lowest terms. We shall restrict the discussion to the case $\overline{S}$ = diagonal and real, the most general $\overline{S}$ being obtainable by an Engelbrecht-Weidenmüller transformation[10]. In this strong-absorption regime, $\overline{|S^{\delta\ell}_{ab}|^2}$ (for $a \neq b$) factorizes as $\xi_a \cdot \xi_b$; for $a = b$ we can define[3] an elastic enhancement factor $W_a$ through $\overline{|S^{\delta\ell}_{aa}|^2} = \xi_a^2\ W_a$. Then the $\xi_a$ are determined entirely[3] by unitarity and the $W$'s. The latter can be written[11] as $W_a = 2 + k\ |\overline{S}_{aa}|^2 + \ldots$ The factor $k$ from the present formalism[11] and from Ref. 3 (which gives a fit to a Monte Carlo calculation) are compared in Table I, where $n$ indicates the number of channels. The agreement in seen to be very reasonable.

TABLE I

| $n$ | (variational principle) | (HRTW) |
|---|---|---|
| 2 | 0.60 | 0.53 |
| 3 | 0.44 | 0.40 |
| 5 | 0.30 | 0.30 |
| 10 | 0.16 | 0.23 |

The cross average $\overline{S^{\delta\ell}_{11}\ S^{\delta\ell *}_{22}}$ is, in this regime, identically zero in Ref. 1, whereas the present method and that of Ref. 3 give a result proportional to $\overline{S}_{11} \cdot \overline{S}_{22}^{*}$; the factor of proportionality[11] is compared in Table II, that shows a very good agreement.

TABLE II

| $n$ | variational method | HRTW |
|---|---|---|
| 2 | 0.200 | 0.197 |
| 5 | 0.050 | 0.055 |
| 10 | 0.015 | 0.015 |

Away from the strong-absorption region, a simple two-equivalent-channel case was considered[7]. The elastic enhancement starts from 2 for complete absorption, it increases as in Ref. 3, but then it goes back to 2 for weak absorption, in disagreement with the results of Refs. 2 and 3 where, despite the error bars, an increase in $W$ from about 2 to about 3 is expected, as we go from strong to weak absorption. There is, however, a restriction which we have not taken into account so far. From the analytical structure of $S$ and if one has ergodicity[12], one can prove[11] that the average of products of $S$-matrix elements (involving only $S$ but not $S^*$) must coincide with the product of the averages of the various factors. One can prove that this condition is exactly fullfilled for Dyson's measure $\mu(dS)$, but is gradually spoiled as we go away from this extreme. However, it can be incorporated as a set of new restrictions, by means of Lagrange multipliers. This has been done[13] for quadratic combinations of $S$-matrix elements in the two-channel problem mentioned above, and the results indeed go in the right direction.

One can also calculate the variance of the cross section[14], var $\sigma_{ab}$, for any number $n$ of channels in the region of strong absorption. If $n \gg 1$, var $\sigma_{ab} \approx (\bar{\sigma}_{ab})^2$ as in Ericson's theory[15], meaning essentially that $Re S_{ab}$ and $Im\, S_{ab}$ become Gaussian for $n \gg 1$. For arbitrary $n$, we predict[14] corrections to this result.

We may thus summarize by saying that starting from the statistical ansatz $\mu(dS)$ and including successively known physical constraints it is possible to obtain a satisfactory theory for the domain of strong absorption, and the two-channel example gives indications how this theory can be completed for the general case.

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