

RELATIVISTIC RELATIVE KINEMATIC QUANTITIES
AND SOME OF THEIR APPLICATIONS

I. Lukačević

Department of Mathematics, Mechanics and Astronomy, Faculty of
Sciences, Beograd

S U M M A R Y

This paper gives, in an abridged form, the essential exposition of relative deformation and rotation tensors. The question about the number of local dimensions of these tensors is considered, especially from the standpoint of the absence of relative deformation in spacetime as a consequence of initial data. A first application is made, of these results, to a MHD fluid without a specified energy tensor (only Maxwell equations of the electromagnetic field are given). A second application is made in cosmology. Several types of cosmological metrics are considered and determined under different conditions on relative deformation or rotation.

We use, in this work, the relative deformation and rotation (or vorticity) introduced in our previous papers ^{7,10}. These quantities extend the well known tensors of proper deformation $\epsilon_{\alpha\beta}$ and rotation $\omega_{\alpha\beta}$, corresponding to a congruence $u^\alpha (u^\alpha u_\alpha = -1)$ of world lines, in the sense that they are obtained by coupling two congruences of world lines, u^α and $\xi^\alpha (u^\alpha u_\alpha = \xi^\alpha \xi_\alpha)$ with their proper spacelike elements. The conclusions are such that they can be indistinctly applied to both special and general relativity.

* * *

Relative kinematic quantities we consider are:

$$v_{\alpha\beta} = L_{\xi}(g_{\alpha\beta} + u_{\alpha}u_{\beta}) = \nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha} + u_{\alpha}\xi^{\gamma}\nabla_{\gamma}u_{\beta} + u_{\beta}\xi^{\gamma}\nabla_{\gamma}u_{\alpha} + \\ + u_{\alpha}u^{\gamma}\nabla_{\beta}\xi_{\gamma} + u_{\beta}u^{\gamma}\nabla_{\alpha}\xi_{\gamma} \quad (1.1)$$

$$\tilde{v}_{\alpha\beta} = \nabla_{\alpha}\xi_{\beta} - \nabla_{\beta}\xi_{\alpha} + u_{\alpha}\xi^{\gamma}\nabla_{\gamma}u_{\beta} - u_{\beta}\xi^{\gamma}\nabla_{\gamma}u_{\alpha} + u_{\alpha}u^{\gamma}\nabla_{\beta}\xi_{\gamma} - u_{\beta}u^{\gamma}\nabla_{\alpha}\xi_{\gamma} \quad (1.2)$$

where L_{ξ} denotes the Lie derivative with respect to ξ^{α} , ∇ and $\tilde{\nabla}$ denoting respectively covariant and partial derivation. Relative kinematical parameters $v_{\alpha\beta}$ and $\tilde{v}_{\alpha\beta}$, correspond to proper ones, already mentioned:

$$v_{\alpha\beta} = \nabla_{\alpha}u_{\beta} + \nabla_{\beta}u_{\alpha} + u_{\alpha}u^{\gamma}\nabla_{\gamma}u_{\beta} + u_{\beta}u^{\gamma}\nabla_{\gamma}u_{\alpha} \quad (1.3)$$

$$\tilde{v}_{\alpha\beta} = \nabla_{\alpha}u_{\beta} - \nabla_{\beta}u_{\alpha} + u_{\alpha}u^{\gamma}\nabla_{\gamma}u_{\beta} - u_{\beta}u^{\gamma}\nabla_{\gamma}u_{\alpha} \quad (1.4)$$

We shall add, in order to be complete, the conjugates of above tensors:

$$\tilde{v}_{\alpha\beta} = L_u(g_{\alpha\beta} + \xi_{\alpha}\xi_{\beta}) = \tilde{\nabla}_{\alpha}\xi_{\beta} + \tilde{\nabla}_{\beta}\xi_{\alpha} + \\ + \xi_{\alpha}u^{\gamma}\tilde{\nabla}_{\gamma}\xi_{\beta} + \xi_{\beta}u^{\gamma}\tilde{\nabla}_{\gamma}\xi_{\alpha} + \xi_{\alpha}\xi^{\gamma}\tilde{\nabla}_{\gamma}u_{\beta} + \xi_{\beta}\xi^{\gamma}\tilde{\nabla}_{\gamma}u_{\alpha}, \quad (1.5)$$

$$\tilde{\tilde{v}}_{\alpha\beta} = \tilde{\nabla}_{\alpha}u_{\beta} - \tilde{\nabla}_{\beta}u_{\alpha} + \xi_{\alpha}u^{\gamma}\tilde{\nabla}_{\gamma}\xi_{\beta} - \xi_{\beta}u^{\gamma}\tilde{\nabla}_{\gamma}\xi_{\alpha} + \\ + \xi_{\alpha}\xi^{\gamma}\tilde{\nabla}_{\beta}u_{\gamma} - \xi_{\beta}\xi^{\gamma}\tilde{\nabla}_{\alpha}u_{\gamma} \quad (1.6)$$

and:

$$\tilde{v}_{\alpha\beta} = \tilde{\nabla}_{\alpha}\xi_{\beta} + \tilde{\nabla}_{\beta}\xi_{\alpha} + \xi_{\alpha}\xi^{\gamma}\tilde{\nabla}_{\gamma}\xi_{\beta} + \xi_{\beta}\xi^{\gamma}\tilde{\nabla}_{\gamma}\xi_{\alpha} \quad (1.7)$$

$$\tilde{\tilde{v}}_{\alpha\beta} = \tilde{\nabla}_{\alpha}\xi_{\beta} - \tilde{\nabla}_{\beta}\xi_{\alpha} + \xi_{\alpha}\xi^{\gamma}\tilde{\nabla}_{\gamma}\xi_{\beta} - \xi_{\beta}\xi^{\gamma}\tilde{\nabla}_{\gamma}\xi_{\alpha} \quad (1.8)$$

The study of the pairs $v_{\alpha\beta}$, $\tilde{v}_{\alpha\beta}$ and $\tilde{v}_{\alpha\beta}$, $\tilde{\tilde{v}}_{\alpha\beta}$ is obviously sufficient, the conclusions for $\tilde{v}_{\alpha\beta}$, $\tilde{\tilde{v}}_{\alpha\beta}$ and $\tilde{v}_{\alpha\beta}$, $\tilde{\tilde{v}}_{\alpha\beta}$ being symmetric, except in some questions concerning relations between these pairs.

The first question that arises is related to the local number of dimensions of these tensors, exceptionnally to the case of the absence of deformation or rotation.

We start with an identity concerning second order derivatives of a tensor (in our case it is the proper spacelike metric tensor) with respect to two different vector fields. This identity reads (cf (1)):

$$L_U L_\xi (g_{\alpha\beta} + u_\alpha u_\beta) - L_\xi L_U (g_{\alpha\beta} + u_\alpha u_\beta) = L_\lambda (g_{\alpha\beta} + u_\alpha u_\beta) \quad (19)$$

where:

$$\lambda^\alpha = L_U \xi^\alpha.$$

We shall restrict ourselves to the case when fields u^α and ξ^α are two-surface forming, i.e.:

$$L_U \xi^\alpha = a u^\alpha + b \xi^\alpha \quad (1.10)$$

where a and b are arbitrary scalar functions. Then, having in mind (1) and (10), relation (9) takes the form:

$$\begin{aligned} L_U \nu_{\alpha\beta} - L_\xi c_{\alpha\beta} &= a c_{\alpha\beta} + b \nu_{\alpha\beta} + \\ &+ (\xi_\alpha + \varepsilon u_\alpha) \partial_\beta b + (\xi_\beta + \varepsilon u_\beta) \partial_\alpha b, \end{aligned} \quad (1.11)$$

where:

$$\varepsilon = g_{\alpha\beta} u^\alpha \xi^\beta$$

After a scalar multiplication of (11) by u^α , we shall have:

$$\begin{aligned} L_U (\nu_{\alpha\beta} u^\beta) - b c_{\alpha\beta} \xi^\beta &= b \nu_{\alpha\beta} u^\beta + \\ &+ (\xi_\alpha + \varepsilon u_\alpha) u^\beta \partial_\beta b, \end{aligned} \quad (1.12)$$

or

$$L_U (\xi_\alpha + \varepsilon u_\alpha) - c_{\alpha\beta} \xi^\beta = b (\xi_\alpha + \varepsilon u_\alpha). \quad (1.13)$$

After another scalar multiplication we draw the conclusion that:

$$\theta u^\alpha u^\beta \nabla_\alpha \xi_\beta - \xi^\alpha \xi^\beta \nabla_\alpha u_\beta = b(\theta^2 - 1) \quad (1.14)$$

wherefrom we are able to determine the scalar b .

1) We draw, first, from (13), the conclusion that for:

$$\sigma_{\alpha\beta} \xi^\beta = 0$$

i.e. for a proper deformation tensor that has locally no components in the two-planes u^α, ξ^α ($\sigma_{\alpha\beta}$ is always orthogonal to u^α), the nullity of b has the consequence that the Lie derivative of $\xi_\alpha + \epsilon u_\alpha$ vanishes. Conversely, for a vanishing Lie derivative of $\xi_\alpha + \epsilon u_\alpha$ and $b = 0$, $\sigma_{\alpha\beta}$ has no components in any u^α, ξ^α plane.

2) For b constant along u^α world lines, with a two dimensional $\sigma_{\alpha\beta}$ as previously, and $(\sigma_{\alpha\beta} u^\beta)_0 = 0$, on an initial hypersurface, we have the consequence, from (12), that:

$$\sigma_{\alpha\beta} u^\beta = 0$$

along world lines u^α that leave the initial hypersurface.

* * *

We shall apply, first, these conclusions to MHD.

Consider a magnetohydrodynamic electromagnetic field in a fluid described by Maxwell equation (cf. ⁶) which are then of the form:

$$\nabla_\alpha (u^\alpha h^\beta - u^\beta h^\alpha) = 0 \quad (2.1)$$

$$\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \nabla_\beta (h_\gamma u_\delta - h_\delta u_\gamma) = j^\alpha \quad (2.2)$$

where u^α is the four velocity, h^α the magnetic field and j^α the electric current. Other constitutive relations can have any form (we do not restrict ourselves even to a perfect fluid) we ascribe to them.

The first group of Maxwell equations (2.1), when developed, tells us, in fact, that streamlines u^α and magnetic field lines h^α form two-surfaces. We shall assume that ξ^α world lines are contained in

these surfaces. Having in mind that then;

$$h^\alpha = h/\sqrt{\theta^2-1} (\xi^\alpha + \theta u^\alpha), \quad h = |h^\alpha|$$

we can write equations (2.1) as:

$$L_\xi u^\beta = \sigma_\alpha^\alpha + u^\alpha \partial_\alpha \ln(h/\sqrt{\theta^2-1}) |(\xi^\beta + \theta u^\beta) + u^\beta u^\alpha u^\gamma \nabla_\alpha \xi_\gamma| \quad (2.3)$$

where σ_α^α is the expansion corresponding to $\sigma_{\alpha\beta}$. We have from (2.3):

$$v_\alpha^\alpha + \xi \sigma_\alpha^\alpha = h^\alpha \partial_\alpha \ln(\sqrt{\xi^2-1}/h), \quad (2.4)$$

where v_α^α is the expansion corresponding to $v_{\alpha\beta}$. Proper incompressibility means $\sigma_\alpha^\alpha = 0$, and relative on $v_\alpha^\alpha = 0$. Then from (2.4):

1) For a MHD fluid having a constant ratio $h/\sqrt{\theta^2-1}$ along magnetic field lines, the relative expansion v_α^α is proportional to the proper expansion σ_α^α , and they are of the same sign with $|v_\alpha^\alpha| \geq |\sigma_\alpha^\alpha|$ (the equality holding only for $v_\alpha^\alpha = \sigma_\alpha^\alpha = 0$).

And from (1.9) and (2.3):

2) When either the proper or the relative deformation tensor is orthogonal to the magnetic field, with the same condition on the other one, given on a spacelike hypersurface Σ or Σ' , then both are orthogonal to the stream and the magnetic field lines along world lines u^α , resp., u'^α , which intersect with Σ , resp., Σ' .

3) For a fluid having $h/\sqrt{\xi^2-1}$ constant in two-surfaces u^α , ξ^α , with a vanishing σ_α^α (or v_α^α relative vorticity tensor $\Omega_{\alpha\beta}$ is orthogonal to the stream lines.

*
* * *

Another field of application for our kinematic parameters is cosmology.

1) Consider the case of de Sitter metric (cf. [5]):

$$c ds^2 = \frac{dr^2}{1-r^2/R^2} + r^2(dv^2 + \sin^2 v ds^2) - c^2(1-r^2/R^2) dt^2$$

with an observer moving along the radial coordinate line, whose world lines are:

$$t' = t + \cdot (r)$$

if we set up the condition:

$$\gamma_{rr} = 0, \quad (3.1)$$

we shall obtain:

$$t' = t + D \ln \left(E \frac{1+r/R}{1-r/R} \right), \quad \dot{t}' = c$$

$$r' = 4c^2 Dt + R^2 \ln \left(E \frac{1+r/R}{1-r/R} \right), \quad \dot{r}' = c$$

the tangent unit vectors ξ^a of t' being:

$$\xi^a = \left(\frac{c}{\sqrt{1-r^2/R^2}}, 0, 0, \frac{-R^2}{c\sqrt{1-r^2/R^2}} \right),$$

$$a = (R^2 - 4c^2 D^2)^{-1}, \quad R^2 \geq c D.$$

The metric of the observer is then:

$$\begin{aligned} c ds^2 = & \left(\exp \left[a^2 (r'^2 - 4c^2 Dt') \right] + E'^{-2} \left[4a^2 E \exp \left[a^2 (r' - \right. \right. \right. \\ & \left. \left. \left. - 4c^2 Dt') \right] (dr'^2 - c^2 R^2 dt'^2) + R^2 \left[\exp \left[a^2 (r' - \right. \right. \right. \right. \\ & \left. \left. \left. - E'^2 (dv'^2 + \sin^2 v' ds'^2) \right] \right). \end{aligned} \quad (3.2)$$

This metric has the property of expansion in the (r', t') directions and of contraction in the (v', s') directions. The world lines t intersect with t' under constant pseudoangles.

2) For the generalized Lemaitre metric:

$$c ds^2 = a^2(t) \left[f(r) dr^2 + r^2 (dv^2 + \sin^2 v ds^2) \right] - c^2 dt^2,$$

we have, for vanishing cyclic components of relative deformation:

$$v_{\dot{\epsilon}\dot{\epsilon}} = 0 \iff v_{\phi\phi} = 0 \Rightarrow a^2(t) = k^2 t + l^2 \quad (3.3)$$

So the expansion of the metric must be, under conditions (3.3), a linear function of time. Then v_{rt} , v_{rr} and Ω_{rt} are the only components of deformation and rotation different from zero. On the other hand, the absence of radial deformation $v_{rr} = 0$ would lead us to the absence of expansion of the metric.

The condition $\tilde{\sigma}_{rr} = 0$, where $\tilde{\sigma}_{\alpha\beta}$ is the conjugate proper deformation (1.7) leads us, after simple but very long calculations, to the conclusion that the metric could not be in expansion, the coefficient in the metric form being constant ($a^2/H = \text{const}$).

3) For a more generalized form of a cosmological metric (cf. Peebles [4]):

$$c ds^2 = A(r,t) dr^2 + B(r,t) (d\phi^2 + \sin^2 \phi d\epsilon^2) - c^2 dt^2$$

we have:

$$v_{rr}, v_{\phi\phi}, v_{\epsilon\epsilon}, v_{rt} \neq 0, \Omega_{rt} \neq 0.$$

Then the condition of the absence of rotation leads us to:

$$\Omega_{rt} = 0 \Rightarrow v_{rt} = 0 \iff A = A(r) \quad (3.4)$$

giving the necessary and sufficient condition for the existence of a relatively nonrotating congruence of world lines ξ^α having a purely spacelike diagonal relative deformation.

REFERENCES

1. K. Yano: The theory of Lie derivatives, North-Holland (1957).
2. J. Synge: Relativity, the special theory, North-Holland (1964).
3. Ph. Greenberg: Jr. Mat. Anal. and Appl. 29, p. 647 (1970)

8

⁴ P.S.E. Peebles: Physical Cosmology, Princeton UP (1971).

⁵ Ch.Moeller: The theory of relativity, Clarendon Press (1972).

⁶ I.Lukačević: Theoretical and Applied Mechanics, Belgrade, Vol. I,
p. 23 (1975).

⁷ I.Lukačević: Publ. Inst. Math. 19 (33), p. 101 (1975)

⁸ M.Bray: C.R. Acad.Sc., A-B 282, p.127, Paris (1976)

⁹ T.H. Date: Ann.Inst.H. Poincaré, XXIV, N^o 4, p.417 (1976)

¹⁰ I.Lukačević: Publ. Inst. Math. 22 (36), p. 175 (1977)