https://doi.org/10.31896/k.29.4
Original scientific paper
Accepted: 1 November 2025

Hiroshi Okumura

A Generalization of the Twin Circles of Archimedes

A Generalization of the Twin Circles of Archimedes

ABSTRACT

We consider the arbelos and generalize Archimedean circles and the twin circles of Archimedes.

Key words: arbelos, Archimedean circle, k-Archimedean circle, twin circles of Archimedes, k-Archimedean twins.

MSC2020: 01A27, 51M04

Poopćenje Arhimedovih kružnica blizanaca

SAŽETAK

U radu proučavamo arbelose i dajemo poopćenje Arhimedovih kružnica i Arhimedovih kružnica blizanaca

Ključne riječi: arbelos, Arhimedova kružnica, k-Arhimedova kružnica, Arhimedove kružnice blizanci, k-Arhimedovi blizanci

1 Introduction

For a point C on the segment AB such that $|BC| = 2a$, $|CA| = 2b$ and $|AB| = 2c$, let α, β and γ be the semicircles of diameters BC, CA and AB, respectively, constructed on the same side of AB. The area formed by the three semicircles is called an arbelos, and the radical axis of α and β is called the axis. The axis divides the arbelos into two curvilinear triangles with congruent incircles of radius ab/c. It has been believed that the two circles were studied by Archimedes, and they are called the twin circles of Archimedes (see Figure 1). Circles of radius ab/c are called Archimedean circles. In this paper we generalize Archimedean circles and the twin circles of Archimedes.

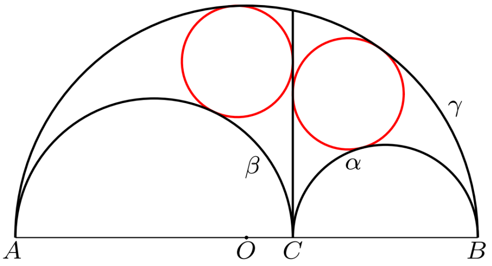

Figure 1.

We use a rectangular coordinate system with origin C such that the farthest point on α from the line AB has coordinates (a,a). The center of γ is denoted by O.

2 k-Archimedean circle

We give a definition of a generalized Archimedean circle.

Definition 1 *Let $w_k = a^2 + kab + b^2$ for a real number k. We say that a circle is k-Archimedean if it has radius*

$$r_k = \frac{abc}{w_k}.$$

The incircle of the arbelos has radius

$$\frac{ab(a+b)}{a^2+ab+b^2} = \frac{abc}{w_1}.$$

Therefore it is 1-Archimedean. The twin circles of Archimedes have radius

$$\frac{ab}{a+b} = \frac{abc}{c^2} = \frac{abc}{w_2}.$$

Therefore they are 2-Archimedean. Hence k-Archimedean circles are generalizations of those circles. 3-Archimedean circles can be found in the following problem in Wasan geometry (see Figure 2):

Problem 1. Three congruent circles of radius r touch the semicircle γ internally so that two of them touch the remaining circle externally and also touches the external common tangent of the semicircles α and β from the side opposite to the point C. Show that the following relation holds:

$$r = \frac{abc}{a^2+3ab+b^2}. \tag{1}$$

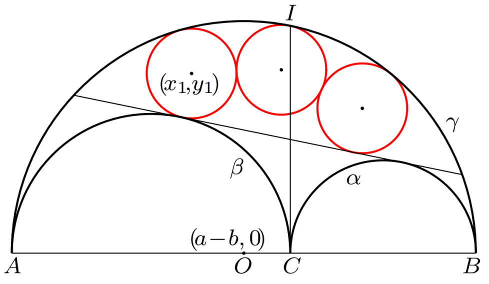

Figure 2.

The problem was proposed by Taguchi in 1817 [3]. Wasan is the Japanese mathematics developed in Edo period. For a brief introduction of Wasan geometry, see [1]. Definition 1 has been made inspired by this problem. It is obvious that r_k is a monotonically decreasing function of k.

3 k-Archimedean twins

In this section we generalize the twin circles of Archimedes. We regard that if t is a perpendicular to AB, then it is represented by the equation $x = t$ with the same symbol t.

Theorem 1 *For a circle δ_a (resp. δ_b) of radius r touching β (resp. α) externally, and γ internally, let t_a (resp. t_b) be the perpendicular to AB touching δ_a (resp. δ_b) from the same side as A (resp. B). Then the circles δ_a and δ_b are k-Archimedean if and only if*

$$t_b - t_a = 2kr. \tag{2}$$

Proof. Let (x_a, y_a) (resp. (x_b, y_b)) be the coordinates of the center of δ_a (resp. δ_b). We have

$$(x_a+b)^2+y_a^2 = (b+r)^2 \text{ and } (x_a-(a-b))^2+y_a^2 = (c-r)^2.$$

Solving the equations for x_a and y_a, we have

$$(x_a, y_a) = \left(r - 2b\left(1-\frac{r}{a}\right), \frac{2\sqrt{bc(a-r)r}}{a}\right). \tag{3}$$

Similarly, we have

$$(x_b, y_b) = \left(-r + 2a\left(1-\frac{r}{b}\right), \frac{2\sqrt{ac(b-r)r}}{b}\right). \tag{4}$$

Therefore we have

$$t_a = -2b\left(1-\frac{r}{a}\right), \quad t_b = 2a\left(1-\frac{r}{b}\right).$$

Hence we have

$$\begin{aligned} \frac{t_b-t_a}{2r} - k &= \frac{a+b}{r} - \left(\frac{a}{b}+\frac{b}{a}\right) - k \\ &= \frac{c}{r} - \frac{a^2+abk+b^2}{ab} = c\left(\frac{1}{r}-\frac{1}{r_k}\right). \end{aligned}$$

Therefore (2) and $r = r_k$ are equivalent. □

We call the two congruent circles δ_a and δ_b in the theorem *the k-Archimedean twins*, which are generalizations of the twin circles of Archimedes. We have the next corollary (see Figure 3).

Corollary 1 *If k is a positive integer in the event of Theorem 1, there are congruent circles $\delta_a = \delta_1, \delta_2, \delta_3, \cdots, \delta_k = \delta_b$ and perpendiculars $t_a = t_0, t_1, t_2, \cdots, t_k = t_b$ to AB such that $t_i - t_{i-1} = 2r$ for $i = 1, 2, \cdots, k$ and t_{i-1} touches the circles δ_{i-1} and δ_i for $i = 2, 3, \cdots, k$.*

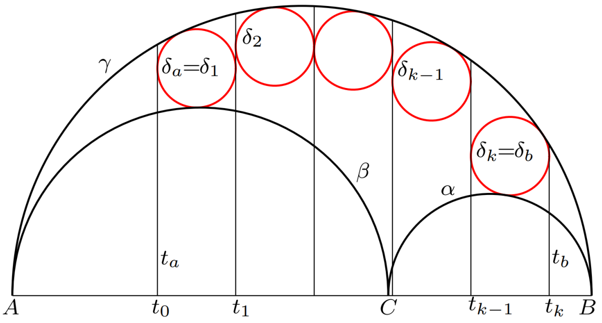

Figure 3: $k = 5$.

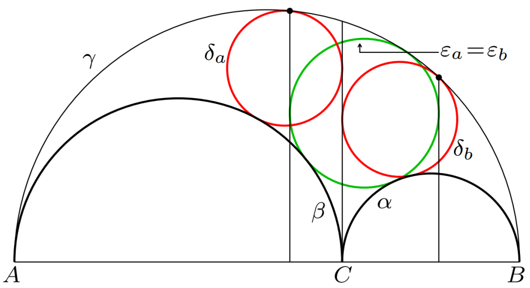

Figure 4: 1-*Archimedean twins and* 2-*Archimedean twins.*

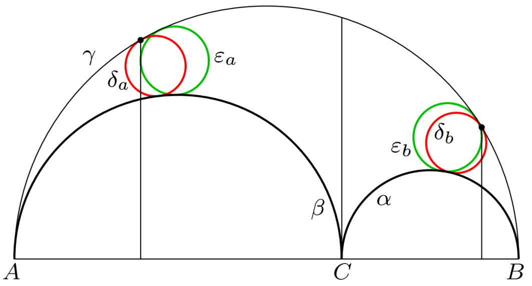

Figure 5: *k-Archimedean twins and $k-1$-Archimedean twins ($k=6$).*

The next theorem shows that $k-1$-Archimedean twins are obtained from k-Archimedean twins, and conversely (see Figures 4 and 5, where 1-Archimedean twins in Figure 4 are overlapping).

Theorem 2 *Let δ_a and ε_a (resp. δ_b and ε_b) be the circles touching β (resp. α) externally, γ internally such that the perpendicular to AB touching ε_a (resp. ε_b) from the same side as A (resp. B) passes through the point of tangency of δ_a (resp. δ_b) and γ. The following statements hold.*
(i) *δ_a (resp. δ_b) is k-Archimedean if and only if ε_a (resp. ε_b) is $k-1$-Archimedean.*
(ii) *δ_a and δ_b are k-Archimedean twins if and only if ε_a and ε_b are $k-1$-Archimedean twins.*

Proof. Let r and x_a be the radius of δ_a and the x-coordinate of its center, respectively. Then

$$x_a = r - 2b\left(1-\frac{r}{a}\right) \tag{5}$$

by (3). Let e be the radius of ε_a. The perpendicular to AB touching ε_a from the same side as A is represented by the equation $x=-2b(1-e/a)$. The point of tangency of γ and δ_a is the external center of similitude of the two circles. Hence it has x-coordinate $(-r(a-b)+cx_a)/(c-r)$. Therefore we have

$$-2b\left(1-\frac{e}{a}\right) = \frac{-r(a-b)+cx_a}{c-r}.$$

Substituting (5) in this equation and solving the resulting equation for $1/e$, we have

$$\frac{1}{e} = \frac{1}{r} - \frac{1}{c}.$$

While we have

$$\frac{1}{c} + \frac{1}{r_{k-1}} = \frac{1}{r_k}.$$

Eliminating $1/c$ from the last two equations, we have

$$\frac{1}{e} - \frac{1}{r_{k-1}} = \frac{1}{r} - \frac{1}{r_k}.$$

Therefore δ_a is k-Archimedean if and only if ε_a is $k-1$-Archimedean. The rest of (i) is proved similarly. The part (ii) is obvious. ☐

4 Maximal k-Archimedean twins

We consider the maximal k-Archimedean twins. We denote the configuration consisting of an arbelos and k-Archimedean twins δ_a and δ_b with their tangents t_a and t_b by $\mathcal{T}_k$. For $\mathcal{T}_k$, the centers of δ_a and δ_b have x-coordinates t_a+r_k and t_b-r_k, respectively. By Theorem 1 we have the followings: If $k=1$ then $t_a+r_k=t_b-r_k$. If $k<1$ then $t_b-r_k<t_a+r_k$, and if $1<k$ then $t_a+r_k<t_b-r_k$ (see Figures 6 and 7).

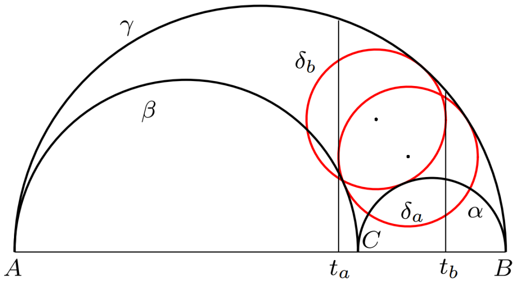

Figure 6: $k<1$, $t_b-r_k<t_a+r_k$.

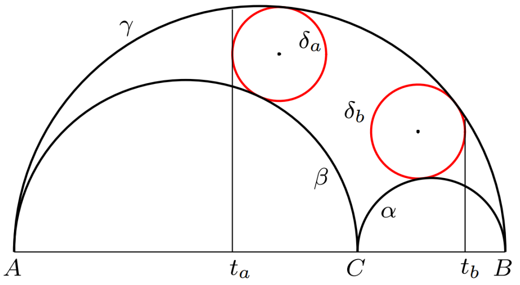

Figure 7: $1<k$, $t_a+r_k<t_b-r_k$.

Assume $a\le b$ for $\mathcal{T}_k$. Then the circles δ_a and δ_b are maximal if δ_a and α overlap (see Figure 9). Solving the equation $r_k=a$ for k in this case, we have

$$k = 1-\frac{a}{b}. \tag{6}$$

Therefore *the k-Archimedean twins exist if and only if $1-a/b\le k$ and the maximal k-Archimedean twins are obtained if* (6) *holds*. Notice that $1-a/b\ge 1-b/a$ in this event. Therefore we can say that *k-Archimedean twins exist if and only if $k\ge\max(1-a/b, 1-b/a)$*. A similar result can also be obtained in the case $a>b$. Therefore we have the following theorem.

Theorem 3 *k-Archimedean twins exist if and only if*

$$k \geq \max\left(1-\frac{a}{b}, 1-\frac{b}{a}\right).$$

The maximal k-Archimedean twins are obtained if and only if

$$k = \max\left(1-\frac{a}{b}, 1-\frac{b}{a}\right).$$

Assume $a \leq b$ and $k = 1-a/b$ for $\mathcal{T}_k$. Then we have $r_k = a$ and

$$\begin{aligned}(x_b, y_b) &= \left(-r_k + 2a\left(1-\frac{r_k}{b}\right), \frac{2\sqrt{ac(b-r_k)r_k}}{b}\right)\\ &= \left(a - \frac{2a^2}{b}, \frac{2a\sqrt{b^2-a^2}}{b}\right)\end{aligned}$$

by (4). While solving the equations $x^2+y^2 = 4a^2$ and $(x-(-b))^2+y^2 = b^2$, we get that the semicircles of center C passing through the point B meets β in the point of coordinates

$$\left(-\frac{2a^2}{b}, \frac{2a\sqrt{b^2-a^2}}{b}\right).$$

Therefore this point is one of the endpoints of the diameter of δ_b parallel to AB (see Figure 8). Since $\delta_a = \alpha$, the axis and t_a overlap. Especially if $a = b$, then $\max(1-a/b, 1-b/a) = 0$. Therefore the configuration $\mathcal{T}_0$ exists, where δ_a and δ_b are the maximal 0-Archimedean twins and overlap with α and β, respectively, and t_a and t_b overlap with the axis (see Figure 9).

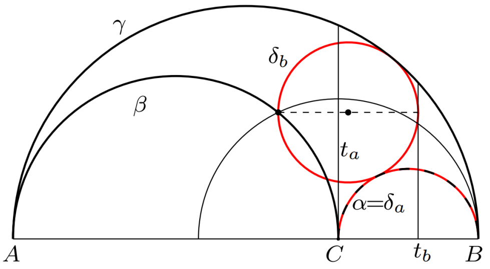

Figure 8: $\mathcal{T}_k$ $(a < b, k = 1-a/b)$.

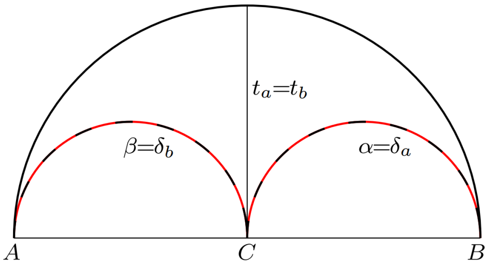

Figure 9: $\mathcal{T}_0$ *$(a = b)$.*

References

[1] Okumura, H., Wasan geometry. In: Sriraman, B. ed., *Handbook of the Mathematics of the Arts and Sciences*, Springer, Cham, 2020, 1–52, `https://doi.org/10.1007/978-3-319-70658-0_122-1`

[2] Okumura, H., Watanabe, M., The twin circles of Archimedes in a skewed arbelos, *Forum Geom* **4** (2004), 229–251.

[3] Saitama Prefectural library ed., The sangaku in Saitama, 1969, Saitama Prefectural Library.

Hiroshi Okumura
orcid.org/0000-0003-4332-0189
e-mail: hokmr@yandex.com

Takahanadai Maebashi Gunma 371-0123, Japan

Appendix: Proof of Problem 1

We give a proof of Problem 1, since it was proposed with no solution (see Figure 2). Let δ_i $(i = 1,2,3)$ be the three congruent circles, where δ_1 and δ_3 touch δ_2 externally. Let (x_i, y_i) be the coordinates of the center of the circle δ_i. The point of intersection of γ and the axis is denoted by I. Let t be the external common tangent of α and β. The line t has an equation ([2]):

$$t(x,y) = (a-b)x - 2\sqrt{ab}y + 2ab = 0.$$

While the point I has coordinates $(0, 2\sqrt{ab})$, because γ is represented by an equation $(x-2a)(x+2b)+y^2 = 0$. Hence the line IO is perpendicular to t. Therefore I coincides with the midpoint of the arc of γ cut by t, i.e., the circle δ_2 touches γ at I. Hence we have $x_2^2 + (y_2 - 2\sqrt{ab})^2 = r^2$ and $(x_2-(a-b))^2 + y_2^2 = (c-r)^2$. Solving the two equations for x_2 and y_2, we have

$$x_2 = \frac{(a-b)r}{c}, \quad y_2 = \frac{2\sqrt{ab}(c-r)}{c}. \tag{7}$$

If the perpendicular from the center of δ_1 to AB meet t in a point of coordinates (x_1, y'), then $t(x_1, y') = 0$, while there is a real number $z > 0$ such that $y_1 = y' + z$. Then $t(x_1, y_1) = t(x_1, y'+z) = t(x_1, y') - 2\sqrt{ab}z = -2\sqrt{ab}z < 0$. Hence we get $t(x_1, y_1) < 0$. Therefore we have $t(x_1, y_1)/c = -r$. We also have $(x_2-x_1)^2 + (y_2-y_1)^2 = (2r)^2$ and $(x_1-(a-b))^2 + y_1^2 = (c-r)^2$. Eliminating x_1 and y_1 from the three equations with (7) and solving the resulting equation for r, we get (1).