

ON THE METRIC AFFINE THEORY OF GRAVITY

Dj. Šijački

B. Kidrič Institute, Vinča

We discuss the basis of a recently proposed Metric Affine theory of gravity. It is a gauge field theory based on the $GA(4,R)$ group and it generalizes the corresponding Einstein-Cartan theory based on the Poincaré group. The basic currents, energy-momentum and hypermomentum, are the sources of the two field equations of gravity. The unitary irreducible representations of the $GA(4,R)$ group offer a possibility of having a different space-time description of leptons and of hadrons. A Lagrangian quadratic in torsion and curvature yields an Einstein-like gravitational interaction, while in the linear approximation one finds a confining potential which applies only to the hadronic matter.

Einstein's General Relativity theory is the most successful theory of the large-scale gravitational interactions including electromagnetic waves. Unfortunately, the classical General Relativity is confronted with difficulties related to the singularity states of infinite mass density, while the corresponding quantum theory is faced with the nonrenormalizability problem.

Rather than to pursue a standard approach to gravity, we choose to treat the matter from a particle physics point of view and rely heavily on the gauge and group theoretical methods. In other words, we prefer to study the basic physical currents (sources of the field equations of gravity) and then to develop the corresponding gauge invariant field theory. We assume that the large-scale gravitational interactions are manifestation of the gravitational interactions of leptons and hadrons.

Now, the space-time properties of leptons are well described

by the full Poincaré group. By gauging the Poincaré group one arrives at the Einstein-Cartan-Sciame-Kibble theory of gravity. There are strong indications that in the world of hadrons one finds additional space-time originated currents beyond the energy-momentum and spin currents of the Poincaré group. Spin-independence (the basic part of the SU(6) symmetry), scaling (as observed in deep inelastic scattering of electrons or neutrinos on nucleons) and linear $m^2(j)$ sequences of rotational excitations (Regge trajectories) indeed represent the full set of noninternal regularities of the hadrons. Thus, we enlarge the Lorentz group to the $\overline{GL}(4,R)$ one. When the translations are taken in account as well one ends up with the General Affine $\overline{GA}(4,R)$ extension of the Poincaré symmetry.

We endow the four-dimensional base space-time manifold at each point by a tangent linear space of local co-frames, the tetrads e^a_ν (anholonomic tetrad (holonomic world) indices are denoted by Latin (Greek) letters with values running from 0 to 3). At each point of the base space-time manifold one can now define an action of the $\overline{GA}(4,R)$ group, which translates and deforms the tetrads of a space-time carrying a local Minkowski metric g_{ab} . The $\overline{GA}(4,R)$ transformations will now be gauged,¹⁻⁵ i.e. they will be allowed local values as in the Weyl or Yang-Mills case. Now, we postulate the invariance of the action function of matter under tetrad translations (4 parameters of T_4) and tetrad deformations (16 parameters of $\overline{GL}(4,R)$). The invariance, under the local $\overline{GA}(4,R)$ transformations with the space-time dependent parameters, can be saved only by introducing into the matter Lagrangian the translational gauge potentials, the tetrads $e^a_\nu(x)$ (actually, $e^a_\nu - \delta^a_\nu$ are the gauge potentials in the sense of internal gauge symmetries), and the deformational gauge potentials, the connection coefficients $\Gamma^b_{a\nu}$. In this way we end up with a metric-affine space-time with an independent connection.

In gauging a space-time symmetry group, generalizing the Poincaré case, one replaces the translation generators P_a by the parallel-transport operators

$$D_a = ie_a^\mu (\partial_\mu - i\Gamma_c^d{}_\mu Q_c^d),$$

with parameters $\epsilon^a = \epsilon^\mu e_\mu^a$, thus producing an anholonomized general coordinate transformations; Q_c^a are the $\overline{GL}(4, R)$ generators. It is convenient to separate the $\overline{GL}(4, R)$ indices in the antisymmetric $[ab]$ Lorentz part and the remaining symmetric (ab) shear and dilation part, and write

$$D_a = ie_a^\mu (\partial_\mu - i\Gamma_{[cd]}^{\quad\quad}{}_\mu Q_{[cd]} - i\Gamma_{(cd)}^{\quad\quad}{}_\mu Q_{(cd)}).$$

For a matter tensor field or for an infinite-component (spinor or tensor) field one has

$$\delta\phi = i(\epsilon^a D_a + \epsilon^{[ab]} Q_{[ab]} + \epsilon^{(ab)} Q_{(ab)})\phi,$$

where the D_a , $Q_{[ab]}$ and $Q_{(ab)}$ are infinite-dimensional matrices⁴ in the latter case. For the lepton Poincaré spinor field ψ in the nonlinear representation,⁴ we list the lowest terms

$$\delta\psi = i(\epsilon^a D_a + \epsilon^{[ab]} \sigma_{[ab]} + \epsilon^{(ab)} Q_{(ab)})\psi + \dots,$$

where $\sigma_{[ab]} = \frac{i}{4}[\gamma_a, \gamma_b]$, and the $Q_{(ab)}$ matrices act on the ten components of g_{cd} . Using these variations we find the matter Noether currents (\mathcal{L} is the matter Lagrangian) resulting from $\overline{GA}(4, R)$ invariance: energy-momentum

$$e \theta_a^\mu = -e_a^\mu + \frac{\partial}{\partial(\partial_\mu \phi)} D_a \phi + \frac{\partial}{\partial(\partial_\mu \psi)} D_a \psi,$$

and hypermomentum, i.e. shear and dilation (intrinsic part)

$$e \tilde{\Upsilon}^{\dot{a}\dot{b}\dot{\mu}} = \frac{\partial \mathcal{L}}{\partial (\partial_{\dot{\mu}} \phi)} Q^{\dot{a}\dot{b}} \phi + \frac{\partial \mathcal{L}}{\partial (\partial_{\dot{\mu}} \psi)} (\sigma^{\dot{a}\dot{b}} + g^{ac} Q_{(cb)} g) \psi,$$

where $\dot{e} = \det(e^{\dot{a}}_{\dot{\mu}})$. The gauge ensures universal coupling to the energy-momentum tensor $\theta^{\dot{a}\dot{b}}$ and to hypermomentum $\tilde{\Upsilon}^{\dot{a}\dot{b}\dot{\mu}}$

$$e \theta^{\dot{a}\dot{b}} = \delta \mathcal{L} / \delta e^{\dot{a}}_{\dot{\mu}}, \quad e \tilde{\Upsilon}^{\dot{a}\dot{b}\dot{\mu}} = \delta \mathcal{L} / \delta \Gamma^{\dot{a}\dot{b}}_{\dot{\mu}}.$$

There is a conclusive difference between space-time and internal gauge symmetries originating from the redefined local translation generators. We find

$$[Q^{\dot{a}}_{\dot{b}}, Q^{\dot{c}}_{\dot{d}}] = i \delta^{\dot{c}}_{\dot{b}} Q^{\dot{a}}_{\dot{d}} - i \delta^{\dot{a}}_{\dot{d}} Q^{\dot{c}}_{\dot{b}},$$

$$[Q^{\dot{a}}_{\dot{b}}, D_{\dot{c}}] = -i \delta^{\dot{a}}_{\dot{c}} D_{\dot{b}}$$

$$[D_{\dot{a}}, D_{\dot{b}}] = i e^{\dot{u}\dot{v}} e^{\dot{c}}_{\dot{b}} (-R^{\dot{c}}_{\dot{u}\dot{v}} D_{\dot{c}} + R^{\dot{c}\dot{d}}_{\dot{c}\dot{u}\dot{v}} Q^{\dot{c}}_{\dot{d}}).$$

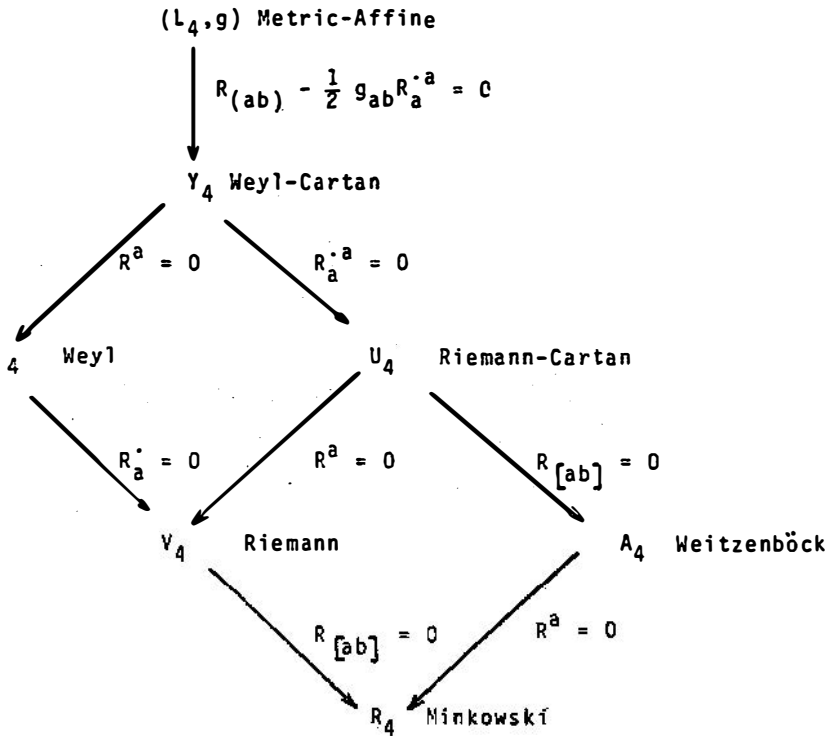
The quantities entering the last expression are the torsion tensor (translation field strength)

$$R^{\dot{a}}_{\dot{\mu}\dot{\nu}} = \partial_{\dot{\mu}} e^{\dot{a}}_{\dot{\nu}} - \partial_{\dot{\nu}} e^{\dot{a}}_{\dot{\mu}} + \Gamma^{\dot{a}}_{\dot{b}\dot{\mu}} e^{\dot{b}}_{\dot{\nu}} - \Gamma^{\dot{a}}_{\dot{b}\dot{\nu}} e^{\dot{b}}_{\dot{\mu}},$$

and the curvature tensor (deformation field strength)

$$R^{\dot{a}\dot{b}}_{\dot{c}\dot{d}\dot{e}\dot{f}} = \partial_{\dot{c}} \Gamma^{\dot{b}}_{\dot{d}\dot{e}\dot{f}} - \partial_{\dot{d}} \Gamma^{\dot{b}}_{\dot{c}\dot{e}\dot{f}} + \Gamma^{\dot{b}}_{\dot{c}\dot{d}} \Gamma^{\dot{c}}_{\dot{e}\dot{f}} - \Gamma^{\dot{b}}_{\dot{c}\dot{e}} \Gamma^{\dot{c}}_{\dot{d}\dot{f}}.$$

The gauge potentials are 1-forms $e^{\dot{a}} = e^{\dot{a}}_{\dot{\mu}} dx^{\dot{\mu}}$ and $\Gamma^{\dot{a}\dot{b}} = \Gamma^{\dot{a}\dot{b}}_{\dot{c}\dot{d}} dx^{\dot{c}} dx^{\dot{d}}$, while the field strengths are 2-forms $R^{\dot{a}} = R^{\dot{a}}_{\dot{b}\dot{c}} dx^{\dot{b}} dx^{\dot{c}}$ and $R^{\dot{a}\dot{b}} = R^{\dot{a}\dot{b}}_{\dot{c}\dot{d}\dot{e}\dot{f}} dx^{\dot{c}} dx^{\dot{d}} \wedge dx^{\dot{e}} dx^{\dot{f}}$.



Classification of (L_4, g) geometries - general affine geometries L_4 with metric g .

Let \mathcal{V} be the Lagrangian density for the gauge fields. As usually, we will assume it to be a function of the gauge potentials and their first derivatives $\mathcal{V} = \mathcal{V}(e^a_{\cdot\mu}, \partial e^a_{\cdot\mu}, \Gamma^{\cdot b}_{a\cdot\mu}, \Gamma^{\cdot b}_{a\cdot\mu})$. Using the canonical momenta

$$\pi_a^{\cdot\mu\nu} = \delta\mathcal{V} / \delta(\partial_\nu e^a_{\cdot\mu}), \quad \pi^{\cdot a\cdot\mu\nu}_b = \delta\mathcal{V} / \delta(\partial_\nu \Gamma^{\cdot b}_{a\cdot\mu}),$$

the $\overline{GA}(4, R)$ gauge field equations can be put in the following form

$$\begin{aligned} D_\nu \pi_a^{\cdot\mu\nu} - \epsilon_a^{\cdot\mu} &= e \theta_a^{\cdot\mu}, \\ D_\nu \pi^{\cdot a\cdot\mu\nu}_b - \epsilon^{\cdot a\cdot\mu}_b &= e \tilde{\Upsilon}^{\cdot a\cdot\mu}_b, \end{aligned}$$

where $\epsilon_a^{\cdot\mu}$ and $\epsilon_{\cdot b}^{a\cdot\mu}$ represent respectively the energy-momentum and the hypermomentum currents of the gauge fields themselves.

Usually, following Hilbert and Einstein, the gravitational Lagrangian is assumed to depend only linearly on curvature; the torsion piece is excluded by assumption. In the spirit of the gauge approach, it seems much more natural to start with a Lagrangian quadratic in torsion and curvature, typically with

$$\mathcal{V} = -\frac{e}{4\ell^2} (R_{\cdot\mu\nu}^a R^{\cdot\mu\nu}_a + 2R_{\cdot b\nu}^b R^{\cdot c\nu}_c) - \frac{e}{4\kappa} R_{\cdot\mu\nu}^a R^{\cdot\mu\nu}_b$$

The leading terms in the gauge field equations have now the structure of 2nd order differentials for e^a and Γ^{ab}

$$\begin{aligned} \partial\partial e^a + \dots + (e, \Gamma, \partial e, \partial\Gamma) \dots \sim \ell^2 \theta^a, \\ \partial\partial\Gamma^{ab} + \dots + (e, \Gamma, \partial e, \partial\Gamma) \dots \sim \kappa \tilde{\gamma}^{ab}. \end{aligned}$$

We study first the gravitational behaviour of macroscopic matter. Now, macroscopic matter can be modeled by a scalar field, which carries non hypermomentum. Accordingly, there is no reason to gauge the deformations. Gauging the translations yields a teleparallelism (Weitzenböck) space-time. Thus the (torsion)² piece in the Lagrangian, for vanishing curvature, leads to a viable macroscopic theory of gravity.

The Γ^{ab} potentials behave analogously to the internal Yang-Mills potentials. They couple to the hadronic matter with a dimensionless coupling constant κ . Furthermore, they obey a field equation which is of a Yang-Mills type. Altogether we expect the deformational potentials to be short-ranged and asymptotically free ($U(4, \mathbb{R})$ non-Abelian symmetry feature), i.e. acting only "inside" hadrons. In linear approximation, for a static source, one finds from the complete

Lagrangian a confinement type potential¹ proportional to κr . We now conjecture that the confinement occurs only for hadrons, while the leptons evade confinement due to nonlinear representation technique.

REFERENCES

- ¹ F.W.Hehl, G.D.Kerlick and P. von der Heyde, Phys. Lett. 63B (1976) 446.
- ² F.W. Hehl, E.A. Lord and Y.Ne'eman, Phys. Rev. D 17 (1978) 428
- ³ E.A. Lord, Phys. Lett. 65A (1978) 1.
- ⁴ Y. Ne'eman and Dj. Šijački, Proc. Natl. Acad. Sci. USA 76 (1979) 561; Ann. Phys. (N.Y.) 120 (1979) 292.
- ⁵ F.W. Hehl and Dj. Šijački, Gen. Relat. Grav. J. 12 (1980) 83.