

### 1.17 Quadrupole channels for the determination of energy and momentum of charged particles

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#### 1. Introduction

Electrostatic and magnetic quadrupole channels can be used as energy or momentum filters.

The principle of such channels is akin to that employed in the RF quadrupole mass spectrometer<sup>1)</sup>. The transverse oscillations of the beam are described by a system of coupled linear differential equations with periodic coefficients. The voltages or the magnetic field in the channel can be adjusted to give stable orbits only in narrow ranges of energy or momentum.

#### 2. Basic equations

The transverse motion of a beam in a quadrupole channel is given by

$$x'' + G(\zeta)x = 0, \quad (1)$$

$$y'' - G(\zeta)y = 0, \quad (2)$$

where  $\zeta$  is the normalized axial coordinate,  $\zeta = 2\pi z/d$  and  $d$  the period of the channel. For an electrostatic channel

$$G(\zeta) = ed^2 U(\zeta) / 8\pi^2 a^2 E, \quad (3)$$

and in the magnetic case

$$G(\zeta) = ed^2 B(\zeta) / 4\pi^2 ap, \quad (4)$$

where  $a$  is the radius of the channel,  $p$  the momentum,  $E$  the energy of the particle and  $U(\zeta)$  and  $B(\zeta)$  are the voltage and magnetic field at the quadrupole tip. In both cases the equations are normalized so that  $G(\zeta + 2\pi) = G(\zeta)$ . The magnetic channel described by equations (1) and (2) is rotated through  $45^\circ$  in respect with the electrostatic channel described by the same equations.

#### 3. Rectangular variation of the field gradient

We choose

$$G(\zeta) = A + QH(\sin \zeta), \quad (5)$$

where  $H$  is Heavyside function.

The regions of stability of equation (1) with the coefficient (5) are given in Strutt<sup>2)</sup>. The first stability region of equations (1) and (2) was studied in great detail in connection with alternating gradient proton-synchrotrons. The intersection of the stability curves for the  $x$  and  $y$  direction is given by  $Q_0 = 0.2764$  and  $A_0 = 0.0595$ . If we energize the quadrupoles in such a way that the ratios of the two magnetic fields or voltages in one period are constant, the ratio  $(Q_0 + A_0) / (Q_0 - A_0)$  is constant, and all momenta or energies fall (in parametric space) on

the straight line  $A_0/Q_0 = \text{const.}$  The slope of this straight line determines the resolution of the apparatus, in the same way in which resolution is varied in a quadrupole mass analyser. The intersection is given by

$$A_0 + Q_0 = ed^2 U_1 / 8 \pi^2 a^2 E, \tag{6}$$

$$A_0 - Q_0 = ed^2 U_2 / 8 \pi^2 a^2 E, \tag{7}$$

for electrostatic quadrupoles, and

$$A_0 + Q_0 = ed^2 B_1 / 4 \pi^2 ap, \tag{8}$$

$$A_0 - Q_0 = ed^2 B_2 / 4 \pi^2 ap, \tag{8}$$

for magnetic quadrupoles.

Equations (6–9) determine the energy, or the momentum of the particle.

#### 4. Succession of $\delta$ functions in the field gradient

As a second example we choose

$$\pi G(\zeta) = A_1 \delta(\zeta) + A_2 (\zeta - \tau), \tag{10}$$

$$0 < \zeta < 2\pi, \quad G(\zeta + 2\pi) = G(\zeta).$$

It is possible to show that the least voltages and magnetic fields are required if  $\tau = \pi$ . The stability chart for this case is given in Fig. 1. The energy and momentum are determined by  $A_1 = 2$  and  $A_2 = -1$ . The straight line passing near this point in the diagram is meant to represent the locus of all energies and momenta when  $A_1/A_2$  is kept constant. If we approximate the  $\delta$  functions with short quadrupoles of length  $\delta$  we have

$$\pi G_1 \Delta = A_1, \tag{11}$$

$$\pi G_2 \Delta = A_2,$$

with  $\Delta = 2\pi\delta/d$ , so that energy and momentum are given by:

$$\frac{edU_1 \delta}{4 a^2 E} = A_1 = (2), \tag{12}$$

$$\frac{edU_2 \delta}{4 a^2 E} = A_2 = (-1), \tag{13}$$

$$\frac{edB_1 \delta}{2 ap} = A_1 = (2), \tag{14}$$

$$\frac{edB_2 \delta}{2 ap} = A_2 = (-1). \tag{15}$$

### 5. Comparison between the electrostatic quadrupole channel and the cylindrical electrostatic analyser

As  $U_1$  is the larger voltage in the period, we take formulae (6) and (12) to compare the voltage necessary to measure a given energy with the voltage required by an electrostatic analyser operating at the same energy

$$E = \frac{ed^2 U}{8 \pi^2 a^2 (A_0 + Q_0)} = \frac{eU_1}{26.52} (d/a)^2 = 3.7707 \times 10^{-2} eU_1 (d/a)^2, \quad (16)$$

for long, and

$$E = \frac{ed \delta U_1}{4 a^2 A_1} = \frac{1}{8} eU_1 \frac{d \delta}{a^2} = 0.125 eU_1 d \delta / a^2, \quad (17)$$

for short quadrupoles.

The quantity  $E/eU_1$  has the character of the constant of the analyser, which, in the case of the cylindrical condenser, is given by:

$$E = \frac{\rho}{2l} eU, \quad (18)$$

where  $\rho$  is the radius of the equilibrium orbit and  $e$  the separation between the analyser plates. It should be noted that formula (18) sets an upper limit to the energy measurable with sectoral analysers. The electric field  $U/l$  is limited by vacuum breakdown at 100 kV/cm, and the largest radii  $\rho$  which are machineable with the necessary tolerances are of the order of 1 meter, so that the largest measurable energy is of the order of 5 MeV for singly charged particles. As both  $\rho$  in equation (18) and  $d$  in equation (16) are proportional to the longitudinal dimensions of the measuring system, we see that formulae (18) and (16) differ fundamentally in that the constant of the analyser is respectively a linear function and a quadratic function of the length of the system and its transverse dimensions. Therefore, at least in principle, it is possible to obtain arbitrary large analyser constants by increasing  $d/a$ . It will be shown later that the increase of  $d/a$  has the adverse effect to reduce the acceptance of the quadrupole channel.

As a numerical example, let us take  $2a = 1$  cm,  $d = 30$  cm. Formula (16) gives  $E/eU_1 = 3.77 \times 10^{-2} \times 3600 = 135.7$ . If we assume that 10 periods suffice, the whole channel would be 3 meters long and capable of analysing singly charged ions having an energy of 1.35 MeV. For short quadrupoles the constant  $E/eU_1$  is approximately  $3 \delta/d$  smaller than the value obtained for long quadrupoles.

### 6. Comparison of magnetic quadrupole channels with homogenous sectoral magnets

Equations (8) and (14) give

$$p = eB_1 \frac{d^2}{4 \pi^2 (A_0 + Q_0)} \quad \text{for long,} \quad (19)$$

and

$$p = eB_1 \frac{d \delta}{2 A_1 a} \quad \text{for short quadrupoles.}$$

As the momentum of an ion in a sectoral magnet is given by

$$p = eB \varrho,$$

where  $\varrho$  is the radius of the equilibrium orbit, it appears to be logical to define an equivalent radius of curvature of the magnetic channel by writing

$$\varrho^* = \frac{d^2}{4 \pi^2 (A_0 + Q_0) a} = 7.541 \times 10^{-2} d^2/a, \quad (21)$$

$$\varrho^* = \frac{d \delta}{2 A_1 a} = 0.25 d \delta/a, \quad (22)$$

for long and short quadrupoles, respectively, with the simple interpretation that  $B_1 \varrho^*$  is the  $B\varrho$  product of the quadrupole channel. In other words, a magnetic quadrupole channel can analyse ions of exactly the same momentum as a sectoral magnet with radius  $\varrho^*$  and magnetic field strength equal to the strongest field in the quadrupole channel.

As a numerical example  $a=1$  cm,  $d=30$  cm, the equivalent radius is  $\varrho=67.5$  cm. In the magnetic case short quadrupoles are of interest because an increase in  $d$  increases  $\varrho^*$  only at the expense of a reduction in acceptance, with the same magnets and supply. As an example  $d=40$  cm,  $\delta=5$  cm,  $a=1$  cm,  $\varrho^*=50$  cm.

### 7. Ion-optical parameters of analysing quadrupole channels

An ion optical device whose input—output matrix is  $N$  has the object—image equation:

$$N_{12} L_1 L_2 + N_{22} L_1 + N_{11} L_2 + N_{21} = 0 \quad (23)$$

Comparing this with the general equation

$$(L_1 - g)(L_2 - h) = f^2, \quad (24)$$

under the condition  $\det N=1$ , we get

$$g = -\frac{N_{11}}{N_{12}}, \quad h = -\frac{N_{22}}{N_{12}},$$

$$f^2 = \frac{1}{N_{12}^2}, \quad (25)$$

so that the ion—optical parameters depend only on the ratios of the elements of the matrix of the device. If the matrix of one period is  $P$  and there are  $n$  periods,  $N=P^n$  both in the  $x$  and  $y$  direction. Denoting by  $k_1$  and  $k_2$  the two eigenvalues of the period matrix, we have

$$P = k_1^n A + k_2^n B, \quad (26)$$

where  $A$  and  $B$  are two singular matrices. In the stable case  $k_1$  and  $k_2$  are unimodular numbers, and Floquet's coefficient is purely imaginary  $\mu = j\beta$  and  $k_1 = \exp(2\pi j\beta)$ ,  $k_2 = \exp(-2\pi j\beta)$ ,  $0 < \beta < 1$ , so that equation (26) can be rewritten in the form

$$P^n = \frac{\sin 2\pi n\beta}{\sin 2\pi\beta} P - \frac{\sin 2\pi(n-1)\beta}{\sin 2\pi\beta} E, \tag{27}$$

where  $E$  is the unit matrix. It is possible to select such values of  $\beta$  that the ion optical elements are the same in both directions. If we chose  $\beta_y = 0.05$  and  $\beta_x = 0.45$  (from Fig. 1) and  $n = 10$ , the first term in (27) vanishes so that the elements of  $P_x^n$  and  $P_y^n$  are proportional, and the focal properties in both directions are equal. There are other combinations, too, and it is necessary to investigate the general case.

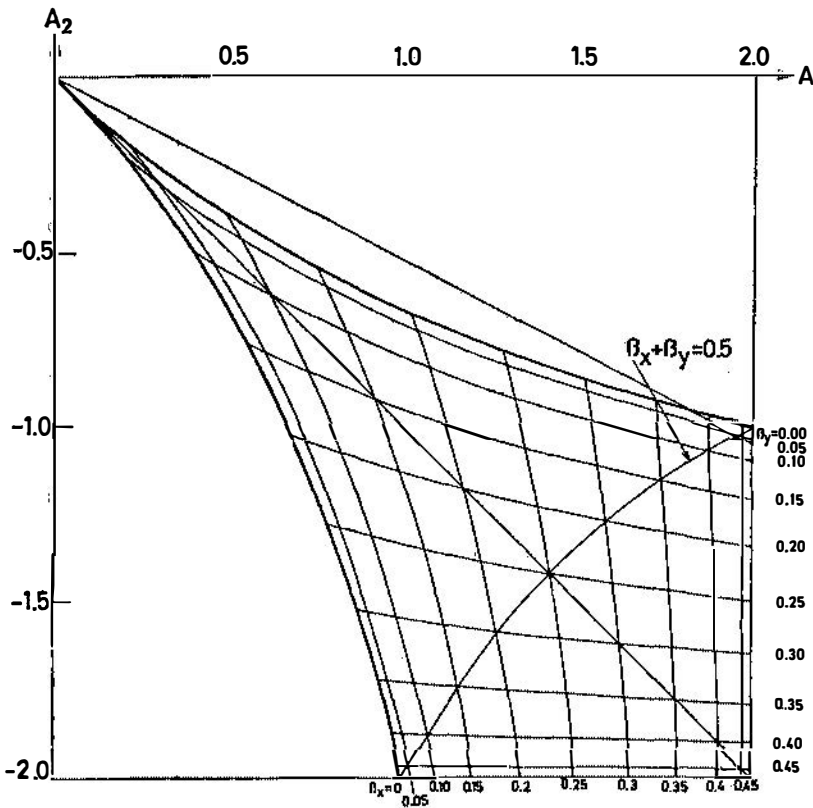


Fig. 1

8. The acceptance of the quadrupole channel

As with the RF quadrupole mass filter, large acceptance is not to be expected when the channel is operated in the corner of the stability region. Acceptance was estimated for short quadrupoles. It is assumed that the channel is infinitely long (a finite channel certainly has larger acceptance). The trajectories between quadrupoles are straight lines, and therefore we choose the quadrupoles as points of

control. If  $x_n$  denotes the  $x$  coordinate at the beginning of one period and  $x_{n+1/2}$  at the middle of the period, we have as a condition of passage  $|x_n| < a$ ,  $|x_{n+1/2}| < a$  for every  $n$  and with  $y=0$ . These conditions can be majorized as follows

$$|x_n| < \frac{1}{|\sin 2\pi\beta|} (|P_{21}| |x'_0| + (1 + |P_{22}|) |x_0|) < a, \tag{28}$$

$$|x_{n+1/2}| < \frac{1}{|\sin 2\pi\beta|} \{(\pi(1 + P_{11}) + (1 + |A_1|) |P_{21}|) |x'_0| + (\pi |P_{12}| + (1 + |A_1|) (1 + |P_{22}|)) |x_0|\} < a. \tag{29}$$

These conditions encompass two tetragonal regions in the phase space of input quantities, such that the corners lie on the axes. If (29) is fulfilled, (28) is fulfilled automatically, and it suffices to work with (29). With the most unfavorable values of  $|A_1|$  and  $|P_{1m}|$  we have

$$|x_{n+1/2}| < \frac{1}{|\sin 2\pi\beta|} (16\pi |x'_0| + 50 |x_0|) < a, \tag{30}$$

so that we get for acceptance

$$2 |x'_0|_{\max} \cdot |x_0|_{\max} = \frac{a^2 \sin^2 2\pi\beta}{4000\pi}, \tag{31}$$

or, after returning to the real  $z$  coordinate:

$$2 \left| \frac{dx_0}{dz} \right|_{\max} |x_0| = \frac{\sin^2 2\pi\beta}{2000} \frac{a^2}{d}. \tag{32}$$

The acceptance is inversely proportional to the period, proportional to the square of the radius of the circle tangent to the quadrupoles and to the square of the sine of Floquet's angle  $2\pi\beta$ . With  $a=1$  cm,  $d=40$  cm,  $\beta=5 \times 10^{-2}$  we get  $10^{-2}$  mm (mrad), a value to be considered as fairly low. It should be noted that the present analysis uses a very rough majorization.

The acceptance obtained in this way is a property of the channel itself and does not include entrance or exit slits.

### 9. Conclusions

It is possible to build electrostatic and magnetic quadrupole channels so that these measure ion energy and momentum. The resolution of these analysers is varied electrically. Double focussing can be achieved by suitable parameter choice. The electric and magnetic fields necessary are realizable and can be reduced arbitrarily at the expense of the acceptance of the device.

### References

- 1) W. Paul, H. P. Renhard and U. von Zahn, *Z. f. Phys.* **152** (1958) 143;
- 2) Strutt, *Physica* **7** (1927) 265.