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Cyclic up-down words

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Abstract

A word $w_1w_2\cdots w_n$ of even length n is said to be cyclic up-down if $w_1 < w_2 > w_3 < \cdots$ and $w_n > w_1$. We find the generating function for the number of cyclic up-down words over $\{1,2,\dots,k\}$, from which a closed-form formula is derived. We also construct a bijection with semi-magic labelings of certain graphs, called cycle-of-loops graphs.

Keywords: *Chebyshev polynomial, cyclic up-down word, generating function, semi-magic labeling*

1 Introduction

A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is said to be alternating if $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$. By a classical result of André [2], if E_n stands for the number of alternating permutations of length n, then

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec(x) + \tan(x).$$

The analogue problem for words appears to have been first studied by Carlitz and Scovill [3]. Let $k \geq 1$ and $n \geq 0$ be two integers and set $[k] = \{1,2,\dots,k\}$. A word over $[k]$ of length n is an element of the set $[k]^n$. A word $w_1w_2\cdots w_n \in [k]^n$ is said to be up-down if $w_1 < w_2 > w_3 < \cdots$. In [3] the generating functions for the number of up-down words over $[k]$ of even and odd length were found. Gao et al. [6] proved via a bijection that the number of up-down words over $[k]$ of length n is equal to the number

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of order ideals of certain posets. Up-down words were studied implicitly by Xin and Zhong [11].

For even n, it is natural to consider cyclic alternating permutations. These are alternating permutations $\sigma_1\sigma_2\cdots\sigma_n$ with the additional requirement that $\sigma_n > \sigma_1$. Cyclic alternating permutations seem to have been first studied by Elkies [5], who proved that the number of cyclic alternating permutations of length $2n$ is equal to nE_{2n-1} (see also sequence A024255 in The On-Line Encyclopedia of Integer Sequences [9]).

This work is concerned with the study of cyclic up-down words, which seem not to have been studied previously. A word $w_1\cdots w_n \in [k]^n$ of even length n is said to be cyclic up-down if it is up-down and, in addition, $w_n > w_1$. Our first main result is an explicit expression for the generating function $A_k(x) = \sum_{n=0}^{\infty} a_{k,2n}x^{2n}$ counting cyclic up-down words over $[k]$, in terms of Chebyshev polynomials of the third and fourth kind (Theorem 3.1). From this we derive a closed-form formula for $a_{k,2n}$ as a trigonometric sum. For $k = 3$, the sum yields a bisection of the Lucas numbers (Example 3.1). In the second part of the paper we construct a bijection between cyclic up-down words and semi-magic labelings of the cycle-of-loops graph LOOP $\times$ C_{2n} with prescribed common sum (Theorem 3.2). Table 2 shows the numbers of cyclic up-down words over $[k]$ for $k = 2, \ldots, 6$, of length $n = 0, 2, \ldots, 14$.

2 Preliminaries

Let

$$F_k(x) = \sum_{n=0}^{\infty} f_{k,2n+1}x^{2n+1}$$

be the generating function for the number of up-down words over $[k]$ of odd length. In [3] it was shown that $F_k(x) = P_k(x)/Q_k(x)$, where

$$P_k(x) = \sum_{i=0}^{k}(-1)^i\binom{k+i}{2i+1}x^{2i+1} \quad \text{and} \quad Q_k(x) = \sum_{i=0}^{k}(-1)^i\binom{k+i-1}{2i}x^{2i}.$$

It is helpful to express the functions $P_k(x)$ and $Q_k(x)$ in terms of the Chebyshev polynomials. These are orthogonal polynomials that are especially important in approximation theory and numerical analysis. Their connection to combinatorics goes back at least to [8]. The Chebyshev polynomials are divided into four families, whose definitions and recursions are summarized in Table 1.

Table 1. The definitions and recursions of the four kinds of the Chebyshev polynomials

Kind	**Definition**	**Recursion**
First	$T_n(\cos\theta) = \cos(n\theta)$	$T_0(x) = 1, \quad T_1(x) = x$ $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$
Second	$U_n(\cos\theta) = \dfrac{\sin((n+1)\theta)}{\sin\theta}$	$U_0(x) = 1, \quad U_1(x) = 2x$ $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$
Third	$V_n(\cos\theta) = \dfrac{\cos((n+\frac{1}{2})\theta)}{\cos(\frac{\theta}{2})}$	$V_0(x) = 1, \quad V_1(x) = 2x - 1$ $V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x)$
Fourth	$W_n(\cos\theta) = \dfrac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})}$	$W_0(x) = 1, \quad W_1(x) = 2x + 1$ $W_{n+1}(x) = 2xW_n(x) - W_{n-1}(x)$

It is well-known (e.g., [1, 22.5.47 and 22.5.48]), that the Chebyshev polynomials of the first and second kind admit the hypergeometric representations

$$T_n(x) = {}_2F_1\left(-n, n; \tfrac{1}{2}; \tfrac{1-x}{2}\right) = n\sum_{i=0}^{n}(-2)^i \frac{(n+i-1)!}{(n-i)!(2i)!}(1-x)^i,$$

$$U_n(x) = (n+1){}_2F_1\left(-n, n+2; \tfrac{3}{2}; \tfrac{1-x}{2}\right) = \sum_{i=0}^{n}(-2)^i \binom{n+i+1}{2i+1}(1-x)^i.$$

It immediately follows that $P_k(x) = xU_{k-1}(1-\frac{x^2}{2})$. Now,

$$\begin{aligned}
Q_k(x) &= \sum_{i=0}^{k-1}(-1)^i\binom{k+i-1}{2i}x^{2i} \\
&= k\sum_{i=0}^{k-1}(-2)^i\frac{(k+i-1)!}{(k-i)!(2i)!}\left(1-\left(1-\frac{x^2}{2}\right)\right)^i \\
&\qquad\qquad - \sum_{i=0}^{k-1}(-2)^i i\frac{(k+i-1)!}{(k-i)!(2i)!}\left(1-\left(1-\frac{x^2}{2}\right)\right)^i \\
&= T_k\left(1-\frac{x^2}{2}\right) - \frac{x}{2k}\frac{d}{dx}T_k\left(1-\frac{x^2}{2}\right) \\
&= U_{k-1}\left(1-\frac{x^2}{2}\right) - U_{k-2}\left(1-\frac{x^2}{2}\right)
\end{aligned}$$

$$= V_{k-1}\left(1-\frac{x^2}{2}\right).$$

Thus,

$$F_k(x) = \frac{xU_{k-1}\left(1-\frac{x^2}{2}\right)}{V_{k-1}\left(1-\frac{x^2}{2}\right)}.$$

It will be convenient in the proof of Theorem 3.1 to modify $F_k(x)$ and replace it with $F_k(x) + \frac{1}{x} - x$. Thus, from now on, $F_k(x)$ denotes this modified generating function. In particular, $f_{k,-1} = 1$ and $f_{k,1} = k-1$. Consequently,

$$F_k(x) = \frac{1}{x} + \frac{xU_{k-2}\left(1-\frac{x^2}{2}\right)}{V_{k-1}\left(1-\frac{x^2}{2}\right)}.$$

We shall also need the following technical result.

Lemma 2.1. *We have*

$$\sum_{i=1}^{k-1} \frac{U_{2i-1}(x)+2i}{V_i(x)V_{i-1}(x)} = \frac{1}{2}\left((2k-1)\frac{W_{k-1}(x)}{V_{k-1}(x)} - 1\right).$$

Proof. Let $\theta = \arccos(x)$. We have

$$\begin{aligned}
&\sum_{i=1}^{k-1} \frac{U_{2i-1}(x)+2i}{V_i(x)V_{i-1}(x)} \\
&= \sum_{i=1}^{k-1} \frac{U_{2i-1}(x)}{V_i(x)V_{i-1}(x)} + \sum_{i=1}^{k-1} \frac{2i}{V_i(x)V_{i-1}(x)} \\
&= \frac{\cos^2(\frac{\theta}{2})}{\sin(\theta)} \sum_{i=1}^{k-1}\left(\frac{\sin(2i\theta)}{\cos((i+\frac{1}{2})\theta)\cos((i-\frac{1}{2})\theta)} + \frac{2i\sin(\theta)}{\cos((i+\frac{1}{2})\theta)\cos((i-\frac{1}{2})\theta)}\right).
\end{aligned} \tag{1}$$

With

$$\begin{aligned}
\sin(2i\theta) &= \sin((i+\tfrac{1}{2})\theta)\cos((i-\tfrac{1}{2})\theta) + \sin((i-\tfrac{1}{2})\theta)\cos((i+\tfrac{1}{2})\theta), \\
\sin(\theta) &= \sin((i+\tfrac{1}{2})\theta)\cos((i-\tfrac{1}{2})\theta) - \sin((i-\tfrac{1}{2})\theta)\cos((i+\tfrac{1}{2})\theta),
\end{aligned}$$

we then have that (1) is equal to

$$\frac{\cos^2(\frac{\theta}{2})}{\sin(\theta)} \sum_{i=1}^{k-1}\Big(\tan((i+\tfrac{1}{2})\theta) + \tan((i-\tfrac{1}{2})\theta) + 2i\tan((i+\tfrac{1}{2})\theta) - 2i\tan((i-\tfrac{1}{2})\theta)\Big)$$

$$= \frac{\cos^2(\frac{\theta}{2})}{\sin(\theta)} \Big((2k-1)\tan((k-\tfrac{1}{2})\theta) - \tan(\tfrac{\theta}{2}) \Big)$$
$$= \frac{1}{2}\left((2k-1)\frac{W_{k-1}(x)}{V_{k-1}(x)} - 1 \right). \qquad \square$$

3 Main Results

Theorem 3.1. *Let $A_k(x) = \sum_{n=0}^{\infty} a_{k,2n}x^{2n}$ be the generating function for the number of cyclic up-down words over $[k]$. Then*

$$A_k(x) = 1 + \frac{x^2}{2(4-x^2)} \left((2k-1)\frac{W_{k-1}\left(1-\frac{x^2}{2}\right)}{V_{k-1}\left(1-\frac{x^2}{2}\right)} - 1 \right). \tag{2}$$

In particular, for $n \geq 1$,

$$a_{k,2n} = \frac{1}{2^{2n}}\sum_{i=1}^{k-1} \csc^{2n}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right) = \frac{1}{2^{2n}}\sum_{i=1}^{k-1} \sec^{2n}\left(\frac{2i\pi}{2k-1}\right). \tag{3}$$

Proof. We count the number of cyclic up-down words over $[k+1]$ according to the number of occurrences and positions of the letter $k+1$. Notice that the letter $k+1$ can be placed only at an even letter index. There are $a_{k,2n}$ cyclic up-down words without any $k+1$ and $nf_{k,2n-1}$ such words with exactly one occurrence of $k+1$. Now assume that there are at least two occurrences of $k+1$ and denote by i and j the smallest and largest index of these $k+1$s, respectively. There are $f_{k+1,2(j-i)-1}$ possibilities for the subword $w_{i+1}\cdots w_{j-1}$ and $f_{k,2n-2(j-i)-1}$ possibilities for the (cyclic) subword $w_{j+1}\cdots w_{2n}w_1\cdots w_{i-1}$. It follows that

$$\begin{aligned} a_{k+1,2n} &= a_{k,2n} + nf_{k,2n-1} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} f_{k+1,2(j-i)-1}f_{k,2n-2(j-i)-1} \\ &= a_{k,2n} + \sum_{i=1}^{n} if_{k+1,2(n-i)-1}f_{k,2i-1}. \end{aligned}$$

Multiplying the equation by x^{2n}, summing over $n \geq 1$, and finally adding 1, we obtain

$$1 + \sum_{n=1}^{\infty} a_{k+1,2n}x^{2n} = 1 + \sum_{n=1}^{\infty} a_{k,2n}x^{2n} + \sum_{n=1}^{\infty}\sum_{i=1}^{n} if_{k+1,2(n-i)-1}f_{k,2i-1}x^{2n}. \tag{4}$$

We have

$$A_{k+1}(x) = 1 + \sum_{n=1}^{\infty} a_{k+1,2n} x^{2n}, \quad A_k(x) = 1 + \sum_{n=1}^{\infty} a_{k,2n} x^{2n}.$$

To identify the double sum term, we calculate

$$\begin{aligned} 2x + \frac{d}{dx}(xF_k(x)) &= 2x + \frac{d}{dx}\left(x\left(\frac{1}{x} - x + \sum_{n=0}^{\infty} f_{k,2n+1}x^{2n+1}\right)\right) \\ &= 2x + \frac{d}{dx}\left(1 - x^2 + \sum_{n=0}^{\infty} f_{k,2n+1}x^{2n+2}\right) \\ &= 2\sum_{n=0}^{\infty}(n+1)f_{k,2n+1}x^{2n+1} \\ &= 2\sum_{n=1}^{\infty} nf_{k,2n-1}x^{2n-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{x^2}{2}F_{k+1}(x)\left(2x + \frac{d}{dx}(xF_k(x))\right) &= x^2F_{k+1}(x)\sum_{n=1}^{\infty} nf_{k,2n-1}x^{2n-1} \\ &= \left(\sum_{n=-1}^{\infty} f_{k+1,2n+1}x^{2n+3}\right)\left(\sum_{n=1}^{\infty} nf_{k,2n-1}x^{2n-1}\right) \\ &= \sum_{m=-1}^{\infty}\sum_{i=1}^{\infty} if_{k+1,2m+1}f_{k,2i-1}x^{2(m+i+1)}. \end{aligned} \quad (5)$$

We have $m + i + 1 \geq 1$ and, for $n \geq 1$ we have $n = m + i + 1$ if and only if $1 \leq i \leq n$ and $m = n - i - 1$. It follows that the coefficient of x^{2n} in (5) is given by

$$\sum_{i=1}^{n} if_{k+1,2(n-i)-1}f_{k,2i-1}.$$

It follows that

$$\frac{x^2}{2}F_{k+1}(x)\left(2x + \frac{d}{dx}(xF_k(x))\right) = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} if_{k+1,2(n-i)-1}f_{k,2i-1}\right)x^{2n},$$

exactly the double sum term in (4). We have therefore shown that

$$A_{k+1}(x) = A_k(x) + \frac{x^2}{2}F_{k+1}(x)\left(2x + \frac{d}{dx}(xF_k(x))\right).$$

Using Lemma 2.1, it follows that

$$\begin{aligned}A_k(x) &= 1 + \frac{x^2}{2}\sum_{i=1}^{k-1} F_{i+1}(x)\left(2x + \frac{d}{dx}\left(xF_i(x)\right)\right)\\ &= 1 + \frac{x^2}{4-x^2}\sum_{i=1}^{k-1}\frac{U_{2i-1}\left(1-\frac{x^2}{2}\right)+2i}{V_i\left(1-\frac{x^2}{2}\right)V_{i-1}\left(1-\frac{x^2}{2}\right)}\\ &= 1 + \frac{x^2}{2(4-x^2)}\left((2k-1)\frac{W_{k-1}\left(1-\frac{x^2}{2}\right)}{V_{k-1}\left(1-\frac{x^2}{2}\right)} - 1\right).\end{aligned}$$

To prove (3), first recall (e.g., [7, (2.7)]) that the roots of $V_{k-1}(x)$ are

$$\alpha_{k,i} = \cos\left(\frac{(i-\frac{1}{2})\pi}{k-\frac{1}{2}}\right), \quad i \in [k-1].$$

Thus, the roots of $V_{k-1}(1-\frac{x^2}{2})$ are $\pm\beta_{k,i}, i \in [k-1]$, where

$$\beta_{k,i} = \sqrt{2(1-\alpha_{k,i})} = 2\sin\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right).$$

By [7, (1.17) and (1.18)],

$$V_{k-1}(x) = U_{k-1}(x) - U_{k-2}(x), \quad W_{k-1}(x) = U_{k-1}(x) + U_{k-2}(x).$$

Thus, $W_{k-1}(x) = V_{k-1}(x) + 2U_{k-2}(x)$ and therefore

$$A_k(x) = 1 + \frac{x^2}{4-x^2}\left(k - 1 + (2k-1)\frac{U_{k-2}\left(1-\frac{x^2}{2}\right)}{V_{k-1}\left(1-\frac{x^2}{2}\right)}\right).$$

In general, if $q(x)$ is polynomial of degree m which factors as $q(x) = (x-\beta_1)\cdots(x-\beta_m)$ and $p(x)$ is a polynomial whose degree is less than m, then

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{m}\frac{p(\beta_i)}{q'(\beta_i)}\frac{1}{x-\beta_i}.$$

By [4, p. 7],

$$(1-x^2)V'_{k-1}(x) = -((k-1)x+\tfrac{1}{2})V_{k-1}(x) + (k-\tfrac{1}{2})V_{k-2}(x).$$

Using $V_{k-1}(\alpha_{k,i}) = 0$, we then have

$$V'_{k-1}(\alpha_{k,i}) = \frac{(1-2k)V_{k-2}(\alpha_{k,i})}{\pm\beta_{k,i}(1+\alpha_{k,i})}.$$

Thus, with the definitions in Table 1 and some elementary trigonometric identities,

$$\begin{aligned}
&(2k-1)\frac{U_{k-2}(1-\frac{x^2}{2})}{V_{k-1}(1-\frac{x^2}{2})} \\
&= \sum_{i=1}^{k-1} \frac{\beta_{k,i}(1+\alpha_{k,i})U_{k-2}(\alpha_{k,i})}{V_{k-2}(\alpha_{k,i})}\left(\frac{1}{x+\beta_{k,i}} - \frac{1}{x-\beta_{k,i}}\right) \\
&= \sum_{i=1}^{k-1} \cos\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)\cot\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)\left(\frac{1}{x+2\sin\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)} - \frac{1}{x-2\sin\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)}\right) \\
&= \sum_{n=0}^{\infty}\sum_{i=1}^{k-1} \frac{\cot^2\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)}{2^{2n}\sin^{2n}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)} x^{2n}.
\end{aligned}$$

For $n \geq 0$, set

$$b_n = \sum_{i=1}^{k-1} \frac{\cot^2\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)}{2^{2n}\sin^{2n}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)}.$$

The function $\frac{x^2}{4-x^2}$ has the expansion $\sum_{m=1}^{\infty}\frac{x^{2m}}{4^m}$. Thus, for $n \geq 1$,

$$\begin{aligned}
a_{k,2n} &= [x^{2n}]\sum_{m=1}^{\infty}\frac{x^{2m}}{4^m}\left(k-1+\sum_{j=0}^{\infty} b_j x^{2j}\right) \\
&= \frac{k-1}{4^n} + \sum_{m=1}^{n}\frac{b_{n-m}}{4^m} = \frac{k-1}{4^n} + \sum_{m=0}^{n-1}\frac{b_m}{4^{n-m}}.
\end{aligned}$$

It follows that

$$a_{k,2n} = \frac{k-1}{4^n} + \sum_{m=0}^{n-1}\frac{1}{4^{n-m}}\sum_{i=1}^{k-1}\frac{\cot^2\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)}{2^{2m}\sin^{2m}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)}$$

$$= \frac{k-1}{2^{2n}} + \frac{1}{2^{2n}} \sum_{i=1}^{k-1} \cot^2\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right) \sum_{m=0}^{n-1} \frac{1}{\sin^{2m}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)}.$$

Set $c_i = 1/\sin^2\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)$. Then $\cot^2\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right) = c_i - 1$ and

$$\sum_{m=0}^{n-1} \frac{1}{\sin^{2m}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)} = \sum_{m=0}^{n-1} c_i^m = \frac{c_i^n - 1}{c_i - 1}.$$

Therefore

$$\cot^2\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right) \sum_{m=0}^{n-1} \frac{1}{\sin^{2m}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)} = \frac{1}{\sin^{2n}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)} - 1.$$

It follows that

$$\begin{aligned} a_{k,2n} &= \frac{1}{2^{2n}}\left((k-1) + \sum_{i=1}^{k-1}\left(\frac{1}{\sin^{2n}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)} - 1\right)\right) \\ &= \frac{1}{2^{2n}} \sum_{i=1}^{k-1} \frac{1}{\sin^{2n}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right)} \\ &= \frac{1}{2^{2n}} \sum_{i=1}^{k-1} \csc^{2n}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right). \end{aligned}$$

Finally, with

$$\sin\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right) = \cos\left(\frac{(k-i)\pi}{2k-1}\right),$$

we obtain

$$\begin{aligned} a_{k,2n} &= \frac{1}{2^{2n}} \sum_{i=1}^{k-1} \csc^{2n}\left(\frac{(i-\frac{1}{2})\pi}{2k-1}\right) \\ &= \frac{1}{2^{2n}} \sum_{i=1}^{k-1} \sec^{2n}\left(\frac{(k-i)\pi}{2k-1}\right) \end{aligned}$$

$$= \frac{1}{2^{2n}} \sum_{i=1}^{k-1} \sec^{2n}\left(\frac{i\pi}{2k-1}\right)$$
$$= \frac{1}{2^{2n}} \sum_{i=1}^{k-1} \sec^{2n}\left(\frac{2i\pi}{2k-1}\right). \qquad \square$$

Table 2. Number of cyclic up-down words

k/n	0	2	4	6	8	10	12	14
2	1	1	1	1	1	1	1	1
3	1	3	7	18	47	123	322	843
4	1	6	26	129	650	3281	16565	83635
5	1	10	70	571	4726	39175	324787	2692756
6	1	15	155	1884	23219	286555	3537032	43659386

Example 3.1. *For $k = 2$ we have*

$$a_{2,2n} = \frac{\sec^{2n}\left(\frac{2\pi}{3}\right)}{2^{2n}} = \frac{(-2)^{2n}}{2^{2n}} = 1,$$

as it should. For $k = 3$ we have

$$a_{3,2n} = \frac{1}{2^{2n}}\left(\sec^{2n}\left(\frac{2\pi}{5}\right) + \sec^{2n}\left(\frac{4\pi}{5}\right)\right)$$
$$= \left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n}$$
$$= L_{2n},$$

where L_{2n} is the $2n$th Lucas number. This seems to give a new combinatorial interpretation to the bisection of the Lucas numbers (cf. A005248 in [9]).

Recall (e.g., [10, (M.1) on p. 1033]), that a semi-magic labeling is a labeling of the edges of a graph by nonnegative integers, such that the sum of the labels on the edges incident to every vertex is the same. In the following theorem we establish a connection between the number of cyclic up-down words and the number of semi-magic labelings of the cycle-of-loops graphs. These graphs are defined by LOOP $\times$ C_m, where LOOP is the 1-vertex-1-loop-edge graph and C_m is the cycle graph of order m. See Figure 1 for an example.

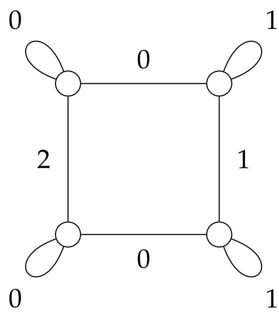

Figure 1. A semi-magic labeling of the graph LOOP $\times$ C_4 with common sum 2

Theorem 3.2. *The number of cyclic up-down words over* $[k]$ *of length* $2n$ *is equal to the number of semi-magic labelings of* LOOP $\times$ C_{2n} *with common sum* $k-2$.

Proof. First, we notice that a semi-magic labeling of LOOP $\times$ C_m is determined by the labels of the cycle C_m, since the labels of the loops must complement the sum of the two respective labels to the common sum. Thus, it suffices to show that the number of cyclic up-down words over $[k]$ of length $2n$ is equal to the number of edge labelings of the cycle C_{2n}, such that the sum of every two consecutive labels is less than or equal to $k-2$. To this end, consider a cyclic up-down word $w_1 \cdots w_{2n} \in [k]^{2n}$ and define $u_1 \cdots u_{2n} \in \{0, 1, \ldots, k-2\}^{2n}$ by

$$u_i = \begin{cases} w_i - 1, & \text{if } i \text{ is odd,} \\ k - w_i, & \text{if } i \text{ is even.} \end{cases}$$

Let $i \in [2n]$ and identify the index $2n+1$ with the index 1. We need to show that $u_i + u_{i+1} \le k-2$. First, assume that i is odd. Then $w_i < w_{i+1}$, i.e., $w_i - w_{i+1} \le -1$. Thus

$$u_i + u_{i+1} = w_i - 1 + k - w_{i+1} \le k-2.$$

The case of even i is similar. Conversely, let $u_1 \cdots u_{2n} \in \{0, 1, \ldots, k-2\}^{2n}$ such that $u_i + u_{i+1} \le k-2$, for every $i \in [2n]$. Define $w_1 \cdots w_{2n} \in [k]^{2n}$ by

$$w_i = \begin{cases} u_i + 1, & \text{if } i \text{ is odd,} \\ k - u_i, & \text{if } i \text{ is even.} \end{cases}$$

We need to show that $w_1 \cdots w_{2n}$ is a cyclic up-down word. Assume that i is odd. We have

$$\begin{aligned} w_i < w_{i+1} &\iff u_i + 1 < k - u_{i+1} \\ &\iff u_i + u_{i+1} < k - 1 \\ &\iff u_i + u_{i+1} \leq k - 2. \end{aligned}$$

The case of even i is similar.

It is easy to see that the two maps are inverse to one another and hence bijections. □

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References

[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions*, Natl. Bur. Stand. Appl. Math. Ser., no. 55, U.S. Govt. Printing Office, Washington, DC, 1970.

[2] D. André, *Sur les permutations alternées*, J. Math. Pures Appl. **7** (1881), 167–184.

[3] L. Carlitz and R. Scoville, *Up-down sequences*, Duke Math. J. **39** (1972), 583–598.

[4] R. N. Desmarais and S. R. Bland, *Tables of Properties of Airfoil Polynomials*, NASA Ref. Publ. **1343**, NASA Langley Research Center, Hampton, VA, September 1995. Available at: `https://ntrs.nasa.gov/api/citations/19960001864/downloads/19960001864.pdf`.

[5] N. D. Elkies, *On the sums* $\sum_{k=-\infty}^{\infty}(4k+1)^{-n}$, Amer. Math. Monthly **110** (2003), 561–573.

[6] A. L. L. Gao, S. Kitaev, and P. B. Zhang, *Pattern-avoiding alternating words*, Discrete Appl. Math. **207** (2016), 56–66.

[7] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman and Hall/CRC, 2002.

[8] L. W. Shapiro, *A combinatorial proof of a Chebyshev polynomial identity*, Discrete Math. **34** (1981), 203–206.

[9] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, OEIS Foundation Inc., `https://oeis.org`.

[10] B. M. Stewart, *Magic graphs*, Canad. J. Math. **18** (1966), 1031–1059.

[11] G. Xin and Y. Zhong, *Proving some conjectures on Kekulé numbers for certain benzenoids by using Chebyshev polynomials*, Adv. Appl. Math. **145** (2023), 102479.