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TIME DEPENDENT HARTREE APPROACH FOR KINK-ANTIKINK ANNIHILATION

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Abstract
Time dependent Hartree approximation for non-linear theories in 1+1 dimensions is derived and applied to the description of the kink-antikink annihilation in ϕ^4 theory. The self-consistent equations are solved in the weak coupling limit: the time evolution of the system is then governed by the classical equation of motion. A long-lived oscillatory state formed after the collision is explained in terms of quasi bound states of the mean field potential.

1. Introduction

The description of baryon-antibaryon annihilation into mesons in terms of the soliton-antisoliton system of non-linear meson theories , e.g. the Skyrme model, is a particularly interesting problem: in such a process two asymptotically topological structures decay into their constituents. As a first step toward a realistic calculation it is instructive to solve a simpler model problem in 1+1 dimensions. In the present paper we study the soliton-antisoliton system within the framework of the Time Dependent Hartree Approximation (TDHA). The Hartree approach in field theory has been used so far to describe the vacuum and the soliton sector of the sine-Gordon and the ϕ^4 theory in 1+1 dimensions[1-4]. In Sect. 2. we show how to extend this approach to the non-stationary case. The S-matrix for the annihilation into a given number of mesons is introduced in Sect. 4. In Sect. 5. we present some results of the kink-antikink annihilation in ϕ^4 theory.

2. Time Dependent Hartree Approximation

The classical Hamiltonian of the ϕ^4 theory in 1+1 dimensions[5]:

$$H = 1/2 \int dx \left[\dot{\phi}(x)^2 + \phi'(x)^2 + \lambda/2\, (\phi(x)^2 - m^2/\lambda)^2 \right], \tag{2.1}$$

has two vacuum solutions, $\phi_v = \pm m/\sqrt{\lambda}$, which break the symmetry of the Hamiltonian, and two static solutions with nontrivial topology: $\phi_{K,\bar{K}}(x) = \pm \phi_v \tanh(m(x-a)/\sqrt{2})$, referred to as "kink" and "antikink", respectively.

To quantize the theory we introduce the field operators as normal mode fluctuations around the vacuum value $\phi_v = -m/\sqrt{\lambda}$:

$$\hat{\Phi}(x) = \hat{\phi}(x) - m/\sqrt{\lambda}, \qquad \hat{\phi}(x) = 1/\sqrt{L} \sum_k 1/\sqrt{2\omega_k} \left[a_k e^{ikx} + a_k^+ e^{-ikx} \right], \tag{2.2}$$

where a_k^+ and a_k are creation and annihilation operators for free mesons with momentum k and $\omega_k^2 = k^2 + \mu^2$, $\mu = \sqrt{2}\, m$ is the physical mass of the meson. The Hamiltonian of the quantized theory is written as the normal ordered product with respect to the operators a_k:

$$H = 1/2 \int dx : \left[\hat{\pi}(x)^2 + \hat{\phi}'(x)^2 + \mu^2 \hat{\phi}(x)^2 + \lambda/2 (\hat{\phi}(x)^4 + 4\phi_v \hat{\phi}(x)^3) \right] : . \tag{2.3}$$

Due to the interaction the meson states are not plane waves but rather distorted waves in a mean field potential $U_{eff}(x)$. The idea of the Hartree approximation is to replace the plane waves in (2.2) by the wave functions $\xi_n(x)$ which satisfy the Klein-Gordon equation:

$$\left[-\partial^2/\partial x^2 + \mu^2 + 2\mu U_{eff}(x) \right] \xi_n(x) = \varepsilon_n^2\, \xi_n(x), \tag{2.4}$$

and expand:

$$\hat{\phi}(x) = \sum_n 1/\sqrt{2\varepsilon_n} \left[c_n \xi_n(x) + c_n^+ \xi_n^*(x) \right].$$

The creation and annihilation operators c_n^+ and c_n are related to the operators a_k^+ and a_k by the canonical transformation[3].

The state vector in the Hartree approximation is written as a coherent state with respect to the reference vacuum $|0\rangle_c$, $c_n|0\rangle_c = 0$:

$$|\Phi\rangle = \exp\{-\sum_n \varepsilon_n |C_n|^2/4\} \exp\{\sum_n \sqrt{\varepsilon_n/2}\, C_n c_n^+\}|0\rangle_c \, , \tag{2.5}$$

where the coefficients C_n are related to the expectation values:

$$\langle \Phi | \hat{\phi}(x) | \Phi \rangle = 1/2 \sum_n \left[C_n \xi_n(x) + C_n^* \xi_n^*(x) \right] , \tag{2.6}$$

$$\langle \Phi | \hat{\pi}(x) | \Phi \rangle = -i/2 \sum_n \varepsilon_n \left[C_n \xi_n(x) - C_n^* \xi_n^*(x) \right] . \tag{2.7}$$

We derive the TDH equations relating the coefficients C_n, the effective potential $U_{eff}(x)$ and the wave functions $\xi_n(x)$ from the variational principle

$$\langle \delta \Phi(t) | i\, \partial/\partial t - H | \Phi(t) \rangle = 0 \, ,$$

allowing the following variations of the state vector $|\Phi(t)\rangle$:

$$\langle \delta \Phi^{(i)}(t)| = \sum_n q_n \langle \Phi(t)| b_n \, , \qquad \langle \delta \Phi^{(ii)}(t)| = \sum_{nm} p_{nm} \langle \Phi(t)| b_n b_m$$

Here $b_n = c_n - \sqrt{\varepsilon_n/2}\, C_n$ with the property $b_n|\Phi(t)\rangle = 0$, q_n and p_{nm} are arbitrary coefficents. The coefficients C_n are time dependent and complex; the equations (2.6) and (2.7) are then independent, and we choose to write the expectation values of $\hat{\phi}(x)$ and $\hat{\pi}(x)$ as $\phi(x,t)$ and $\dot{\phi}(x,t)$ respectively. In the following we shall see that this choice establishes a link between the TDHA and the classical equation of motion for the field $\phi(x,t)$.

The first variation (i) then gives[6]:

$$\ddot{\phi}(x,t) - \phi''(x,t) + \mu^2 \phi(x,t) + \lambda[\phi(x,t)^3 + 3\phi_v \phi(x,t)^2 + 3\,\Delta(x)(\phi(x,t) + \phi_v) = 0 \, , \tag{2.8}$$

where the quantum fluctuating part $\Delta(x)$ is given by:

$$\Delta(x) = \sum_n |\xi_n(x)|^2/2\varepsilon_n - 1/L \sum_k 1/2\omega_k \quad . \tag{2.9}$$

The second variation (ii) gives the identical result as in the static case: the functions $\xi_n(x)$ satisfy the Klein-Gordon equation (2.4) with the potential:

$$2\mu U_{eff}(x) = 3\lambda[\phi(x,t)^2 + 2\phi_v\phi(x,t) + \Delta(x)] \ . \tag{2.10}$$

The self-consistent equations (2.8) and (2.4) are coupled: the effective potential is expressed in terms of the field $\phi(x,t)$; at the same time $\Delta(x)$ contains the solution of (2.4). In the weak coupling limit, $(\lambda/\mu^2 << 1)$, $\Delta(x)$ is a small correction and the equation (2.8) reduces to the classical equation of motion, and (2.4), to the classical equation[7] for small oscillations around $\phi(x,t)$. The coefficients C_n do not enter explicitely in the equations; they can be expressed as:

$$C_n(t) = \int \phi(x,t)\xi_n^*(x)dx + (i/\varepsilon_n)\int \dot{\phi}(x,t)\xi_n^*(x)dx \ . \tag{2.11}$$

3. The Hartree vacuum

The Hamiltonian (2.3) has been normal ordered with respect to the vacuum $|0\rangle_a$ which is specified by the expectation value of the field $-m/\sqrt{\lambda}$ and the meson mass $\sqrt{2}\,m$. We do not know a priory whether this vacuum (the "classical vacuum") is the solution of the self-consistent equations with the lowest energy; another vacuum, $|0\rangle_\mu$, specified by an expectation value ϕ_v and a meson mass μ, may have lower energy and thus represent the true vacuum of the Hartree approximation. (Because of the translational invariance the basis states in (2.4) have to be plane waves; they may differ only in the meson mass.) The energy of the state $|0\rangle_\mu$ is most easily evaluated by changing the normal ordering with respect to the mass $\sqrt{2}\,m$ to that with respect to μ[8]:

$$E_\mu = {}_\mu\langle 0|:H:_{\sqrt{2}m}|0\rangle_\mu = \int dx\{[(\mu^2-2m^2) + (3\lambda\phi_v^2-m^2)\ln(2m^2/\mu^2) + + 3\lambda\ln^2(2m^2/\mu^2)/8\pi]/8\pi + \lambda(\phi_v^2 - m^2/\lambda)^2/4\} \ .$$

In the minimum, ϕ_v and μ obey the following equations: $y = 1 + 3\lambda \ln z/(4\pi m^2)$ and $1/z + 3\lambda \ln z/(4\pi m^2 z) = 1$, where $z = \mu^2/2m^2$ and $y = \phi_v{}^2/(m^2/\lambda)$. The solution with a finite energy exits for $\lambda/2m^2 < 2\pi/3$ with $z = y = 1$, i.e., the Hartree vacuum corresponds to the "classical vacuum" for sufficiently small coupling strengths.

4. The S-matrix for the soliton-antisoliton annihilation

Our aim is to calculate the probabilities for the kink-antikink system to decay into a given number of mesons. The initial state $|\Psi_0\rangle$ of such a process describes the non-interacting kink and antikink at large distance, and the final states are taken in the form:

$$|N,k_1,k_2,\ldots k_N\rangle = a^+_{k1}a^+_{k2}\ldots a^+_{kN}|0\rangle_a$$

The S matrix for the annihilation into N mesons with momenta $k_1,\ldots k_N$ is then defined as:

$$S(0;N,k_1\ldots k_N) = \langle N,k_1,\ldots k_N|S|\Psi_0\rangle = \langle N,k_1,\ldots k_N|\Psi_{K\bar{K}}(t\to\infty)\rangle$$

We shall determine the time evolution of the state $\Psi_{K,\bar{K}}(t)$ in the TDHA. At $t\to\infty$, the self-consistent potential (2.10) vanishes and the basis states $\xi_n(x)$ become plane waves and the S-matrix can immediately be evaluated:

$$S(0;N,k_1,\ldots k_N) = e^{-\langle N(\infty)\rangle/2}\sqrt{\omega_1/2}\,C_{k1}(\infty)\ldots\sqrt{\omega_N/2}\,C_{kN}(\infty). \qquad (4.1)$$

By $C_k(\infty)$ we mean the values of the coefficients (2.11) at $t\to\infty$. $\langle N\rangle$ is the average number of mesons in the coherent state $\langle N(\infty)\rangle = \int dk\omega_k/2\,|C_k(\infty)|^2$.

The probability for the annihilation into N mesons is then:

$$\sigma_N = A/N!\int[\omega_1/2\,|C_{k1}(\infty)|^2\ldots\omega_N/2\,|C_{kN}(\infty)|^2]\,\delta(\sum_i\omega_i-E)\,\delta(\sum_i k_i)\,dk_1\ldots dk_N\,.$$

The normalization constant A is chosen so that $\sum_N\sigma_N = 1$.

5. Numerical simulation of the kink-antikink annihilation

The time evolution of the kink-antikink system in the TDHA is governed by the self-consistent equations (2.8) and (2.4) with the self-consistent potential (2.10). To solve them, one would have to make small steps in time, in each step find the self-consistent solution, and procede until the asymptotic conditions are reached, i.e. the potential becomes sufficiently small compared to the meson mass μ. The main difficulty in carrying out this program lies in solving (2.4) to determine the basis states $\xi_n(x)$; during and after the collision, the effective potential spreads over a large region which makes the numerical calculation extremely tedious.

We therefore suggest an approximate method which consists in neglecting the term $\Delta(x)$ resulting from quantum fluctuations, i.e. solving the classical equation of motion [9]. This approximation is valid in the weak coupling limit, or equivalently, in the limit of a large kink mass compared to the mass of the meson. (For illustration we choose this ratio to be about that of the nucleon mass and the pion mass, so that λ/μ^2 is close to 1/20.) Solving the classical equation of motion is much simpler than the self-consistent procedure described above, since the basis states do not enter in the calculation of the time evolution. The coefficients C_k of the coherent state describing the outgoing meson radiation far from the source where the effective potential vanishes, can be determined by performing either the spatial Fourier transform or the Fourier transform in time of the fields $\phi(x,t)$ and $\dot{\phi}(x,t)$.

The most significant result of the numerical analysis is that the kink-antikink annihilation is a slow process, i.e. it evolves on time scales large compared to μ^{-1}, which does not allow to explore the limit $t \to \infty$ numerically. Nevertheless, the behaviour of the system can well be understood by carrying out a precise Fourier analysis of the outgoing meson radiation spectrum in relatively short time intervals. One notices that there are only narrow peaks in that spectrum far away from the source (Fig. 1). Such a spectrum is explained by assuming that the peaks appear at the frequencies which are combinations of two principal frequencies, one at the energy around $0.8\,\mu$ and the other slightly below $1.0\,\mu$.

The appearence of principal frequencies below the meson treshold can be well understood in terms of the mean field picture. The nonlinear interactions between mesons generate an atractive mean field potential. In one dimension, any arbitrarily weak atractive potential has at least one bound state. The dominant frequency ω_0 found in the numerical calculation can therefore be identified with the energy of the lowest bound state. The residual interaction between mesons contains terms of higher order in the field $\phi(x,t)$, so that meson processes such as the ones illustrated in Fig. 2 occur: The non-linear interaction excites states with energies $\omega = N\omega_1 + M\omega_0$ for $N,M \in Z$.

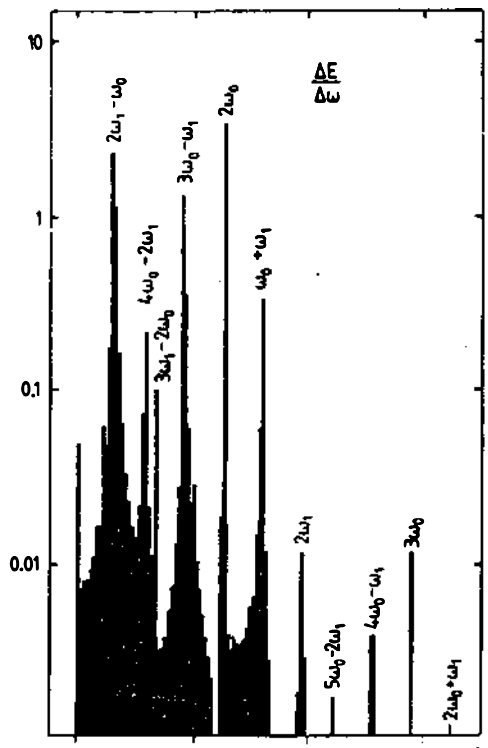

Fig. 1

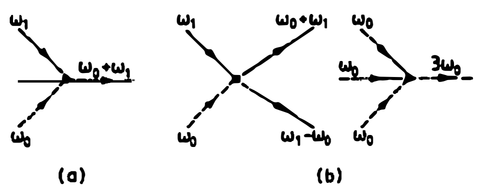

Fig. 2

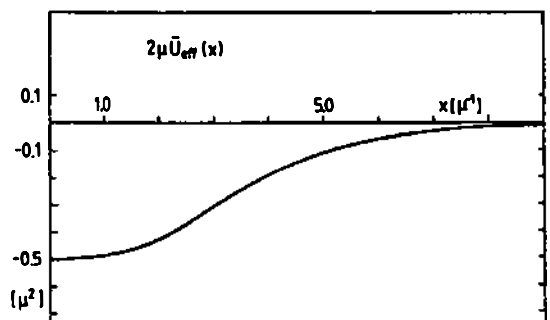

Fig. 3

To check the consistency of this interpretation we calculated the time average of the potential $U_{eff}(x,t)$ in (2.10) for several time intervals. A typical shape is shown in Fig. 3. Numerical solutions of the Klein-Gordon equation with the time averaged potential give two bound states, one at the energy $0.8\,\mu$ and the other slightly below $1.0\,\mu$ in agreement with the frequencies of the principal modes.

We can discuss qualitatively the effect of including the quantum fluctuating part $\Delta(x)$ (2.9) in the effective potential. A rough estimate of the quantum correction to $U_{eff}(x,t)$ shows that the main contribution to $\Delta(x)$ comes from the bound state solutions $\xi_0(x)$ and $\xi_1(x)$:

$$2\mu U_{eff}(x)^{(quantum)} = 3\lambda\hbar\,(\xi_0(x)^2/\omega_0 + \xi_1(x)^2/\omega_1)\ ,$$

which represents a correction of the order of λ/μ^2 to the time averaged potential. The eigenvalues ω_0 and ω_1 are then shifted by the same order of magnitude. For sufficiently small λ/μ^2, e.g. for our "physical" choice $\lambda/\mu^2 = 1/20$, the qualitative picture of the kink-antikink annihilation process does not change.

We can also give an estimate how the system behaves after longer time periods for which the numerical calculation is not feasible. The numerical analysis shows that the second mode with the frequency ω_1 decays faster than the one with ω_0. After sufficiently long time, only the dominant peak at $2\omega_0$ remains in the spectrum. Its position moves slowly towards higher frequencies until it reaches $\omega = 2\mu$. The spectrum of the emitted mesons is then expected to look like the one in Fig. 4 (a), leading to an approximate distribution of annihilation events shown in Fig. 4 (b).

Fig. 4(a)

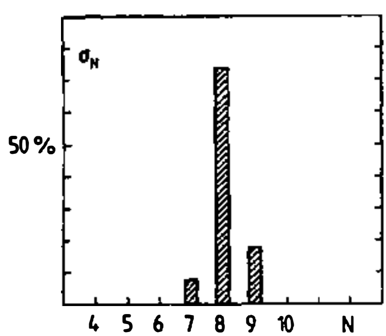

Fig. 4(b)

5. Conclusions

Let us draw some conclusions with reference to the physically interesting case in 3+1 dimensions. The extremely long life time of the oscillatory quasi-bound state found in the kink-antikink annihilation has clearly no counterpart in real baryon-antibaryon systems. We have understood such a long life time in terms of bound states in the mean field potential. In the one-dimensional case, such a situation is very likely to happen, since bound states appear for any arbitrarily weak attractive potential. In three dimensions, however, the attractive potential must have a certain minimal strength to support a bound state. It is an interesting question whether a realistic non-linear meson theory with topological solitons generates a sufficiently strong mean field potential to produce bound states during the nucleon-antinucleon annihilation. If this is not the case, then annihilation is a fast process with a broad continuous energy spectrum of outgoing mesons. On the other hand, if the model has excited states below the meson-nucleon threshold already for a single nucleon (similarly as ϕ^4 theory in 1+1 dimensions), then one expects bound states to appear also in the mean field potential during annihilation leading to the formation of a long lived quasi-bound state.

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