

HAMILTONIAN FORMULATION OF POINCARÉ GAUGE THEORY OF GRAVITY

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Using Dirac's method for systems with constraints, Einstein-Cartan form of Poincaré gauge theory of spinor field is put into Hamiltonian form. This is done in time gauge, treating tetrads and connection coefficients as independent variables.

In order to get the unified description of all fundamental interactions in nature, it is natural to treat gravity as a Poincaré gauge theory<sup>1,2</sup>. Einstein-Cartan form of such theory is the simplest extension of Einstein general relativity. In such theory torsion does not propagate. The Hamiltonian formulation of Einstein-Cartan theory is necessary in canonical as well in the path-integral quantization procedure. Besides, the Hamiltonian formulation of the theory with constraints, which was developed by Dirac<sup>3</sup>, results in a deeper understanding of canonical structure and physical degrees of freedom of the theory.

We shall apply Dirac's systematic method to find Hamiltonian, treating all gauge potentials as independent fields. The advantage of our approach (this is so called the first order formalism) is that it is unavoidable when more general Lagrangians than R are used.

For the total action for the spinor field (described by  $\psi$  and  $\bar{\psi}$ ) in interaction with gravitational field (described by gauge potentials  $h_k{}^\mu$  and  $A^j{}_\mu$ ) we choose<sup>1</sup>

$$S = \int d^4x (\mathcal{L}_g + \mathcal{L}_m) \equiv \int d^4x \mathcal{L} \quad (1)$$

where the matter Lagrangian is given by

$$\mathcal{E}_m \equiv \frac{b}{2} |h_k^\mu (\bar{\psi}^i \gamma^{k+\alpha} \psi) - 2m\bar{\psi} \psi - \frac{1}{2} \epsilon_{ijkl} A^{ij}{}_\mu h^{k\mu} \bar{\psi} \gamma^i \gamma^5 \psi| \quad (2)$$

and gravitational Lagrangian is proportional to scalar curvature  $R$ , i.e. it corresponds to the Einstein-Cartan theory

$$\begin{aligned} \mathcal{E}_g \equiv & \frac{b}{\kappa} |h_{ij}^\mu h_j^\nu| \left( -\frac{b_{,\nu}}{b} A^{ij}{}_\mu + A^i{}_{k\nu} A^{kj}{}_\mu \right) - \\ & - (h_i^\mu h_j^\nu)_{,\nu} A^{ij}{}_\mu + \left( \frac{b}{\kappa} h_i^\mu h_j^\nu A^{ij}{}_\mu \right)_{,\nu} \end{aligned} \quad (3)$$

There,  $\kappa$  is Einstein gravitational constant and  $b_k$  is determinant formed by inverse tetrad fields  $b^k{}_\mu$ . The last term in Eq. (3) is 4-divergence and can be omitted. We shall prove that this choice of Lagrangian gives correct expression for the total energy of the system (see Eq.(28)). The total Lagrangian  $\mathcal{E}$  can be written in the form  $\mathcal{E} = \mathcal{E}(0) + \mathcal{E}(1)$ , where  $\mathcal{E}(0)$  is independent on the velocities, and  $\mathcal{E}(1)$  is linear and homogeneous in velocities. Because of this structure, canonical Hamiltonian  $\mathcal{H}$  will be simply equal to  $-\mathcal{E}(0)$ . In order to obtain the simpler form of the theory and give the time coordinate a physical meaning, we impose the time gauge conditions<sup>4</sup>

$$h_a^0 = 0 \quad (4)$$

(indices  $a, b, c, d$  and  $\alpha, \beta, \gamma, \delta$  take on values 1,2,3). After this, we are left with 13 tetrad components  $h_0^\mu$  and  $h_a^\alpha$  (fields  $b_a^\alpha$  are now inverse to the  $h_a^\alpha$  while  $b^0_0 = 1/h_0^0$ ,  $b^0_\alpha = 0$  and  $b^a = b^0_0 b_a^\alpha h_0^\alpha$ ). Using the time gauge conditions (1) becomes

$$\mathcal{E}(1) = \frac{K}{\kappa} \left\{ A^{a0}{}_\alpha (h_a^\alpha b_\beta^b - \epsilon_a^b \delta_\beta^a) h_{b,0}^\beta + \frac{\kappa}{2} \bar{\psi}^i \gamma^{0+\alpha} \psi \right\} \quad (5)$$

where  $K \equiv b h_0^0 = \det(b_a^\alpha)$ . Defining canonical momenta in the usual way

$$\delta \mathcal{E} = \pi^0{}_\mu \delta h_0^\mu + \pi^a{}_\alpha \delta h_a^\alpha + \pi_{ij}{}^\mu \delta A^{ij}{}_\mu + \pi \delta \psi + (\delta \psi)_0 \pi \quad (6)$$

we easily find the primary constraints

$$\phi_{\alpha}^a \equiv A^{a0}_{\alpha} + \frac{\kappa}{K} (\pi^a_{\alpha} - \frac{1}{2} b^a_{\alpha} h_b^{\beta} \pi^b_{\beta}) \approx 0 \quad (7)$$

$$\phi_{\mu}^0 \equiv \pi^0_{\mu} \approx 0, \quad \phi_{ij}^{\mu} \equiv \pi_{ij}^{\mu} \approx 0 \quad (8,9)$$

$$\bar{\phi} \equiv \bar{\pi} - \frac{\kappa}{2} \bar{\psi} i\gamma^0 \approx 0, \quad \varphi \equiv \pi + \frac{\kappa}{2} i\gamma^0 \psi \approx 0 \quad (10)$$

Cononical Hamiltonian in the time gauge can be written in the following form

$$\mathcal{H} = \mathcal{H}^0 + b^0_0 \mathcal{H}_L + b^0_0 h_0^{\alpha} \mathcal{H}_{\alpha} + D^{\alpha, \alpha} \quad (11)$$

where  $D^{\alpha, \alpha}$  is 3-divergence

$$D^{\alpha, \alpha} \equiv \left| \frac{1}{\kappa} \kappa b^0_0 (2h_a^{\alpha} h^{\beta} |A^{0a}_{\beta} + h_a^{\alpha} h_b^{\beta} A^{ba}_{\beta}) \right|_{;\alpha} \quad (12)$$

and  $\mathcal{H}^0$ ,  $\mathcal{H}_L$ , and  $\mathcal{H}_{\alpha}$  are independent on  $h_0^{\mu}$

$$\mathcal{H}^0 \equiv \frac{\kappa}{\kappa} \left\{ A^{a0}_0 h_b^{\alpha} \chi^b_a + h_a^{\alpha} A^a_{b0} \phi^b_{\alpha} - \frac{1}{2} A^{ab}_0 \chi_{ab}^0 \right\} \quad (13)$$

$$\mathcal{H}_{\beta} \equiv \frac{\kappa}{\kappa} \left\{ 2h_a^{\alpha} (A^{a0}_{|\beta, \alpha|} + A^a_{b|\alpha} A^{b0}_{\beta|}) - \frac{\kappa}{2} \bar{\psi} i\gamma^0 \bar{\sigma}_{\beta}^{\alpha} \psi + \frac{\kappa}{4} \epsilon_{0abc} A^{ab}_{\beta} J^c \right\} \quad (14)$$

$$\mathcal{H}_L \equiv \frac{\kappa}{\kappa} \left\{ h_a^{\alpha} h_b^{\beta} |A^{ab}_{\beta, \alpha} + A^a_{k\alpha} A^{kb}_{\beta}| - \frac{\kappa}{2} h_a^{\alpha} \psi i\gamma^{\alpha} \bar{\sigma}_{\alpha}^{\alpha} \psi - \frac{\kappa}{4} \epsilon_{0abc} h^{a\alpha} (2A^{0b}_{\alpha} J^c + A^{bc}_{\alpha} J^0) + m\kappa \bar{\psi} \psi \right\} \quad (15)$$

where the expressions for  $\chi_{ab}$  and  $\chi_{ab}^0$  do not depend of  $A^{ij}_{\nu}$ .

Now we define the total Hamiltonian

$$\mathcal{H}_T \equiv \mathcal{H} + u_0^{\mu} \phi_{\mu}^0 + u_a^{\alpha} \phi^a_{\alpha} + u^{ij}_{\mu} \phi_{ij}^{\mu} + \bar{u} \phi + \bar{\psi} u \quad (16)$$

where  $u$ -s are undetermined multipliers. Consistency conditions

imposed on constraints  $\phi^a_{\alpha}$  and  $\bar{\phi}$ , serve to determine  $u^{a0}_{\alpha}$ ,  $u$  and  $\bar{u}$  while  $\phi_{ij}^{\mu}$  give new, secondary constraints

$$\chi_{ab}^0 \equiv \pi_{a\alpha} h_b^{\alpha} - \pi_{b\alpha} h_a^{\alpha} - \frac{\kappa}{2} \epsilon_{0abc} J^c \approx 0 \quad (17)$$

$$\chi_a \equiv A^0_{a0} + b^0_{\alpha} h_{\alpha} A^0_{a\alpha} - h_a (b^0_{\alpha})_{,\alpha} = 0 \quad (18)$$

$$\chi_{ab\alpha} \equiv A_{ab\alpha} - A_{ab} + \frac{\kappa}{4} \epsilon_{\alpha abc} b^c_{\alpha} J^0 = 0 \quad (19)$$

where

$$L_{ab\alpha} \equiv b^c_{\alpha} (C_{cab} + C_{bca} - C_{abc}), \quad (20)$$

$$C_{abc} h_a^{\alpha} h_b^{\beta} b_{c\beta,\alpha}$$

and

$$J^k \equiv i\psi_Y^k \gamma^5 \psi \quad (21)$$

Finally, consistency conditions for  $\epsilon^0_{ab}$  lead to

$$\mathcal{H}_L = 0, \quad \mathcal{H}_\alpha = 0 \quad (22)$$

Consistency conditions for Eq. (17-19) and (22) do not give new secondary constraints.

It is easy to check that  $\chi_{ab}^0$  do not commute with  $\phi$  and  $\bar{\phi}$  as claimed in the literature<sup>5</sup>. But if we replace  $J$  given by (21) with

$$J^C \equiv \frac{i}{\kappa} (\bar{\psi}_Y^C \gamma^5 \gamma^0 \psi - \bar{\pi}_Y^0 \gamma^C \gamma^5 \psi) \approx J^C \quad (23)$$

we do have the first class condition which corresponds to the gauge freedom of tetrads rotation in  $x^0 = \text{const}$  plane. A similar procedure is necessary to represent constraints (22) as the first class constraints. They correspond to the freedom of orthogonal and tangential displacement of  $x^0 = \text{const}$  plane. We will not write these expressions because the difference between them and the constraints (17) and (22) vanish after elimination of unphysical degrees of freedom, i.e. after using remaining set of constraints (except  $\epsilon_{ab}^0$ ) as strong equations.

Remaining set of constraints, without  $\epsilon_{ab}^0$ , is of the second class and it cannot combine to give the first class constraints. So, we can use iterative property of Dirac brackets and compute preliminary Dirac brackets for constraints  $\pi_{ij\mu} \approx 0$  ( $i, j, \mu \neq a, b, 0$ ),

$\phi^a_\alpha$ ,  $\chi_{ab\ \alpha}$  and  $\chi_a$ . Using these equations as strong equations for elimination of  $A^{ij}_\mu$  ( $i, j, \mu \neq a, b, 0$ ) and corresponding momenta from the theory, we observe that preliminary Dirac brackets are simply Poisson brackets for the remaining set of variables. This proves that the equivalence between the first order and the second order Lagrangians in gravity still holds for corresponding Hamiltonian formulation. Finally, using constraints  $\phi \approx 0$  and  $\bar{\phi} \approx 0$  to eliminate  $\bar{\phi}$  and  $\pi$  from all expressions, and to calculate Dirac brackets, we obtain

$$\{\psi(x), \bar{\pi}(y)\}^* = \frac{1}{2} \delta(\vec{x}-\vec{y}), \quad (24)$$

$$\{\pi^a_\alpha(x), \psi(y)\}^* = \frac{1}{2} b^a_\alpha \psi(x) \delta(\vec{x}-\vec{y}), \quad (25)$$

$$\{\pi^a_\alpha(x), \bar{\psi}(y)\}^* = -\frac{1}{2} b^a_\alpha \bar{\pi}(x) \delta(\vec{x}-\vec{y}), \quad (26)$$

All the other Dirac brackets for the set of variables we are left with:

$$h_0^\mu, \pi_0^\mu, h_a^\alpha, \pi_a^\alpha, A^{ab}_0, \pi^{ab}_0, \psi \text{ and } \bar{\pi}$$

are standard ones, i.e. simply Poisson brackets.

After this elimination, the total Hamiltonian (16) is given by

$$\begin{aligned} \mathcal{H}_T = & b^0_0 \mathcal{H}_L + b^0_0 h_0^\alpha \chi_\alpha - \frac{1}{2} A^{ab}_0 J_{ab} + \\ & + u_0^\mu \pi_0^\mu + u^{ab}_0 \pi_{ab}^0 + D^\alpha_{,\alpha} \end{aligned} \quad (27)$$

This expression is essentially equal to the extended Hamiltonian, as defined by Dirac. Notice that our  $\mathcal{H}_T$  does not vanish weakly (as it does in <sup>4,5</sup>) because of the divergence term. This term has a clear physical meaning. If the tetrad coefficients at spatial infinity correspond to Schwarzschild metric

$$h_0^0 \rightarrow 1 - \frac{M}{r}; \quad h_a^\alpha \rightarrow \delta_a^\alpha - \frac{M}{r} \frac{x^\alpha x^\beta}{r^2} \delta_{a\beta}$$

where  $r^2 = -\eta_{\alpha\beta} x^\alpha x^\beta$  and  $M$  is the total mass of the system, we easily find

$$H_T = \int d^3x D^\alpha_{,\alpha} \equiv \int d^3x D^\alpha_{,\alpha} = M \quad (28)$$

as it should be. We also find that besides  $h_0^\mu$ ,  $A^{ab}_0$  can vary arbitrarily in time. This difference between our results and those obtained in <sup>4,5</sup> vanish in the space of physical variables.

To summarize, we develop a complete Hamiltonian scheme and prove that elimination of gauge fields  $A$  do not alter Dirac brackets. This fact is not so obvious when such elimination is done in Lagrangian form of the theory<sup>3</sup>.

#### REFERENCES:

1. T.W.B.Kibble, J.Math.Phys. 2(1961), 212
2. M.Bлагоjević et al., Preprint TF - 80/01
3. P.A.M.Dirac, Can.J.Math. 2(1950), 129
4. P.A.M.Dirac, in Recent Developments in General Relativity (PWN-Polish Scientific Publications, Warsaw; 1962), p. 191
5. M.Kasuya, Prog.Theor.Phys. 60, (1978) 167