THE LEVER PROBLEM IN SPECIAL RELATIVITY

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The introduction of the special theory of relativity profoundly altered the concepts of space and time. It influenced all parts of physics in which these concepts are used, particularly mechanics. The mathematical formalism in which special relativity is most naturally expressed is four-dimensional space-time. At the beginning physicists did not accept this formalism but rather insisted on time and three-dimensional space. This lead sometimes to apparent inconsistencies that were considered as "paradoxes". An instructive example of this kind is the Lewis-Tolman "paradox" or the problem of the stressed lever. This short contribution gives a solution of the problem in a manifestly four-dimensional manner.

Take a rigid right-angled lever at rest in its proper inertial frame of reference S (Fig.1). The equilibrium conditions for forces and torques read

$$\sum_a F_{ia} = 0 \qquad \sum_a (x_{ia}F_{ja} - x_{ja}F_{ia}) = 0, \quad i,j = 1,2,3, \quad a = 1,2..N$$

The index a refers to N forces and their points of application and the torque is defined with respect to the pivot at the origin O. In our case the second equation reads: $x_1F_2 - x_2F_1 = 0$.

Consider this lever in an inertial frame of reference S' moving with velocity v in the direction of the common x and x' axes. The longitudinal and the transversal dimensions of the lever are transformed, respectively, according to

$$x_1' = x_1/\gamma \qquad x_i' = x_i, \quad i = 2,3$$

and the longitudinal and transversal components of the forces, respectively, according to

$$F_1' = F_1, \qquad F_i' = F_i/\gamma, \; i = 2,3$$

with $\gamma = (1-v^2/c^2)^{-1/2}$. Therewith the equilibrium conditions in frame S' take the form

$$\sum_a F_{ia}' = 0, \; \sum_a (x_{ia}' F_{ja}' - x_{ja} F_{ia}') \; x_1 F_2/\gamma^2 - x_2 F_1 =$$

$$= - v^2 x_2 F_1/c^2 \neq 0$$

As equilibrium should be an invariant state Tolman and Lewis concluded that this result is a paradox.[1] Laue gave a solution, known nowadays as synchronous.[2] The torque in frame S' is considered with respect to the origin O'. In this frame there is a flow of energy vF_1 from the point O of the lever to the point A and a corresponding flow of mass vF_1/c^2. The arm of this flow with respect to O' is increasing with the rate -v. Thus, its angular momentum with respect to O' is increasing with a rate $-v^2 x_2 F_1/c^2$. This is the time derivative of "hidden" angular momentum which compensates the net torque.

For more than fifty years Laue's solution remained unchalenged. But then Arzelies, Cavalleri and Salgarelli, Butler, and Aranoff objected that the energy flow is not a very useful concept and that the synchronous solution is not in the spirit of the four-dimensional formulation of special relativity.[3] After a couple of unsuccessful attempts[4] we introduced pivot events which enabled us to find a manifestly four - dimensional, asynchronous solution.[5]

In a covariant approach we start with a point particle of proper mass m and introduce the angular momentum tensor

$$J_{\mu\nu} = m(x_\mu - X_\mu)\, u_\nu - m(x_\nu - X_\nu)\, u_\mu \qquad \mu,\nu = 0,1,2,3$$

and the torque tensor

$$M_{\mu\nu} = (x_\mu - X_\mu)\, f_\nu - (x_\nu - X_\nu)\, f_\mu$$

x_μ is the four-vector of an event on the world line of the particle,
X is the corresponding event on the world line of the pivot,
$u_\mu = dx_\mu/d\tau$ is the four-velocity, $f_\mu = m du_\mu/d\tau$ the four-force and τ the proper time of the particle. If the angular momentum law $dJ_{\mu\nu}/d\tau = M_{\mu\nu}$ should be valid, pivot events should conform with the equation

$$u_\mu dX_\nu/d\tau = u_\nu dX_\mu/d\tau$$

The usual choice for the pivot event is $X_\mu = (0,0,0,0)$. An alternative choice, most appropriate in relativistic statics (and some other parts of mechanics) is

$$dX_\mu/d\tau = u_\mu \qquad X_\mu = X_\mu(0) + \int_0^\tau u_\mu d\tau$$

It leads to a manifestly four-dimensional form of the equilibrium conditions. For a particle we have in its proper frame $X_\mu = (c\tau,0,0,0) = (ct,0,0,0)$. Therewith the four-force arm in the proper frame S is

$$x_\mu - X_\mu = (0, x_1, x_2, x_3)$$

and in the improper frame S'

$$x_\mu' - X_\mu' = (-\gamma v x_1, \gamma x_1, x_2, x_3)$$

Thus, the longitudinal spatial component x_1 is dilatated to γx_1.

The events x_μ' and X_μ' in frame S' are evidently asynchronous but, thanks to our definition of the pivot events, time does not enter explicitly either the proper four-force arm or the improper one.

Going over from a particle to the rigid body one assumes that forces and torques acting at discrete points on a body transform as forces and torques acting on particles. In the proper frame S where all parts of the rigid body are at rest

$$f_0 = 0,\ f_i = F_i \text{ and } M_{0i} = 0,\ M_{ij} = x_i F_j - x_j F_i$$

So the relativistic equilibrium conditions in the proper frame $\sum_a f_{\mu a} = 0$, $\sum_a M_{\mu\nu a} = 0$ read as in nonrelativistic mechanics: $\sum_a F_{ia} = 0$, $\sum_a M_{ija} = 0$, In the improper frame S' Lorentz transformation gives:

$$f_0' = \gamma v F_1,\ f_1' = \gamma F_1,\ f_2' = F_2,\ f_3' = F_3$$

and

$$M_{o1}' = 0,\ M_{oi}' = -\gamma v M_{oi},\ \gamma M_{1i}' = M_{1i},\ M_{23}' = M_{23}$$

It can be directly seen that therewith the equilibrium conditions in the improper frame S' have also the natural four-dimensional form $\sum_a f_{\mu a}' = 0$, $\sum_a M_{\mu\nu a}' = 0$.
The whole procedure is indeed in the spirit of special relativity and at the same time bears as much resemblance to nonrelativistic mechanics as possible.

REFERENCES

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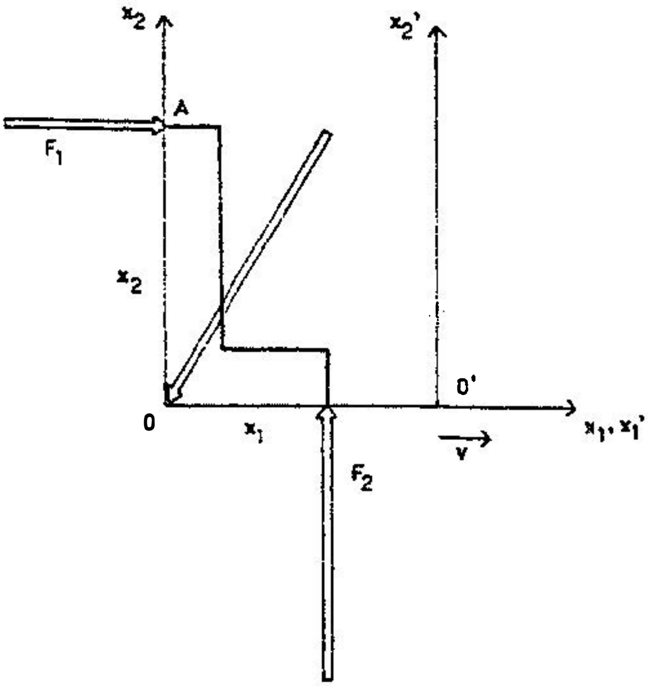

<u>Figure 1.</u> The stressed right-angled lever with the proper frame of reference S and the improper one S'.