FIZIKA 19, (1987), Supplement 2, 51-55
(Proc. Workshop on Mesonic Degrees of Freedom in Hadrons, Bled/Ljubljana, June/July 1987)

Solving the sigma model with quarks using a generalized hedgehog ansatz [1]

M. Fiolhais[a], K. Goeke[b,c] and J.N. Urbano[a]

[a] Departamento de Física da Universidade, P-3000 Coimbra, Portugal.

[b] Institut für Kernphysik, Kernforschungsanlage Jülich GmbH, D-5170 Jülich, West Germany.

[c] Institut für Theoretische Kernphysik, Universität Bonn, D-5300 Bonn, West Germany.

Abstract

Approximate solutions of the chiral soliton model based on the σ– model with quarks are presented, starting from a generalized hedgehog ansatz. The Peierls-Yoccoz projection method is used to obtain states with good angular momentum and isospin quantum numbers. Several properties of the nucleon are calculated. Due to the generalization of the hedgehog, significant improvements in virial theorems and the Goldberger–Treiman relation have been found.

1. Introduction

The idea that QCD leads to spontaneously broken chiral symmetry and that this is the most important feature in the low energy domain, suggested a model for the nucleon and the delta based on the linear σ- model of Gell-Mann and Lévy[1)] involving quarks, a sigma and a pion field[2,3)]. The Lagrangian density reads

$$\hat{\mathcal{L}} = \hat{\bar{q}}(i\gamma^\mu\partial_\mu)\hat{q} + \frac{1}{2}\partial^\mu\hat{\sigma}\partial_\mu\hat{\sigma} + \frac{1}{2}\partial^\mu\hat{\vec{\pi}}.\partial_\mu\hat{\vec{\pi}} - g\,\hat{\bar{q}}(\hat{\sigma} + i\vec{\tau}.\hat{\vec{\pi}}\gamma_5)\hat{q} - \frac{\lambda^2}{4}(\hat{\sigma}^2 + \hat{\vec{\pi}}^2 - \nu^2)^2 + m_\pi^2 f_\pi\hat{\sigma}. \qquad (1)$$

The vacuum expectation values of the pion and sigma fields are zero and $f_\pi = 0.093\,\mathrm{GeV}$ respectively. The chiral symmetry is explicitly broken by the last term in (1) where $m_\pi = 0.138\,\mathrm{GeV}$ is the pion mass. The parameters λ and ν are related to the meson masses and the pion decay constant by $m_\sigma^2 = 2\lambda^2 f_\pi^2 + m_\pi^2$ and $\nu^2 = f_\pi^2 - m_\pi^2/\lambda^2$. For $m_\sigma > 0.7\,\mathrm{GeV}$ the results are almost independent of this parameter and, as in ref. 2), we take $m_\sigma = 1.2\,\mathrm{GeV}$. Basically g is the only adjustable parameter of the model.

This model has been recently considered by several authors to provide approximate solutions with well defined angular momentum and isospin quantum numbers[2,4–8)]. The goodness of the different approximations can be tested by checking virial theorems which are fulfilled by the exact solutions.

The aim of this study is to construct approximate solutions for the Lagrangian (1) consistent with the Goldberger-Treiman (GT) relation and satisfying a virial theorem associated with the evaluation of the pion–nucleon coupling constant $g_{\pi NN}$. These are not fulfilled by the approximate solutions based on a coherent-pair approach[7)] or using projection techniques applied to pure hedgehog structures[5)]. Since the projection

[1] Work supported in part by JNICT, Lisbon, by the Bundesministerium für Forschung und Technologie, Bonn and by NATO Grant RG 85/0217.

techniques seem to be an appropriate method to obtain eigenstates of J^2, T^2, J_z and T_3 in chiral models[9], the simplest way to try to overcome the above mentioned difficulties is to relax the rigid hedgehog stucture.

Here we start from a generalization of the hedgehog ansatz allowing for single particle quark states with grand-spin zero and one and we apply the Peierls-Yoccoz method[10] in order to extract the states with good quantum numbers of spin and isospin. As it is well-known, this method, widely used in conventional nuclear physics[11], is nothing but the generator coordinate method applied to restore the symmetries violated by mean-field solutions. It is a quantal method and therefore it requires a quantum mechanical state to describe the system. Usually the simplest choice is the coherent state[12].

2. The generalized hedgehog

In the present calculation only the three valence quarks occupying the same orbital-spin-isospin state are considered, with the wave function:

$$< \boldsymbol{r} \mid q_{gh} > = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} u(r) \\ i\, v(r)\, \boldsymbol{\sigma}.\hat{\boldsymbol{r}} \end{pmatrix} \mid \chi_{gh} > . \tag{2}$$

For the spin–flavour function $\mid \chi_{gh} >$ we take

$$\mid \chi_{gh} >= \cos\eta \ \mid u \downarrow> - \sin\eta \ \mid d \uparrow>, \tag{3}$$

where η is a variational parameter. The state (3) obviously reduce to the standard hedgehog for $\eta = 45^o$. The motivation for the choice of (3) can be found in ref. 13). Using a variational calculation as in ref. 13), it can be shown that the pion field compatible with equations (2) and (3) yields a coherent state for the pions $\mid \Pi_{gh} >$ with the property

$$< \Pi_{gh} \mid \hat{\pi}_t \mid \Pi_{gh} > = \begin{cases} \frac{x}{r}\, \phi(r) & t = 1 \\ \frac{y}{r}\, \phi(r) & t = 2 \\ \frac{z}{r}\, \varphi(r) & t = 3, \end{cases} \tag{4}$$

where $\hat{\pi}_t$ is the cartesian component t of the pion field operator. Similarly the form (2) suggests for the sigmas a coherent state $\mid \Sigma >$ with spherically symmetric amplitudes, yielding a classical field configuration depending only on the radial coordinate:

$$< \Sigma \mid \hat{\sigma} \mid \Sigma >= \sigma(r). \tag{5}$$

The coherent states are, among the quantum mechanical states, the ones which minimize the uncertainty relation. By definition they are eigenstates of the annihilation operators appearing in the expansions for the field operator and its conjugate momentum in a given orthonormal basis. There is a certain arbitrariness in the choice of the basis, although the exact solutions do not depend on that. Here we considered an expansion in plane waves with the frequencies $\omega(k)$ corresponding to the free field dispersion relation $\omega^2 = k^2 + m^2$, where m is the meson mass. The classical fields are extracted as expectation values of the quantum field operators, as in (4) and (5), between the coherent states whose amplitudes are obtained variationally.

Assuming for the quarks a product state, the Fock state for the generalized hedgehog (GH) baryon is given by[8]

$$\mid \psi_{gh} >=\mid q^3_{gh} > \mid \Pi_{gh} > \mid \Sigma > . \tag{6}$$

This state has not a definite grand-spin but it is still an eigenstate of the third component of the grand-spin operator with eigenvalue equal to zero.

The model states for the physical particles are obtained by projecting $| \psi_{gh} >$ on good angular momentum and isospin. Contrary to the standard hedgehog[4,5)], here separate projections on J and T are necessary. The projected states are given by

$$| \psi_{gh}^{JT} >= \sum_{K,K_T} g_{KK_T} P^J_{MK} P^T_{M_T K_T} | \psi_{gh} >, \tag{7}$$

where the coefficients g_{KK_T} are the lowest energy solutions of the following Griffin–Hill–Wheller discrete equation[11)]

$$\sum_{K'K'_T} (h^{JT}_{KK'K_TK'_T} - E^{JT} n^{JT}_{KK'K_TK'_T}) g_{K'K'_T} = 0, \tag{8}$$

with the kernels $h^{JT}_{KK'K_TK'_T} = < \psi_{gh} | \hat{H} P^J_{KK'} P^T_{K_TK'_T} | \psi_{gh} >$ and $n^{JT}_{KK'K_TK'_T} = < \psi_{gh} | P^J_{KK'} P^T_{K_TK'_T} | \psi_{gh} >$, where $P^J_{MK} = \frac{2J+1}{8\pi^2} \int d\Omega \mathcal{D}^J_{MK}{}^*(\Omega) R(\Omega)$. In the evaluation of projected quantities, important numerical simplifications occur due to the axial symmetry of (6) in grand–spin space: the six-fold integration over the Euler angles can be reduced in such a way that only four, namely over e.g. α, β, γ, for spin and $\tilde{\beta}$, for isospin, are necessary. According to the findings of ref. 13), we notice that the generalization of the hedgehog only makes sense if the calculation goes beyond the mean field approximation as, for instance, in the present case with the use of projection techniques.

3. Results

The expectation value of the Hamiltonian calculated with the ansatz (7) is the mean field energy. The solitonic mean field equations result from the variational principle $< \delta\psi_{gh} | \hat{H} | \psi_{gh} >= 0$ and the corresponding energy shows a minimum at $\eta = 45^o$. By projecting these mean field solutions the energies of the nucleon and the delta exhibit minima about $\eta = 20^o$. This method we call projection after variation (GH – PAV) since only the parameter η is varied after the projection. A better procedure would consist in performing the variations with respect to the radial fields and η after the projection. This leads to a set of five coupled integro–differential equations which we did not solve. Instead we performed a partial projection before variation (GH – PBV) by keeping the radial dependence of the mean fields $\phi(r)$ and $\varphi(r)$ obtained for the different angles η but varying the overall strengths after the projection. Actually it turned out that $\varphi(r)$ remains almost unchanged. The variation after the projection is therefore limited to the plane (η, Λ) where Λ is the variational parameter introduced by $\phi(r) \rightarrow \Lambda\phi(r)$, while the u, v, σ, ϕ and φ are obtained in the mean field approximation for a given η. This simplification is motivated by the fact that for the standard hedgehog that degree of freedom is solely responsable for practically the total energy changes in going from a PAV to a PBV calculation. For the nucleon the value of Λ is about 1.3, being almost independent of the parameters of the model in the relevant ranges.

As we have mentioned in the Introduction we studied the quality of the various approximate solutions by looking at the violation of the virial theorem associated with $g_{\pi NN}$ and with the GT relation, $g_{\pi NN} f_\pi = g_A \, M_N$, where g_A is the axial vector coupling constant and M_N the physical nucleon mass. The first virial theorem states that for an *exact* solution the pion-nucleon coupling constant can be determined either using the "source" operator,

$$\frac{g_{\pi NN}}{2\,M_N} = < N| : \int d\boldsymbol{r}\, z \hat{j}^3_\pi(\boldsymbol{r}) : |N >, \tag{9}$$

or the "field" operator,

$$\frac{g'_{\pi NN}}{2\,M_N} = m^2_\pi < N| : \int d\boldsymbol{r}\, z \hat{\pi}_3(\boldsymbol{r}) : |N >, \tag{10}$$

where $|N>$ stands for the state with the quantum numbers $J = T = M = -M_T = \frac{1}{2}$. Both expressions should lead to the same value[5]. Using the GT relation we also determined $g_{\pi NN}^{(GT)} = g_A M_N / f_\pi$ using the calculated value for g_A. Introducing $g_{\pi NN}^{(av)}$ as the arithmetic average of the $g_{\pi NN}$'s obtained from (9) and (10), we define a global violation as

$$\mathcal{V} = \frac{g_{\pi NN} - g'_{\pi NN}}{g_{\pi NN}^{(av)}} + 2\frac{1.08 g_{\pi NN}^{(GT)} - g_{\pi NN}^{(av)}}{1.08 g_{\pi NN}^{(GT)} + g_{\pi NN}^{(av)}}. \tag{11}$$

The factor 1.08 is introduced to correct for the slight explicit violation of the chiral symmetry observed experimentally. Also the pion-nucleon coupling constant determined through (9) and (10) must be multiplied by a factor 1.05 estimated by Birse[5] to correct for the fact that the experimental value is obtained for $q^2 = m_\pi^2$ rather then $q^2 = 0$. Table 1 shows the global violation (11). For comparison we also give the results obtained in PAV and PBV calculations from the standard hedgehog (HH). The extent to which $\mathcal{V}$ goes to zero provides a test of the quality of the approximation used. In contrast to all other approaches, in (GH - PBV) the virial theorems come out rather well fulfilled, although the values of g_A and $g_{\pi NN}$ come out too large compared to experiment.

Method	g_A	$g_{\pi NN}$	$g'_{\pi NN}$	$g_{\pi NN}^{(GT)}$	$\mathcal{V}$ (%)
HH–PAV	1.72	23.29	14.92	17.34	42
HH–PBV	1.78	16.94	12.80	17.95	56
GH–PAV	1.69	23.70	16.84	17.04	24
GH–PBV	1.75	17.55	17.85	17.65	6
Exp.	1.23	13.60	13.60	12.50	0

Table 1: *List of the values for g_A, the pion-nucleon coupling constant, and the global violations of virial theorems as given by expression (11). The methods and the different ways to calculate the pion-nucleon coupling constant are explained in the text. For each method the quark–meson coupling constant g was adjusted to obtain the proper nucleon mass.*

We show in Table 2 several values obtained for some nucleon and delta observables, using the (GH - PBV) method. There is a reasonable agreement with experiment, in particular for the mean-square charge radii of the proton and the neutron and for the magnetic moment of the proton.

The delta – nucleon splitting is half of the experimental value, leaving room for a residual chromo-magnetic interaction.

In this study we concluded that the generalization of the hedgehog structure is necessary to fulfil two virial theorems. Nevertheless g_A, $g_{\pi NN}$, on the one hand, and the absolute value of the magnetic moment of the neutron, on the other, come out too large compared to experiment. These discrepancies might be associated with the fact that we are neglecting the efects of the sea quarks. It is also desirable to look at those points in the framework of an extension of the present model to include vector

	GH–PBV	*Exp*
E_N (GeV)	0.938	0.938
$E_\Delta - E_N$ (GeV)	0.154	0.294
$< r^2_{ch} >_p$ (fm^2)	0.64	0.65
$< r^2_{ch} >_n$ (fm^2)	-0.11	-0.12
μ_p (n.m.)	2.76	2.79
μ_n (n.m.)	-2.40	-1.91

Table 2: *Results for some nucleon and delta observables using the GH–PBV method. The quark - meson coupling constant,* $g = 5.0$, *has been chosen to give the nucleon mass.*

mesons as suggested by Broniowski and Banerjee[14], treating quantum mechanically all the mesons involved, as in the present work.

References

1) M. Gell-Mann and M. Lévy, *Nuovo Cimento* **16** (1960) 705.

2) M.C. Birse and M.K. Banerjee, *Phys. Lett.* **136 B** (1984) 284; *Phys. Rev.* **D 31** (1985) 118.

3) S. Kahana, G. Ripka and V. Soni, *Nucl. Phys.* **A 415** (1984) 351.

4) B. Golli and M. Rosina, *Phys. Lett.* **165 B** (1985) 347.

5) M.C. Birse, *Phys. Rev.* **D 33** (1986) 1934.

6) Th.D. Cohen and W. Broniowski, *Phys. Rev.* **D 34** (1986) 3472.

7) K. Goeke, M. Harvey, U.-J. Wiese, F. Grümmer and J.N. Urbano, *Z. Physik* **A 326** (1987) 339.

8) M. Fiolhais, A. Nippe, K. Goeke, F. Grümmer and J.N. Urbano, *Phys. Lett.*, in press.

9) M. Fiolhais and M. Rosina, *Portg. Phys.* **17** (1986) 49.

10) R.E. Peierls and J. Yoccoz, *Proc. Roy. Soc. London* **A 70** (1957) 381.

11) P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, New York, 1980).

12) J. da Providência, *Nucl. Phys.* **B 57** (1973) 536. J. da Providência and J.N. Urbano, *Phys. Rev.* **D 18** (1978) 4208.

13) M. Fiolhais, J.N. Urbano and K. Goeke, *Phys. Lett.* **150 B** (1985) 253. K. Goeke, J.N. Urbano, M. Fiolhais and M. Harvey, *Phys. Lett.* **164 B** (1985) 249.

14) W. Broniowski and M.K. Banerjee, *Phys. Rev.* **D 34** (1986) 849.