ON AXIOMATIZING QUANTUM LOGIC

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The usual formulation of the so called quantum logic (cf.[2]) starts with the partially ordered ensemble of closed subspaces of a Hilbert space $\mathcal{H}$. The intersections ($\cap$) and direct sums ($\oplus$) of closed subspaces of $\mathcal{H}$ have properties of meets ($\cap$) and joints ($\cup$), respectively. Hence, the poset of closed subspaces of $\mathcal{H}$ is a lattice. If for any closed subspace $\mathcal{A}$ of $\mathcal{H}$ we define $\mathcal{A}^{\perp}$ as the closure of the subspace of all vectors in $\mathcal{H}$ that are orthogonal to any vector of $\mathcal{A}$, we obtain an orthocomplemented lattice which, in the general case, is orthomodular rather than distributive, i.e.

$$\mathcal{A}\cap(\mathcal{A}^{\perp}\ominus\ (\mathcal{A}\cap\mathcal{B})) = \mathcal{A}\cap\mathcal{B}$$

By OML we denote the orthomodular lattice of closed subspaces of $\mathcal{H}$, and by oml any orthomodular lattice.

OML or oml is usually identified with the quantum logic (QL). However, oml is not an axiomatized logical system in the proper sense of the word. In order to have such a system we need (1) a formal language and (2) a set of axioms and rules.

The best-known logical system in the propositional language is the classical propositional calculus (PC). From an algebraic point of wiew, PC is an algebra free in the class of formulas, with propositional variables as free generators. It is well-known that in a sense PC is equivalent to the two-element Boolean algebra, and in another to the countably infinite Boolean algebra (the so called Lindenbaum-Tarski algebra).

How should QL be constructed in the propostitional language such that QL be equivalent to oml in the same way as PC is equivalent to the Boolean algebra? Every such attempt is faced with the following problem.

It is well known that in any existing logical system the connective $\Rightarrow$(implication) plays a twofold role : it is both an operation and a relation. For example, the formula $A \Rightarrow B$ of PC is interpreted in the Boolean algebra as an element of that algebra, and, in case that $A \Rightarrow B$ is a theorem of PC, $\Rightarrow$ is a relation (B is derivable from A). The operational properties of $\Rightarrow$ are in close connection with the relational properties. Here the rules of inference play important roles, in particular modus ponens.

The twofold role of implication is harmless in PC (and in Boolean algebra) as well as in other propositional systems. Nevertheless, in oml this is not the case. If in oml we define $a \Rightarrow b =_{Df} a^{\perp} \cup b$, where a and b are elements of a non-distributive oml, then $\Rightarrow$ is not transitive (cf.[4]). If we add the new axiom $a \Rightarrow b = a^{\perp} \cup b$ to any oml, this reduces it to a Boolean algebra. It has been shown that in oml there is no operation having most of desirable properties of an implication. The best approximation is perhaps Dishkant's definition $a \Rightarrow b =_{Df} (a^{\perp} \cap b^{\perp}) \cup b$ (cf.[3]).

Thus, in OML (and in any other oml) there is no possibility of defining $\Rightarrow$ via lattice operations and we are left with the partial ordering as a relation of implication. Of course, we may accept the view that nothing more is needed. However, in such a case QL is hardly a logical system, since the main logical concept - that of implication - is reduced to the partial ordering.

Can implication be introduced in QL (or in any oml) as an operation, independently of the lattice operations? The best candidate for such an operation is one of the implications given axiomatically in the so called relavance logics (cf[1]). Such a connective is independent of the remaining connectives. In the sequel we shall concentrate on the system R of relevance logic. It is known that the axiom-schema of distribution, viz.

$$A \& (B \vee C) \rightleftarrows (A \& B) \vee (A \& C)$$

is independent of the remaining schemata (here &, v, and $\rightleftarrows$ stand for conjunction, disjunction, and equivalence, respectively). It is also known that the corresponding Lindenbaum-Tarski algebra is a distributive lattice. Let us omit the axiom-schema of distribution, and let us adjoin the axiom-schema of orthomodularity (OM):

$$A \& (\bar{A} \vee (A \& B)) \rightarrow B,$$

where ‾ and → stand for negation and implication, respectively. This system is denoted by QR. It is easy to show that if → is neglected, the corresponding Lindenbaum-Tarski algebra is an OML.

QR is a genuine aciomatic logical system in the propositional language. How QR and oml are related? It is difficult to interpret the whole QR in an oml for the following two reasons: first, it is not clear how the fact that a formula A is a theorem of QR should be interpreted in an oml, and second, it is not clear how → should be interpreted in oml.

In order to circumvent the first difficulty we shall adapt QR; in fact, we shall define another propositional system QR' weaker than QR. In order to take care of the second difficulty we shall extend the concept of oml, defining an algebraic structure that will allow an interpretation of QR'. We shall deal with the second problem first.

Let S be a non-empty set, let $\leq$ be a partial ordering of S, and let $\Rightarrow$, $\cap$, $\cup$, and $^\perp$ be operations of S such that $\langle S, \leq, \cap, \cup, ^\perp \rangle$ is an oml, and $\Rightarrow$ satisfies the following conditions, for all a,b,c$\in$S:

$$a \leq (a \Rightarrow b) \Rightarrow b$$

$$a \Rightarrow (a \Rightarrow b) \leq a \Rightarrow b$$

$$a \Rightarrow b \leq (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$$
$$(a \Rightarrow b) \cap (a \Rightarrow c) \leq a \Rightarrow (b \cap c)$$
$$(a \Rightarrow c) \cap (b \Rightarrow c) \leq (a \cup b) \Rightarrow c$$
$$a \Rightarrow a^{\perp} \leq a^{\perp}$$
$$a \Rightarrow b^{\perp} \leq b \Rightarrow a^{\perp}$$

and if $a \leq b$ and $a \Rightarrow b \leq c \Rightarrow b$, then $c \leq d$.

The algebra $QI = \langle S, \leq, \Rightarrow, \cap, \cup, {}^{\perp} \rangle$ is the intended extension of an oml.

Now we define QR^-. Delete from QR the rules of modus ponens and adjunction, and adjoin the following rules: from $A \rightarrow B$ and $(A \rightarrow B) \rightarrow (C \rightarrow D)$ to infer $C \rightarrow D$; from $A \rightarrow B$ and $A \rightarrow C$ to infer $A \rightarrow (B \& C)$; and from $A \rightarrow C$ and $B \rightarrow C$ to infer $(A \vee B) \rightarrow C$.

In is easy to see that any theorem of QR^- is of the form $A \rightarrow B$. It is also easy to interpret QR^- in QI. The variables range over S, and $\&$, $\vee$, $^{-}$ in QR^- are interpreted as $\cap$, $\cup$, ${}^{\perp}$, respectively. For $\rightarrow$ in QR^- we have a twofold interpretation in QI. If $\rightarrow$ is the main connective in a formula, then $\rightarrow$ becomes $\leq$ in QI; otherwise, $\rightarrow$ becomes $\Rightarrow$ in QI.

QR^- is an adequate axiomatization of QI (i.e. QR^- is consistent and complete with respect to QI).

It is obvious that the first problem is solved, for if A is a theorem of QR^-, then it has the form $B \rightarrow C$, and is interpreted in QI as $i(A) \leq i(B)$, where $i(A)$ and $i(B)$ are elements of QI.

As to the second problem, it is solved in the sense that QI is well defined. However, the question as to whether there is a structure $QI^{*} = \langle S, \leq, \Rightarrow, \cap, \oplus, {}^{\perp} \rangle$ such that $\langle S, \leq, \cap, \oplus, {}^{\perp} \rangle$ is OML and satisfies the conditions in QI remains open.

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