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LAGRANGIANS WITH HIGHER ORDER DERIVATIVES AND A STATIC POTENTIAL FOR THE QUARKS AT LARGE DISTANCES

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ABSTRACT:

We investigated the action with higher order derivatives of classical fields and the corresponding field equations. The static solutions are due to the choice of the parameters at a constant term and at terms with first and higher order derivatives in the Lagrangian, of the Coulomb type, of the Yukawa type or of the type usually used for quark (confining) interaction in hadrons or the type of potential barrier allowing tunneling of the particles.

1. INTRODUCTION

Usually Lagrangians are considered which are functions of fields and their first derivatives only and which lead to field equations of the type:

$$(\Box^2 - k^2)A = -j \quad , \tag{1}$$

where j is the source of the field A, which can (together with j) be a scalar, a vector or a tensor. Choosing $k^2 = 0$, $A = (\phi, 0,0,0)$, and $j = (e\delta^3(\vec{r}-\vec{r}_0), 0,0,0)$ the solution is the Coulomb potential, while $k^2 \neq 0$ corresponds to meson fields with a mass $m = k$[+]. In both cases the solutions have a local character.

[+](We shall use in this paper the units in which $\hbar = c = 1$).

Lagrangians with higher order derivatives were discussed in the literature (1 - 5) mainly as a possibility to eliminate infinities appearing in classical or quantum field theories. The authors of reference (3) concluded, for example, that when working with Lagrangians which have higher order derivatives the positive definiteness of the energy and strict causality are difficult, if at all possible, to achieve. We shall not comment in this letter on the two mentioned very important points of field theory with higher order derivatives. We shall rather present static solutions of some (inhomogeneous) equations of motion which are partial differential equations of higher orders and comment on them. We shall show that for a special choice of parameters in the Lagrange density the static potential rises with the distance from the source monotonically, having the form usually used for quarks in hadron physics, while different choices of the parameters in Lagrangian density can force the potential to decrease again at large distances which means that quarks, interacting with such a potential, can tunnel through the barrier.

2. LAGRANGIAN DENSITIES AND EQUATIONS OF MOTION

We choose the following action for vector fields:

$$I = \int \mathcal{L}\left(A^\mu, \partial_\nu A^\mu, \partial_\nu \partial_\rho A^\mu, \ldots \partial_{\nu(k)} A^\mu, \ldots\right) d^4x, \quad (1)$$

where we have used the notation:

$$A^\mu = g^{\mu\nu} A_\nu, \quad A_\nu \equiv (A^0, -\vec{A}), \quad g^{\mu\nu} = g_{\mu\nu} = 0 \text{ if } \mu \neq \nu,$$
$$g_{00} = 1 = -g_{\nu\nu}, \quad g^{\mu\nu} g_{\nu\sigma} = g^\mu{}_\sigma = \delta^\mu{}_\sigma \text{ and } \partial_{\nu(k)} A^\mu = \frac{\partial^k A^\mu}{\partial x^{\nu_1} \partial x^{\nu_2} \ldots \partial x^{\nu_k}}.$$

The Lagrange density is equal to:

$$\mathcal{L} = \sum_{i=1}^{N_f} a_i \left(\partial^{\nu_1} \partial^{\nu_2} \ldots \partial^{\nu_i} A_\mu\right)\left(\partial_{\nu_1} \partial_{\nu_2} \ldots \partial_{\nu_i} A^\mu\right)$$
$$+ a_0 A_\mu A^\mu + g_0 A_\mu j^\mu + \sum_{i=1}^{N_c} G_i \left(\partial^{\nu_1} \partial^{\nu_2} \ldots \partial^{\nu_i} A_\mu \partial_{\nu_1} \ldots \partial_{\nu_i} j^\mu\right) \quad (2)$$

We can rewrite the Lagrangian in a more compact way as follows:

$$\mathcal{L} = \alpha_0 A_\mu A^\mu + \alpha A_\mu \prod_{i=1}^{N_f} (\Box^2 - \alpha_i^2) A^\mu + g_0 A_\mu A^\mu + A_\mu g \prod_{i=1}^{N_c} (\Box^2 - g_i) j^\mu \quad (2a)$$

where $\Box^2 = \partial_\mu \partial^\mu$ and the coefficients a_i and G_i are related to α, α_i, g, g_i. The numbers N_j and N_c determine the highest order derivative of the vector field and the current, respectively.

Defining the dynamics of the fields by an action principle

$$\delta \int \mathcal{L}\, d^4x = 0 \tag{3}$$

we obtain the Euler - Lagrange equations of the form:

$$\frac{\partial \mathcal{L}}{\partial A^{\mu}} - \sum_{i=1}^{N_f} (-1)^{i-1}\, \partial_{\nu_1} \dots \partial_{\nu_i} \left(\frac{\partial \mathcal{L}}{\partial (\partial^{\nu_1} \dots \partial^{\nu_i} A^{\mu})} \right) = 0 \tag{4}$$

Using the Lagrange density from eq. (2a) one obtains:

$$\Big[d_0 + d \prod_{i=1}^{N_f} (\Box^2 - d_i^2)\Big] A^{\mu} = -\Big[g_0 + g \prod_{i=1}^{N_c} (\Box^2 - g_i)\Big] j^{\mu} \tag{4a}$$

3. SOLUTIONS FOR SPECIAL CHOICES OF PARAMETERS

If one chooses $N_f = 1$, $N_c = 0$, $\alpha_0 = 0$, $g_0/\alpha = 1$, one obtains eq. (1), which for $\alpha_1 = 0$, represents the equation of motion in electrodynamics, as was already mentioned, while for $\alpha_1 \neq 0$ the equation for mesons with $m = \alpha_1$ follows. In this letter we shall study only the static solutions of the equations (4a) of a point source $j^{\mu} = (\delta^3(\vec{r} - \vec{r}_0), 0, 0, 0)$. We take therefore $A^{\mu} = (\phi, 0, 0, 0)$ and we look for the solution of eq. (4a) in the form:

$$\phi = \frac{1}{(2\pi)^{3/2}} \int F(\vec{k})\, e^{i\vec{k}(\vec{r} - \vec{r}_0)}\, d^3\vec{k} \tag{5}$$

Inserting $\delta^3(\vec{r} - \vec{r}_0) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}(\vec{r} - \vec{r}_0)}\, d^3\vec{k}$ and ϕ from eq. (5) into eq. (4a), one obtains:

$$F(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\; \frac{-\Big[g_0 + g \prod_{i=1}^{N_c} (-1)(k^2 + g_i)\Big]}{d_0 + d \prod_{i=1}^{N_f} (-1)(k^2 + d_i^2)} \;. \tag{6}$$

We put the expression (6) into eq. (5) and perform the integration over the angle $d\Omega$. Then it follows that

$$\phi = \frac{1}{2\pi^2} \int k^2 dk\, \frac{\sin(k|\vec{r} - \vec{r}_0|)}{k|\vec{r} - \vec{r}_0|}\; \frac{(-1)\Big[g_0 + g \prod_{i=1}^{N_c} (-1)(k^2 + g_i)\Big]}{d_0 + d \prod_{i=1}^{N_f} (-1)(k^2 + d_i^2)} \tag{7}$$

The Cauchy principal value of the integral in eq. (7) converges when $(\vec{r}-\vec{r}_0)$ goes to zero as soon as $N_f \geq N_c + 1$, and $\alpha \neq 0$.

To this particular solution of the inhomogeneous eq. (4a) a general solution of the homogeneous equation should be added to obtain the general solution of the inhomogeneous equation (4a). We shall perform the integration of eq. (7) for some special choices of the parameters appearing in eq. (7).

1. $g=0,\ \alpha_0=0,\ \alpha_i=0,\ i=1,\dots N_f,\ \alpha=1,\ g_0\neq 0$

The equation of motion becomes

$$(-1)^{N_f}\quad \nabla^{2N_f}\phi=-\delta^3(\vec{r}-\vec{r}_0)\,g_0 \tag{8a}$$

In this case from eq. (7) follows:

$$\phi=\frac{g_0(-1)^{N_f}}{4\pi\ (2N_f-2)!}\ |\vec{r}-\vec{r}_0|^{(2N_f-3)} \tag{8b}$$

For $N_f = 1$ the electrostatic potential follows, for $N_f = 2$, the potential is linear for $N_f = 3$ the potential is cubic, and so on. If the general solution of the corresponding homogeneous equation is added to eq. (8b) for $N_f = 2$, for example, additional terms of the form $\frac{a}{|\vec{r}-\vec{r}_0|}+b|\vec{r}-\vec{r}_0|+c|\vec{r}-\vec{r}_0|^2+d$ appear.

2. $g=0,\ \alpha_0=-m^2,\ \alpha=1,\ N_f=1,\ \alpha_1=0,\ g_0\neq 0$

To the equation of motion

$$(\nabla^2-m^2)\phi=\ g_0\,\delta^3(\vec{r}-\vec{r}_0) \tag{9a}$$

corresponds the Yukawa solution:

$$\phi=\frac{g_0}{4\pi|\vec{r}-\vec{r}_0|}e^{-m|\vec{r}-\vec{r}_0|} \tag{9b}$$

3. $g_0\neq 0,\ g\neq 0,\ N_c=1,\ g_1=0,\ \alpha_0=0,\ \alpha=1,\ N_f=2,\ \alpha_1=\alpha_2=0$

The equation

$$\nabla^4\phi=-(g_0\ -\ g\nabla^2)\,\delta^3(\vec{r}-\vec{r}_0) \tag{10a}$$

has as the solution just the potential usually used as the potential between two quarks in hadron physics:

$$\phi = \frac{1}{4\pi}\left(\frac{-g}{|\vec{r}-\vec{r}_0|} + \frac{g_0}{2}|\vec{r}-\vec{r}_0|\right) \tag{10b}$$

which at small distances simulates the usually commented one gluon exchange, while at large distances it assures confinement. But similar solution follows also from eq. (8a) if the general solution of the homogeneous eq. (10 a) is added to the solution (8b) of the inhomogeneous equation.

If eq. (4a) reads :

$$\nabla^6\phi = -\left[g_0 + g g_1 \nabla^2 + g \nabla^4\right]\delta^3(\vec{r}-\vec{r}_0)$$

to the two terms of the solution for ϕ of eq. (10b) the term proportional to $|\vec{r}-\vec{r}_0|^3$ is added. We note that in cases where on the left hand side of eq. (4a) only one term ∇^{2n} appears, with $n \gtrsim 2$, the potential is monotonically rising. But the potential is not necessarily a monotonic function of the distance, if terms with different higher order derivatives and a constant term appear on the left hand side of eq. (4a). We shall present two such cases.

4) $\alpha_0 = 0$, $\alpha = 1$, $N_f = 2$, $d_1^2 = \pm b^2$, $d_2^2 = -B^2$, $g = 0$, $g_0 \neq 0$, $B \neq b$, $B > 0$, $b > 0$.

Then the equation:

$$\left(\nabla^4 \pm (b^2 \mp B^2)\nabla^2 \mp b^2 B^2\right)\phi_\pm = -g_0\delta^3(\vec{r}-\vec{r}_0) \tag{11a}$$

has the solution:

$$\phi_+ = -\frac{g_0}{4\pi(B^2+b^2)|\vec{r}-\vec{r}_0|}\left[\cos(b|\vec{r}-\vec{r}_0|) - e^{-B|\vec{r}-\vec{r}_0|}\right]$$

$$\phi_- = \frac{-g_0}{4\pi(B^2-b^2)|\vec{r}-\vec{r}_0|}\left[e^{-b|\vec{r}-\vec{r}_0|} - e^{-B|\vec{r}-\vec{r}_0|}\right] \tag{11b}$$

The two potentials are presented in Fig. 1

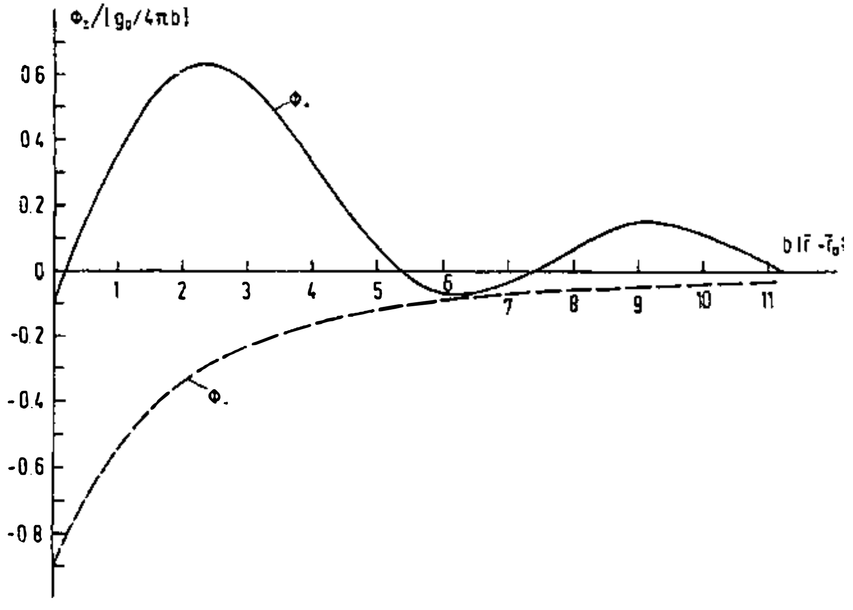

Fig. 1. The two curves present the two potentials Φ_+ (full line) and Φ_- (dotted line) as a function of $|\vec{r}-\vec{r}_0|\, b$. We chose $\frac{B}{b} = 0.1$, While Φ_+ is periodic, Φ_- is a monotonic function.

While Φ_+ is a periodic potential with its strength decreasing as $(|\vec{r}-\vec{r}_0|)^{-1}$, the potential Φ_- is a monotonic function. Both potentials have no singularity at $\vec{r}=\vec{r}_0$, which is also true for all potentials for which equations without derivatives of currents appear on the right hand side of eq. (4a).

5. We shall present the solution of eq. (4a) with $N_f = 3$, $g = 0$, $\alpha_0 = 0$, $\alpha = 1$, $\alpha_1^2 = \pm B^2$, $\alpha_2^2 = \pm b^2$, $\alpha_3 = 0$ and $B \neq b$, $b > 0$, $B > 0$:

$$\left[\nabla^6 \pm (B^2+b^2)\nabla^4 + B^2 b^2 \nabla^2\right]\Phi_\pm = g_0 \delta^3(\vec{r}-\vec{r}_0) \qquad (12a)$$

The solutions:

$$\phi_+ = \frac{-g_0}{4\pi b^2 B^2 (b^2 - B^2)|\vec{r}-\vec{r}_0|}\left[B^2(\cos|\vec{r}-\vec{r}_0|b - 1) - b^2(\cos(B|\vec{r}-\vec{r}_0|) - 1)\right] \tag{12b}$$

$$\phi_- = \frac{-g_0}{4\pi b^2 B^2 (b^2 - B^2)|\vec{r}-\vec{r}_0|}\left[B^2(e^{-b|\vec{r}-\vec{r}_0|} - 1) - b^2(e^{-B|\vec{r}-\vec{r}_0|} - 1)\right]$$

are presented in Fig. 2. While ϕ_+ is oscilatory with an amplitude decreasing again as $(|\vec{r} - \vec{r}_0|)^{-1}$, ϕ_- is a monotonic function.

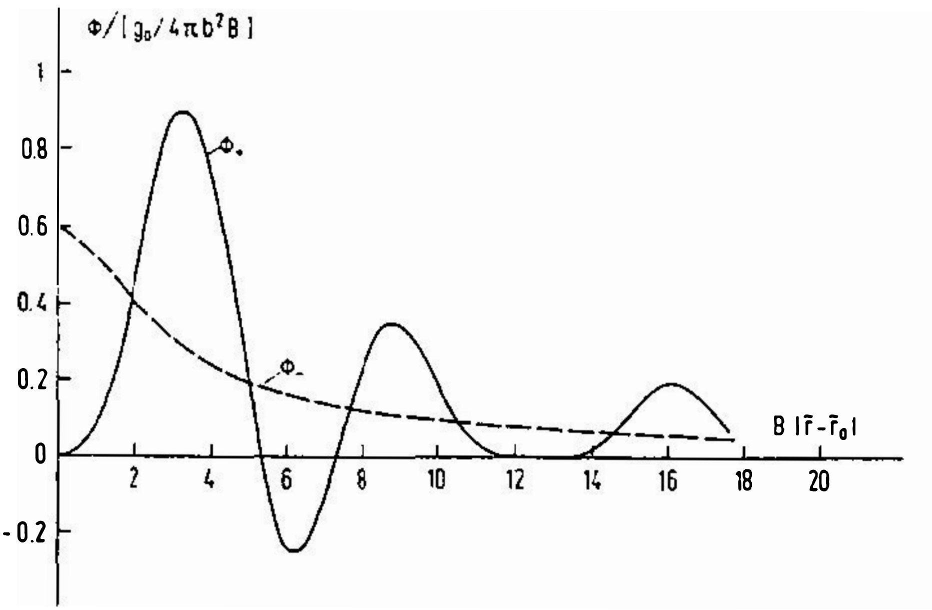

Fig. 2. Potentials ϕ_+ (full line) and ϕ_- (dotted line) from eq. (12b) as a function of the distance from the point source. We chose $\frac{b}{B} = 1.5$

6. Similar behaviour is also displayed by the potential which is the solution of equation:

$$(\nabla^4 + b^2\nabla^2)\phi = -g_0\delta^3(\vec{r}-\vec{r}_0) \tag{13a}$$

The potential is oscilatory

$$\phi = \frac{g_0}{4\pi b^2 |\vec{r}-\vec{r}_0|}\left(1 - \cos(b|\vec{r}-\vec{r}_0|)\right) \tag{13b}$$

and is presented in Fig. 3.

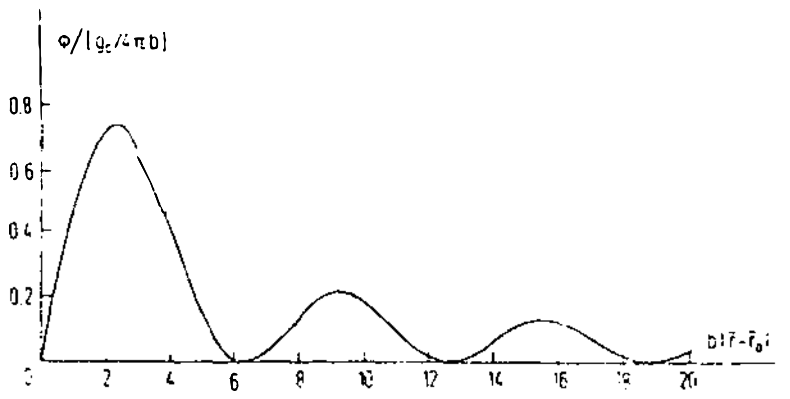

Fig. 3. The potential Φ from eq. (13b) is presented as a function of the distance from the source.

4. CONCLUSION

We have seen that Lagrangian densities with higher order derivatives with respect to fields only, have solutions of the nonhomogeneous equations with no singularities at the origin. This fact can be very promising for classical and quantum field theories. A long time ago (in 1950) Pais and Uhlenbeck in their paper (3) presented a detailed study of the classical and quantum field theories with higher order derivatives, and fourteen years later Barut and Mullen added their work (4) on the same subject. Still the advantage of such theories is not yet evident and additional work is needed to clarify the convergence, the positive definiteness of the free field energy and the causal behaviour of the state describing a physical system. Even less evident are terms with first and higher order derivatives of a current on the right hand side of eq. (4a), particularly because the singular behaviour of the potential at the origin again appears. We used them to reproduce the quark interaction (without adding the general solution of the homogeneous equation), but as yet we have not studied in depth the meaning of such terms. We would like to point out that terms with higher order derivatives suggest that confinement is not necessarily absolute. For the special choice of parameters the potentials behave like a barrier which allows tunnelling. One can estimate the shape of the barrier by choosing the parameters of the potential in such a way that the properties of hadrons will be reproduced. We determined the two parameters of the potential from example 6 in such a way that the system $c\bar{c}$ has approximately the correct spectrum. One then obtains the decay time for $c\bar{c}$ as $\tau \sim 10^{2700}$ s, which is enormous in comparison with the life time of the universe. By choosing different Lagrangian densities one can change the barrier. In any case, the periodicity of the barrier remains for the finite number of higher order derivatives. Such a periodic structure of the potential would suggest the crystalline structure of the universe, the meaning of which should be studied.

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