# POINCARE GAUGE THEORY OF GRAVITY

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The idea of gravity as a gauge theory of Poincaré group was first elaborated by Kibble,[1] who extended Utiyama's work[2] on gravity as the gauge theory of Lorentz group. After that, various treatments of gravity as Poincaré gauge theory appeared.[3] In order to treat gravity along the lines accepted for other fundamental interactions, we shall present Poincaré gauge approach to the theory of gravity which has a very close resemblance to the usual procedure for gauging internal symmetries. Its relation to some other Poincaré gauge approaches will also be discussed.

The starting point is to consider a Minkowski space-time $R_4$, at each point of which one can define a local Lorentz reference frame $e_i(x)$, called tetrad. Using cartesian coordinates $x^\mu$, one can choose the tetrad in such a way that it coincides with coordinate induced bases $e_\mu(x)$, $e_i(x) = \delta_i{}^\mu e_\mu(x)$.[3] We shall adopt convention that Latin indices i,j,k,... refer to local Lorentz frames, whereas Greek indices $\mu,\nu,\lambda,\ldots$ refer to coordinate basis od space time. Let a physical system be described by a Lagrangian density $(\psi(x), \partial_k\psi(x))$, where $\psi(x)$ is a set of field variables which realises a representation of Poincaré group and $\partial_k = \delta_k{}^\mu\partial_\mu$ is partial derivative with respect to tetrad.[3] Under a global Poincaré transformation defined by ten constant infinitesimal parameters $\omega^{\mu\nu} = -\omega^{\nu\mu}$ and $\varepsilon^\mu$,

$$\delta x^\mu = \omega^\mu{}_\nu x^\nu + \epsilon^\mu \qquad (1)$$

(Minkowski metric $\eta_{\mu\nu}$ is used to lower indices), the matter field $\psi(x)$ and its derivative $\partial_k\psi(x)$ undergo the following changes

$$\delta_0\psi = (\tfrac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + \epsilon^\nu P_\nu)\psi \qquad (2a)$$

$$\delta_0\partial_k\psi = (\tfrac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + \epsilon^\nu P_\nu)\partial_k\psi - \omega^\nu{}_k\partial_\nu\psi \qquad (2b)$$

where $P_\nu = -\partial_\nu$ and $M_{\mu\nu} = S_{\mu\nu} + (x_\mu\partial_\nu - x_\mu\partial_\mu)$ are group generators, $S_{\mu\nu}$ is spinor part of $M_{\mu\nu}$, $\omega^\nu = \delta_k{}^\mu\omega^\nu{}_\mu$ and $\delta_0\psi \equiv \psi'(x) - \psi(x)$ defines form variation of matter field $\psi$.[1] The invariance of the action integral under Poincaré transformation, given by Eqs. (1) and (2a), can be expressed in the form of the following condition on Lagrangian density:

$$\delta_0\mathcal{L} + \mathcal{L},_\mu\xi^\mu + \mathcal{L}\xi^\mu,_\mu = 0 \qquad (3)$$

where $\xi^\mu \equiv \omega^\mu{}_\nu\chi^\nu + \epsilon^\mu$ and

$$\delta_0\mathcal{L} \equiv \mathcal{L}(\psi+\delta_0\psi,\ \partial_k\psi + \delta_0\partial_k\psi) - \mathcal{L}(\ldots_K \ldots$$

We note that for global transformations $\xi^\mu,_\mu = \omega^\mu{}_\mu = 0$, due to the antisymmetry of $\omega^{\mu\nu}$.

If one now generalizes Poincaré transformation (1) and (2a) by assuming that ten group parameters are not constants but space--time dependent functions, the condition (3) fails to be fulfilled, and the action integral is no longer invariant. The reason for this is twofold. First, the partial derivative $\partial_k\psi$ is not transformed covariantly, i.e. as in Eq. (2b). The second reason is that $\xi^\mu,_\mu \neq 0$. The violation of local gauge invariance can be compensated by changing Lagrangian density in a suitable manner.

Let us first find Lagrangian density ’ obeying the invariance condition (3) with $\xi^\mu{}_{,\mu} = 0$. This is achieved by defining

$$\mathcal{L}' = \mathcal{L}(\psi, D_k\psi) \tag{4}$$

where $D_k\psi$ is a covariant derivative of $\psi$, which transforms under local Poincaré transformations as $\partial_k\psi$ did under global ones, Eq. (2b),

$$\delta_0 D_k\psi = (\tfrac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + \varepsilon^\nu P_\nu) D_k\psi - \omega^i{}_k D_i\psi \tag{5}$$

where $\omega^i{}_k = \delta^i{}_\mu \delta_k{}^\nu \omega^\mu{}_\nu$. To construct $D_k\Psi$ we have to introduce gauge potentials $A_k$.

$$D_k\psi = (\partial_k + A_k)\psi \tag{6}$$

$$A_k = A^\mu{}_k P_\mu + \tfrac{1}{2}A^{\mu\nu}{}_k M_{\mu\nu} \tag{7}$$

Condition (5) determines the gauge transformation of $A_k$ and $A^{\mu\nu}{}_k$. The form of these transformations allows us to treat $\xi^\mu$ and $\omega^{\lambda\delta}$, instead of $\varepsilon^\mu$ and $\omega^{\lambda\delta}$, as independents parameters.

Motivated by this independence of $\xi^\mu$ and $\omega^{\lambda\delta}$, we now introduce, instead of $A^\mu$, the translation gauge potential $\bar{A}^\mu{}_k$ corresponding to the parameter $\xi^\mu$.

$$\bar{A}^\mu{}_k = A^\mu{}_k + A^{\mu\sigma}{}_k x_\sigma \tag{8}$$

The gauge transformation of $\bar{A}^\mu{}_k$ follows from the corresponding transformations of $A^\mu{}_k$ and $A^{\mu\nu}{}_k$. Rewriten in terms of $h_k{}^\mu = \delta_k^\mu - \bar{A}^\mu{}_k$, which is not the gauge potential, this transformation law reads

$$\delta_0 h_k{}^\mu = h_k{}^\tau \partial_\tau \xi^\mu - \omega^s{}_k h_s{}^\mu - \xi^\lambda \partial_\lambda h_k{}^\mu . \tag{9}$$

Up to now, we have defined the covariant derivative $D_k\psi$ by introducing gauge potentials, and derived their transformation properties from the requirement of the covariance of $D_k\psi$. Lagrangian density $\mathcal{L}'$, Eq. (4), is thereby determined. The procedure follows very closely the basic ideas of gauge approach to internal symmetries.[1,2] In the second step of restoring local invariance of the theory, we have to take care of the fact that $\xi^\mu{}_{,\mu} \neq 0$ in Eq. (3). This can be done by defining the matter Lagrangian as[1]

$$\mathcal{L}_M(\psi,h,A) = b\mathcal{L}(\psi,D_k\psi), \qquad (10)$$

where $b = \det b^k{}_\mu$, $b^k{}_\mu$ being the inverse of the translation gauge fields $h_k{}^\mu$

$$b^k{}_\mu h_k{}^\nu = \delta_\mu{}^\nu, \qquad b^k{}_\mu h_i{}^\mu = \delta^k{}_i. \qquad (11)$$

In order to have clear geometrical interpretation of transformations (1), (2a) and new gauge potentials $\bar{A}^\mu{}_k$ and $A^{\lambda\delta}{}_k$, we shall now generalize our previous convention concerning the use and meaning of Latin and Greek indices. First, any Greek index of gauge potential should be related to the transformation properties with respect to general coordinate transformation $\xi^\mu(x)$, whereas any Latin index should describe transformation properties related to the local Lorentz rotation, defined by parameters $\omega^{ij}(x)$. According to the transformation law (9) the use of Latin and Greek indices in $h_k{}^\mu$ is in agreeement with this convention. From the transformation properties of $A^{\lambda\delta}{}_k$ , one can see that all indices should be Latin ones,

$$\delta_0 A^{ij}{}_k = -h_k{}^\tau\partial_\tau\omega^{ij} - \omega^s{}_k A^{ij}{}_s + \omega^i{}_s A^{sj} + \omega^j{}_s A^{is}{}_k - \xi^\lambda\partial_\lambda A^{ij}{}_k \qquad (12)$$

Geometrically, $h_k{}^\mu$ and $A^{ij}{}_k$ can be identified with tetrad and connection coefficients of Riemann-Cartan space-time $U_4$.[3] In a second step we generalize index convention so that it holds also for $U_4$.

The difference between Latin and Greak indices becomes now essential, in contrast to the case of Minkowski space-time.

To compare our results with Kibble's[1], we note that he started by considering Minkovski space-time in which infinitesimal coordinate transformations (1) induce total change of the matter field given by

$$\delta\psi = \frac{1}{2}\,\omega^{ij}S_{ij}\psi \tag{13}$$

where $\delta\psi = \psi'(x') - \psi(x) = \delta_0\psi - \delta x^\mu\partial_\mu\psi$. Going to a localization of group parameters, the total translation $\xi^\mu(x)$, which is interpreted as general coordinate transformation, and local rotations $\omega^{ij}$, are treated from the very beginning as independent sets of parameters. The gauge potentials which are introduced into the theory, have transformation properties which are essentially described by our Eqs. (9) and (12), justifying thereby aposteriori the asssumed independence of $\xi^u(x)$ and $\omega^{ij}(x)$. The use of form variation $\delta_0\psi$ instead of $\delta\psi$ brings our approach more closely to the spirit of gauge theories of internal symmetries, since $\delta_0\psi$ realizes the representation of space-tima symmetry group, whereas $\delta\psi$ does not. The transformation properties of $h_k{}^\mu$ and $A^{ij}{}_k$ are equal to those of the corresponding Kibble's fields, and covariant derivatives $D_k\psi$ are the same. The two formulations are thus seen to be equivalent.

Hehl, von der Heyde, Kerlick and Nester (HHKN) [3] started from the very beginning with the condition that matter field transformation $\delta_0\psi$ in an active interpretation, should have definite geometrical meaning via so called rigidity condition. They required that Poincaré translation generator $-P_i = \delta_i{}^\mu\partial_\mu$ should be changed everywhere into $D_i$ after localization of symmetry, in order to preserve the operational significance of translation. In that way, expression for $\delta_0\psi$ should be changed into

$$\delta_0^* \psi = (\tfrac{1}{2}\omega^{ij}S_{ij} - \xi^{\nu}D_{\nu})\psi \tag{14}$$

where $D_\nu = b^k{}_\nu D_k$. By comparing expression (14) with (2a) one finds the relation

$$\delta_0^* \psi = \delta_0\psi - \xi^\nu \tfrac{1}{2} A^{ij}{}_\nu S_{ij} \psi \tag{15}$$

($A^{ij}{}_\nu = b^k{}_\nu A^{ij}{}_k$) which shows that $\delta_0^* \psi$ and $\delta_0\psi$ differ by a local Lorentz rotation of matter field, caracterized by parameter $\Delta\omega^{ij} = -\xi^\nu A^{ij}{}_\nu$. Our formulation, which is invariant under local Poincaré transformation (1) and (2a), and therefore under any additional local Lorentz transformation, is also invariant under new total variation $\delta_0{}^*\psi$. Since the reversed line of argument is also correct, the equivalence of HHKN formulation and our is thus established.

We note that the geometric interpretation of gauge potentials in our approach is formulated according to their derived transformation properties. The HHNK approach has a convenient geometrical form, since geometrical language is introduced from the very beginning.

All the three formulations are thus seen to be essentially equivalent Poincaré gauge approaches, although different in form. Therefore, the difference between Kibble's [1] and HHKN[3] approach is much less important than it is claimed in Ref. 3.

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