GLASNIK MATEMATIČKI
Vol. 60(80)(2025), 327 – 352

PRESCRIBED WEINGARTEN CURVATURE EQUATIONS IN WARPED PRODUCT MANIFOLDS

Ya Gao, Chenyang Liu and Jing Mao*
Capital Normal University and Hubei University, China

Abstract. In this paper, under suitable settings, we can obtain the existence of solutions to a class of prescribed Weingarten curvature equations in *warped product manifolds* of special type by the standard degree theory based on the *a priori estimates* for the solutions. This is to say that the existence of closed hypersurface (which is graphic with respect to the base manifold and whose k-Weingarten curvature satisfies some constraint) in a given warped product manifold of special type can be assured.

1. Introduction

Throughout this paper, let (M^n, g) be a compact Riemannian n-manifold with the metric g, and let I be an (unbounded or bounded) interval in $\mathbb{R}$. Clearly, $\bar{M} := I \times_f M^n$ is actually the $(n+1)$-dimensional warped product manifold (sometimes, for simplicity, just say *warped product*) endowed with the following metric

$$\bar{g} = dt^2 + f^2(t)g, \tag{1.1}$$

where $f : I \to \mathbb{R}^+$ is a positive differential function defined on I. Given a differentiable function $u : M^n \to I$, its graph actually corresponds to the following graphic hypersurface

$$\mathcal{G} = \{X(x) = (u(x), x) | x \in M^n\} \tag{1.2}$$

in $\bar{M}$. Equivalently, we can say that $\mathcal{G}$ is graphic w.r.t. *the base manifold* M^n. Denote by $\bar{\nabla}$, D the Riemannian connections on $\bar{M}$ and M^n, respectively. Let $\{e_i\}_{i=1,2,\ldots,n}$ be an orthonormal frame field in M^n. Then one can find

---

2020 *Mathematics Subject Classification.* 53C42, 35J60.
Key words and phrases. Prescribed Weingarten curvature equations, k-convex, star-shaped, warped product manifolds.
*Corresponding author.

an orthonormal frame field $\{\bar{e}_\alpha\}_{\alpha=0,1,\ldots,n}$ in $\bar{M}$ such that $\bar{e}_\alpha=(1/f)e_i$, $1\leq \alpha=i\leq n$ and $\bar{e}_0=\partial/\partial t$. The existence of the frame fields can always be assured in the tangent space of a prescribed point. Denote by[1] $u_i:=D_iu$, $u_{ij}:=D_jD_iu$, and $u_{ijk}:=D_kD_jD_iu$ the covariant derivatives of u w.r.t. the metric g. Clearly, the tangent vectors of $\mathcal{G}$ are given by

$$X_i=(Du,1)=e_i+u_i\partial/\partial t=f\bar{e}_i+u_i\bar{e}_0, \qquad i=1,2,\ldots,n.$$

Let $\langle\cdot,\cdot\rangle$ be the inner product w.r.t. the metric $\bar{g}$. Then the induced metric $\widetilde{g}$ on $\mathcal{G}$ has the form

$$\widetilde{g}_{ij}=\langle X_i,X_j\rangle=f^2\delta_{ij}+u_iu_j,$$

its inverse is given by

$$\widetilde{g}^{ij}=\frac{1}{f^2}\left(\delta^{ij}-\frac{u^iu^j}{f^2+|Du|^2}\right),$$

where $u^i=g^{ij}u_j=\delta^{ij}u_j=u_i$ and $|Du|^2=u^iu_i=\sum\limits_{i=1}^{n}u_i^2$. Of course, in this paper we use the Einstein summation convention – repeated superscripts and subscripts should be made summation[2]. The outward unit normal vector field of $\mathcal{G}$ is given by

$$\nu=\frac{1}{\sqrt{f^2+|Du|^2}}\left(f\frac{\partial}{\partial t}-u^i\frac{e_i}{f}\right)=\frac{1}{\sqrt{f^2+|Du|^2}}\left(f\bar{e}_0-u^i\bar{e}_i\right), \tag{1.3}$$

and the component h_{ij} of the second fundamental form A of $\mathcal{G}$ is computed as follows

$$h_{ij}=-\langle\bar{\nabla}_{X_j}X_i,\nu\rangle=\frac{1}{\sqrt{f^2+|Du|^2}}\left(-fu_{ij}+2f'u_iu_j+f^2f'\delta_{ij}\right). \tag{1.4}$$

One can also see [3, Subsection 2.2] for the computations of the above geometric quantities. Denote by $\lambda_1,\lambda_2,\ldots,\lambda_n$ the principal curvatures of $\mathcal{G}$, which are actually the eigenvalues of the matrix $(h_{ij})_{n\times n}$ w.r.t. the metric $\widetilde{g}$. The so-called *k-th Weingarten curvature* at $X(x)=(u(x),x)\in\mathcal{G}$ is defined as

$$\sigma_k(\lambda_1,\lambda_2,\cdots,\lambda_n)=\sum_{1\leq i_1<i_2<\cdots<i_k\leq n}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}. \tag{1.5}$$

$V=f(t)\frac{\partial}{\partial t}$ is the position vector field[3] of the hypersurface $\mathcal{G}$ in $\bar{M}$, and clearly, for any $x\in M^n$, $V|_x$ is a one-to-one correspondence with $X(x)$. Let $\nu(V)$ be the outward unit normal vector field along the hypersurface $\mathcal{G}$ and

---

[1] Clearly, for accuracy, here D_iu should be $D_{e_i}u$. In the sequel, without confusion and if needed, we wish to simplify covariant derivatives like this. In this setting, $u_{ij}:=D_jD_iu$, $u_{ijk}:=D_kD_jD_iu$ mean $u_{ij}=D_{e_j}D_{e_i}u$ and $u_{ijk}=D_{e_k}D_{e_j}D_{e_i}u$, respectively. We will also simplify covariant derivatives on $\mathcal{G}$ and $\bar{M}$ similarly if necessary.

[2] In this setting, repeated Latin letters should be made summation from 1 to n.

[3] In $\mathbb{R}^{n+1}$ or the hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$, there is no need to define the vector field V since these two spaces are two-points homogeneous and global coordinate system can be set up, and then $X(x)$ can be seen as the position vector directly.

$\lambda(V) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be the principal curvatures of $\mathcal{G}$ at V. Define the annulus domain $\bar{M}^+_- \subset \bar{M}$ as follows

$$\bar{M}^+_- := \{(t, x) \in \bar{M} | r_1 \le t \le r_2\}$$

with $r_1 < r_2$. In this paper, we consider the following Weingarten curvature equation

$$\sigma_k(\lambda(V)) = \sum_{l=0}^{k-1} \alpha_l(u(x), x)\sigma_l(\lambda(V)), \quad \forall V \in \mathcal{G}, \quad 2 \le k \le n, \tag{1.6}$$

where $\{\alpha_l(u(x), x)\}_{l=0}^{k-1}$ are given smooth functions defined on $\mathcal{G}$. The k-th Weingarten curvature $\sigma_k(\lambda(V))$ is also called k-th mean curvature. Besides, when $k = 1$, 2 and n, $\sigma_k(\lambda(V))$ corresponds to the mean curvature, the scalar curvature and the Gaussian curvature of $\mathcal{G}$ at V.

We also need the following conception.

Definition 1.1. *For $1 \le k \le n$, let Γ_k be a cone in $\mathbb{R}^n$ determined by*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_l(\lambda) > 0, \ l = 1, 2, \ldots, k\}.$$

A smooth graphic hypersurface $\mathcal{G} \subset \bar{M}$ is called k-admissible if at every position vector $V \in \mathcal{G}$, $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_k$.

For (1.6), we can prove the following theorem.

Theorem 1.2. *Let M^n be a compact Riemannian n-manifold ($n \ge 3$) and $\bar{M} = I \times_f M^n$, with the metric (1.1), be the warped product manifold defined as before. Assume that the warping function f is positive C^2 differential, $f' > 0$ and $\alpha_l(u(x), x) \in C^\infty(I \times M^n)$ are positive functions for all $0 \le l \le k-1$. Suppose that*

$$\sigma_k(e)\left(\frac{f'}{f}\right)^k \ge \sum_{l=0}^{k-1} \alpha_l(u, x)\sigma_l(e)\left(\frac{f'}{f}\right)^l \qquad \text{for } u \ge r_2, \tag{1.7}$$

$$\sigma_k(e)\left(\frac{f'}{f}\right)^k \le \sum_{l=0}^{k-1} \alpha_l(u, x)\sigma_l(e)\left(\frac{f'}{f}\right)^l \qquad \text{for } 0 < u \le r_1, \tag{1.8}$$

and

$$\frac{\partial}{\partial u}\left[f^{k-l}(u)\alpha_l(u, x)\right] \le 0 \quad \text{for } r_1 < u < r_2, \ 0 \le l \le k-1, \tag{1.9}$$

where $[r_1, r_2] \subset I$, $e = (1, 1, \ldots, 1)$. Then there exists a smooth k-admissible, closed graphic hypersurface $\mathcal{G}$ contained in the interior of the annulus $\bar{M}^+_-$ and satisfying (1.6).

Remark 1.3. (1) The proof of Theorem 1.2 can be discussed in two cases: $f'' \le 0$ or $f'' > 0$, where f is the warping function defined as before. After careful calculations in Section 4, we find that an important ingredient in C^1

estimate is to keep $(\varphi f')'$ non-positive. In the case $f'' \leq 0$, this ingredient can be assured because of the assumptions made on the auxiliary function φ (see Section 3 for details). However, in the case $f'' > 0$, one cannot guarantee that $(\varphi f')'$ is positive or negative if the same assumptions were made on φ. In this situation, we need to do some change, that is, we define a *new* φ as $\varphi = \tilde{k}(f')^{-1}$, where $\tilde{k}$ is a positive constant and satisfies $f'(r_1) < \tilde{k} < f'(r_2)$, and then this function φ satisfies $\varphi(u) > 0$, $\varphi(u) > 1$ for $u \leq r_1$, $\varphi(u) < 1$ for $u \geq r_2$, $\varphi'(u) < 0$. Thus $(\varphi f')' \leq 0$ can also be obtained by this change, and then in the case $f'' > 0$ the rest of the argument for Theorem 1.2 would go back to that of the case $f'' \leq 0$ – please see Section 4 for details.

(2) The k-admissible and the graphic properties of the hypersurface $\mathcal{G}$ make sure that (1.6) is a single scalar second-order elliptic PDE of the graphic function u, which is the cornerstone of the a prior estimates given below. If furthermore M^n is convex, then M^n is diffeomorphic to $\mathbb{S}^n$ (i.e. the Euclidean unit n-sphere), $\mathcal{G}$ is also a graphic hypersurface over $\mathbb{S}^n$ and should be starshaped. In this setting, Theorem 1.2 degenerates into the following:

- FACT 1. *Under the assumptions of Theorem 1.2, if furthermore M^n is convex, then there exists a smooth k-admissible, starshaped closed hypersurface $\mathcal{G}$ contained in the interior of the annulus $\bar{M}^{+}_{-}$ and satisfying (1.6).*

(3) We refer readers to, e.g., [21, Appendix A], [23, pp. 204-211 and Chapter 7] for an introduction to the notion and properties of warped product manifolds. Submanifolds in warped product manifolds have nice geometric properties and interesting results can be expected – see, e.g., several nice eigenvalue estimates for the drifting Laplacian and the nonlinear p-Laplacian on minimal submanifolds in warped product manifolds of prescribed type have been shown in [18, Sections 3-5].

(4) The equation (1.6) is actually a combination of elementary symmetric functions of eigenvalues of a given $(0, 2)$-type tensor. Equations of this type are important not only in the study of PDEs but also in the study of many important geometric problems. For instance, if $\lambda(V)$ in (1.6) were replaced by eigenvalues of the Hessian D^2u of a graphic function u defined over a bounded $(k-1)$-convex domain $\Omega \subset \mathbb{R}^n$, Krylov [15] studied the corresponding PDE

$$\sigma_k(D^2u(x)) = \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(D^2u(x)), \quad \forall x \in \Omega, \tag{1.10}$$

with a prescribed Dirichlet boundary condition (DBC for short) and coefficients $\alpha_l(x) \geq 0$ for all $0 \leq l \leq k-1$, and observed that the natural admissible cone to make equation elliptic is Γ_k; Guan-Zhang [12] showed that comparing with Krylov's this observation, for the admissible solution of (1.6) with prescribed DBC in the sense that $\lambda(D^2u) \in \Gamma_{k-1}$, there is no sign requirement for the coefficient function of $\alpha_{k-1}(x)$. Moreover, they also investigated the solvability of the following fully nonlinear elliptic equation

$$\sigma_k(D^2u + uI) = \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(D^2u + uI), \quad \forall x \in \mathbb{S}^n,$$

for some unknown function $u : \mathbb{S}^n \to \mathbb{R}$ defined over $\mathbb{S}^n$, where $\alpha_l(x)$, $0 \le l \le k-2$, are positive functions; Fu-Yau [7, 8] proposed an equation of this type in the study of the Hull-Strominger system in theoretical physics; Phong-Picard-Zhang investigated the Fu-Yau equation and its generalization in series works [25, 26, 27]. Inspired by Krylov's and Guan-Zhang's works [12, 15], Chen-Shang-Tu [2] considered the following equation

$$\sigma_k(\kappa(X)) = \sum_{l=0}^{k-1} \alpha_l(X)\sigma_l(\kappa(X)), \quad \forall X \in \mathcal{M} \subset \mathbb{R}^{n+1}, \quad 2 \le k \le n \tag{1.11}$$

on an embedded, closed starshaped n-hypersurface $\mathcal{M}$, $n \ge 3$, where $\kappa(X)$ are principal curvatures of $\mathcal{M}$ at X, and $\alpha_l(x)$, $0 \le l \le k-1$, are positive functions defined over $\mathcal{M}$. Under the k-convexity for $\mathcal{M}$ and several other growth assumptions (see [2, Theorem 1.1]), they can show the existence of solutions to (1.11). This result has been generalized by Shang-Tu [28] to the situation that the ambient space $\mathbb{R}^{n+1}$ was replaced by the hyperbolic space $\mathbb{H}^{n+1}$. Recently, Chen-Tu-Xiang [4] studied the equation

$$\sigma_k(\kappa(V)) = \psi(V, \nu(V)), \qquad \forall V \in \mathcal{G}, \tag{1.12}$$

where as before $\mathcal{G} \subset \bar{M} := I \times_f M^n$, with f a positive C^2 differential function defined on $I \subset \mathbb{R}$, is a graphic hypersurface (defined as (1.2)) in the warped product manifold $\bar{M}$, $\sigma_k(\cdot)$ denotes the elementary symmetric function, V and $\nu(V)$ are the position vector field, the outward unit normal vector field along the hypersurface $\mathcal{G}$ respectively. Besides, $\kappa(V) = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ stand for the principal curvatures of hypersurface $\mathcal{G}$ at V. If the function $\psi(\cdot,\cdot)$ and the warping function f satisfy some growth assumptions, by applying the degree theory, they can prove the existence of $C^{4,\alpha}$-solution to (1.12) in the case $k \ge n-2$, provided $\mathcal{G}$ is k-convex and starshaped.

If $M^n = \mathbb{S}^n$, $I = (0, \ell)$ with $0 < \ell \le \infty$, putting a one-point compactification topology by identifying all pairs $\{0\} \times \mathbb{S}^n$ with a single point p^* to $\bar{M}$ (see, e.g., [6, page 705] for this notion) and requiring that $f(0) = 0$, $f'(0) = 1$, then the warped product manifold $\bar{M}$ becomes the spherically symmetric manifold $\widetilde{M} := [0, \ell) \times_f \mathbb{S}^n$. The single point p^* is called the *base point* of $\widetilde{M}$. Applying Fact 1 in Remark 1.3 directly, one has the following statement.

Corollary 1.4. *Under the assumptions of Theorem 1.2 with additionally $M^n = \mathbb{S}^n$, $I = (0, \ell)$ with $0 < \ell \le \infty$, one-point compactification topology imposed, $f(0) = 0$ and $f'(0) = 1$, then there exists a smooth k-admissible, starshaped (w.r.t. the base point p^*), closed hypersurface $\mathcal{G}$ contained in the interior of the annulus $\bar{M}^+_- \subset \widetilde{M}$ and satisfying (1.6).*

REMARK 1.5. (1) If furthermore the warping function f satisfies $f''(t) + Kf(t) = 0$ for some constant K, i.e. the Jacobi equation, then

$$f(t) = \begin{cases} \sin(\sqrt{K}t)/\sqrt{K}, & K > 0,\ \ell = \pi/2\sqrt{K}, \\ t, & K = 0,\ \ell = \infty, \\ \sinh(\sqrt{-K}t)/\sqrt{-K}, & K < 0,\ \ell = \infty, \end{cases}$$

and moreover, in this setting, $\widetilde{M}$ corresponds to $\mathbb{S}^{n+1}(1/\sqrt{K})$ (i.e., the Euclidean $(n+1)$-sphere with radius $1/\sqrt{K}$) with the antipodal point of p^* missed, since we need to make sure that $f' > 0$, so we can only get the case of $\ell = \pi/2\sqrt{K}$, $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}(K)$ (i.e., the hyperbolic $(n+1)$-space with constant curvature $K < 0$), respectively. From this, one can see that spherically symmetric manifolds cover space forms as a special case and actually they were called *generalized space forms* by Katz and Kondo [13].

(2) Clearly, our Corollary 1.4 covers Chen-Shang-Tu's and Shang-Tu's main results in [2, 28] (mentioned in (4) of Remark 1.3) as special cases.

(3) Spherically symmetric manifolds have nice symmetry in non-radial direction, which leads to the fact that one can use this kind of manifolds as model space in the study of comparison theorems. In fact, Prof. J. Mao and his collaborators have used spherically symmetric manifolds as model space to successfully obtain Cheng-type eigenvalue comparison theorems for the first Dirichlet eigenvalue of the Laplacian on complete manifolds with radial (Ricci and sectional) curvatures bounded, Escobar-type eigenvalue comparison theorem for the first nonzero Steklov eigenvalue of the Laplacian on complete manifolds with radial sectional curvature bounded from above, heat kernel and volume comparison theorems for complete manifolds with suitable curvature constraints, and so on – see [6, 19, 20, 22, 30] for details.

This paper is organized as follows. In Section 2, we will list some useful formulas including several basic properties of σ_k, structure equations for hypersurfaces in warped product manifolds. A priori estimates (including C^0, C^1 and C^2 estimates) for solutions to (1.6) will be shown continuously in Sections 3-5. We wish to mention that the calculation about prior estimates is performed at a fixed point on M^n, and so the sign of the prescribed function restricted to this point is fixed. In Section 6, by applying the degree theory, together with the a priori estimates obtained, we can prove the existence of solutions to prescribed Weingarten curvature equations of type (1.6).

2. SOME USEFUL FORMULAE

Except the setting of notations in Section 1, denote by $\bar{\nabla}$, ∇ the Riemannian connections on $\bar{M}$ and $\mathcal{G}$, respectively. The curvature tensors in $\bar{M}$ and $\mathcal{G}$ will be denoted by $\bar{R}$ and R, respectively. Let $\{E_0 = \nu, E_1, \ldots, E_n\}$ be an orthonormal frame field in $\mathcal{G}$ and $\{\omega_0, \omega_1, \ldots, \omega_n\}$ be its associated dual frame field. The connection forms $\{\omega_{ij}\}$ and curvature forms $\{\Omega_{ij}\}$ in $\mathcal{G}$ satisfy the structure equations

$$d\omega_i - \sum_i \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l.$$

The coefficients h_{ij}, $1 \le i, j \le n$, of the second fundamental form are given by Weingarten equation

$$\omega_{i0} = \sum_j h_{ij}\omega_j. \tag{2.1}$$

The covariant derivatives of the second fundamental form h_{ij} in $\mathcal{G}$ are given by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_l h_{il}\omega_{lj} + \sum_l h_{lj}\omega_{li},$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{lj} + \sum_l h_{ijl}\omega_{lk}.$$

The Codazzi equation is

$$h_{ijk} - h_{ikj} = -\bar{R}_{0ijk}, \tag{2.2}$$

and the Ricci identity can be obtained as follows.

LEMMA 2.1 (see also [3, Lemma 2.2]). *Let $X(x)$ be a point of $\mathcal{G}$ and $\{E_0 = \nu, E_1, \dots, E_n\}$ be an adapted frame field such that each E_i is a principal direction and $\omega_i^k = 0$ at $X(x)$. Let (h_{ij}) be the second quadratic form of $\mathcal{G}$. Then, at the point $X(x)$, we have*

$$\begin{aligned} h_{llii} = {} & h_{iill} - h_{lm}(h_{mi}h_{il} - h_{ml}h_{ii}) - h_{mi}(h_{mi}h_{ll} - h_{ml}h_{li}) \\ & + \bar{R}_{0iil;l} - 2h_{ml}\bar{R}_{miil} + h_{il}\bar{R}_{0i0l} + h_{ll}\bar{R}_{0ii0} \\ & + \bar{R}_{0lil;i} - 2h_{mi}\bar{R}_{mlil} + h_{ii}\bar{R}_{0l0l} + h_{li}\bar{R}_{0li0}. \end{aligned} \tag{2.3}$$

As mentioned in Section 1, one can suitably choose local coordinates such that $\{e_i\}_{i=1,2,\dots,n}$ is an orthonormal frame field in M^n, and then one can find an orthonormal frame field $\{\bar{e}_\alpha\}_{\alpha=0,1,\dots,n}$ in $\bar{M}$ such that $\bar{e}_i = (1/f)e_i$, $1 \le \alpha = i \le n$ and $\bar{e}_0 = \partial/\partial t$. Correspondingly, the associated dual frame field of $\{\bar{e}_\alpha\}_{\alpha=0,1,\dots,n}$ should be $\{\bar{\theta}_\alpha\}_{\alpha=0,1,\dots,n}$ with $\bar{\theta}_i = f\theta_i$, $1 \le i \le n$, and $\bar{\theta}_0 = dt$. Clearly, $\{\theta_i\}_{i=1,\dots,n}$ is the dual frame field of the orthonormal frame field $\{e_i\}_{i=1,2,\dots,n}$. We have the following fact.

LEMMA 2.2 (see [3]). *On the leaf M_t of the warped product manifold $\bar{M} = I \times_f M^n$, the curvature satisfies*

$$\bar{R}_{ijk0} = 0 \tag{2.4}$$

and the principal curvature is given by

$$\kappa(t) = \frac{f'(t)}{f(t)}, \tag{2.5}$$

where the outward unit normal vector $\bar{e}_0 = \frac{\partial}{\partial t}$ *is chosen for each leaf* M_t.

REMARK 2.3. In fact, the leaf M_t can also be seen as a closed graphic hypersurface in $\bar{M}$, which corresponds to the graph of some constant function, i.e. $u = const.$. Besides, we refer readers to [3, Section 2] or [24] for the geometry of hypersurfaces in warped product manifolds if necessary.

Consider two functions $\tau : \mathcal{G} \to \mathbb{R}$ and $\Lambda : \mathcal{G} \to \mathbb{R}$ given by

$$\tau = f\langle \nu, \bar{e}_0 \rangle = \langle V, \nu \rangle, \qquad \Lambda = \int_0^u f(s)ds, \tag{2.6}$$

where $V = f\bar{e}_0 = f\frac{\partial}{\partial t}$ is the position vector field and ν is the outward unit normal vector field. Then we have the following statement.

LEMMA 2.4 (see [1]). *The gradient vector fields of the functions* τ *and* Λ *are*

$$\nabla_{E_i}\Lambda = f\left\langle \bar{e}_0, E_i \right\rangle, \tag{2.7}$$

$$\nabla_{E_i}\tau = \sum_j \nabla_{E_j}\Lambda h_{ij}, \tag{2.8}$$

and the second order derivatives of τ *and* Λ *are given by*

$$\nabla^2_{E_i,E_j}\Lambda = -\tau h_{ij} + f' g_{ij}, \tag{2.9}$$

$$\nabla^2_{E_i,E_j}\tau = -\tau \sum_k h_{ik}h_{kj} + f' h_{ij} + \sum_k (h_{ijk} + \bar{R}_{0ijk})\nabla_{E_k}\Lambda. \tag{2.10}$$

The following Newton-Maclaurin inequality will be used frequently (see, e.g., [17, 29]).

LEMMA 2.5. *Let* $\lambda \in \mathbb{R}^n$. *For* $0 \le l \le k \le n,\ r > s \ge 0,\ k \ge r,\ l \ge s$, *we have*

$$k(n-l+1)\sigma_{l-1}(\lambda)\sigma_k(\lambda) \le l(n-k+1)\sigma_l(\lambda)\sigma_{k-1}$$

and

$$\left[\frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{k-l}} \le \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}}, \qquad \text{for } \lambda \in \Gamma_k.$$

At end, we also need the following truth to ensure the ellipticity of (3.1).

LEMMA 2.6. *Let* $\mathcal{G} = \{(u(x), x)\,|\,x \in M^n\}$ *be a smooth* $(k-1)$*-admissible closed hypersurface in* $\bar{M}$ *and* $\alpha_l(u,x) \ge 0$ *for any* $x \in M^n$ *and* $0 \le l \le k-2$. *Then the operator*

$$G\left(h_{ij}(V), u, x\right) := \frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} - \sum_{l=0}^{k-2} \alpha_l(u,x) \frac{\sigma_l(\lambda(V))}{\sigma_{k-1}(\lambda(V))}$$

is elliptic and concave with respect to $h_{ij}(V)$.

PROOF. The proof is almost the same with the one of [12, Proposition 2.2], and we prefer to omit here. ∎

3. C^0 ESTIMATE

We consider the family of equations for $0 \leq t \leq 1$,

$$\frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} - \sum_{l=0}^{k-2} t\alpha_l(u,x)\frac{\sigma_l(\lambda(V))}{\sigma_{k-1}(\lambda(V))} - \alpha_{k-1}(u,x,t) = 0, \tag{3.1}$$

where

$$\alpha_{k-1}(u,x,t) := t\alpha_{k-1}(u,x) + (1-t)\varphi(u)\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'}{f},$$

and φ is a positive function defined on I and satisfying the following conditions:

(a) $\varphi(u) > 0$;
(b) $\varphi(u) > 1$ for $u \leq r_1$;
(c) $\varphi(u) < 1$ for $u \geq r_2$;
(d) $\varphi'(u) < 0$.

LEMMA 3.1 (C^0 ESTIMATE). *Assume that* $0 \leq \alpha_l(u,x) \in C^\infty(I \times M^n)$. *Under the assumptions* (1.7) *and* (1.8) *mentioned in Theorem 1.2, if* $\mathcal{G} = \{(u(x),x) | x \in M^n\} \subset \bar{M}$ *is a smooth* $(k-1)$*-admissible, closed graphic hypersurface satisfying the curvature equation* (3.1) *for a given* $t \in [0,1]$, *then*

$$r_1 \leq u(x) \leq r_2, \qquad \forall x \in M^n.$$

PROOF. Assume that $u(x)$ attains its maximum at $x_0 \in M^n$ and $u(x_0) \geq r_2$. Then from (1.4), one has

$$h^i_j = \frac{1}{v}\left(f'\delta^i_j - \frac{1}{f}u_{ij} + \frac{f'u_j u_i}{v^2} + \sum_{k=1}^n \frac{u_{jk}u_k u_i}{fv^2}\right),$$

where $v = \sqrt{f^2 + |\nabla u|^2}$, which implies

$$h^i_j(x_0) = \frac{1}{f}\left(f'\delta^i_j - \frac{u_{ij}}{f}\right) \geq \frac{f'}{f}\delta^i_j.$$

Note that $\frac{\sigma_k}{\sigma_{k-1}}$ and $\frac{\sigma_{k-1}}{\sigma_l}$ with $0 \leq l \leq k-2$ are concave in Γ_{k-1}. Thus,

$$\frac{\sigma_k}{\sigma_{k-1}}(h^i_j) \geq \frac{\sigma_k}{\sigma_{k-1}}\left(\frac{f'}{f}\delta^i_j\right) + \frac{\sigma_k}{\sigma_{k-1}}\left(-\frac{1}{f^2}u_{ij}\right) \geq \frac{\sigma_k}{\sigma_{k-1}}\left(\frac{f'}{f}\delta^i_j\right).$$

Therefore, it follows that

$$\frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} \geq \frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'}{f}.$$

Similarly, one can get

$$\frac{\sigma_l(\lambda(V))}{\sigma_{k-1}(\lambda(V))} \le \frac{\sigma_l(e)}{\sigma_{k-1}(e)} \left(\frac{f}{f'}\right)^{k-l-1}.$$

Combining with the above two inequalities, we have

$$\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'}{f} - \sum_{l=0}^{k-2} t\alpha_l(u,x)\frac{\sigma_l(e)}{\sigma_{k-1}(e)}\left(\frac{f}{f'}\right)^{k-l-1} \le \alpha_{k-1}(u,x,t).$$

Clearly, if $t=0$, the above inequality is contradict with (3.1). When $0 < t \le 1$, we can obtain

$$\begin{aligned}
\alpha_{k-1}(u,x) &= \left(1-\frac{1}{t}\right)\varphi\frac{f'}{f}\frac{\sigma_k(e)}{\sigma_{k-1}(e)} + \frac{1}{t}\alpha_{k-1}(x,u,t) \\
&\ge \left(\frac{1}{t}\frac{f'}{f} + \left(1-\frac{1}{t}\right)\varphi\frac{f'}{f}\right)\frac{\sigma_k(e)}{\sigma_{k-1}(e)} \\
&\quad - \sum_{l=0}^{k-2}\alpha_l(u,x)\frac{\sigma_l(e)}{\sigma_{k-1}(e)}\left(\frac{f}{f'}\right)^{k-l-1} \\
&> \frac{f'}{f}\frac{\sigma_k(e)}{\sigma_{k-1}(e)} - \sum_{l=0}^{k-2}\alpha_l(u,x)\frac{\sigma_l(e)}{\sigma_{k-1}(e)}\left(\frac{f}{f'}\right)^{k-l-1},
\end{aligned}$$

which is contradiction with

$$\frac{f'}{f}\frac{\sigma_k(e)}{\sigma_{k-1}(e)} - \sum_{l=0}^{k-2}\alpha_l(u,x)\frac{\sigma_l(e)}{\sigma_{k-1}(e)}\left(\frac{f}{f'}\right)^{k-l-1} \ge \alpha_{k-1}(u,x)$$

in view of (1.7) and the condition $\varphi(u) < 1$ for $u \ge r_2$. This shows $\sup u \le r_2$. Similarly, we can obtain $\inf u \ge r_1$ in view of (1.8) and the condition $\varphi(u) > 1$ for $u \le r_1$. Our proof is finished. ∎

Now, we can prove the following uniqueness result.

LEMMA 3.2. *For $t = 0$, there exists a unique admissible solution of* (3.1), *namely $\mathcal{G}_0 = \{(u(x),x) \in \bar{M} | u(x) = u_0\}$, where u_0 is the unique solution of $\varphi(u_0) = 1$.*

PROOF. Let $\mathcal{G}_0$ be a solution of (3.1), and then for $t = 0$,

$$\frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} - \varphi(u)\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'}{f} = 0.$$

Assume that $u(x)$ attains its maximum $u_{\max}$ at $x_0 \in M^n$. Then one has

$$\frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} \ge \frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'}{f},$$

which implies

$$\varphi(u_{\max}) \ge 1.$$

Similarly, the minimum $u_{\min}$ of $u(x)$ satisfies

$$\varphi(u_{\min}) \leq 1.$$

Since φ is a decreasing function, we obtain

$$\varphi(u_{\max}) = \varphi(u_{\min}) = 1,$$

which implies that $u(x) = u_0$ for any $(u(x), x) \in \mathcal{G}_0$, with u_0 the unique solution of $\varphi(u_0) = 1$. □

4. C^1 ESTIMATE

We can rewrite (3.1) as follows:

$$G(h_{ij}(V), u, x, t) = \frac{\sigma_k(\kappa(V))}{\sigma_{k-1}(\kappa(V))} - \sum_{l=0}^{k-2} t\alpha_l(u, x) \frac{\sigma_l(\kappa(V))}{\sigma_{k-1}(\kappa(V))} = \alpha_{k-1}(u, x, t).$$

For convenience, we will simplify notations as follows:

$$G_k(h_{ij}(V)) := \frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))}, \qquad G_l(h_{ij}(V)) =: -\frac{\sigma_l(\lambda(V))}{\sigma_{k-1}(\lambda(V))},$$

and

$$G^{ij}(\lambda(V)) := \frac{\partial G}{\partial h_{ij}}, \quad G^{ij,rs}(\lambda(V)) := \frac{\partial^2 G}{\partial h_{ij} \partial h_{rs}}.$$

LEMMA 4.1 (C^1 ESTIMATE). *Assume that $k \geq 2$ and*

$$\alpha_l(u, x) \geq c_l > 0, \qquad \forall x \in M^n$$

for $0 \leq l \leq k-1$. Under the assumption (1.9)*, if the smooth $(k-1)$-admissible, closed graphic hypersurface $\mathcal{G}$ satisfies* (1.6) *and u has positive upper and lower bounds, then there exists a constant C depending on n, k, c_l, $|\alpha_l|_{C^1}$, the C^0 bound of f and the curvature tensor $\bar{R}$, the minimum and maximum values of u such that*

$$|\nabla u(x)| \leq C, \qquad \forall x \in M^n.$$

PROOF. First, we know from (1.3) and (2.6) that

$$\tau = \frac{f^2(u)}{\sqrt{f^2(u) + |Du|^2}}.$$

It is sufficient to obtain a positive lower bound of τ. Define

$$\psi := -\log \tau + \gamma(\Lambda),$$

where $\gamma(t)$ is a function chosen later. Assume that x_0 is the maximum value point of ψ. If V is parallel to the normal direction ν of $\mathcal{G}$ at x_0, our result holds since $\langle V, \nu \rangle = |V|$. So, we assume that V is not parallel to the normal direction ν at x_0, we may choose the local orthonormal frame $\{E_1, \ldots, E_n\}$ on $\mathcal{G}$ satisfying

$$\langle V, E_1\rangle \neq 0 \quad \text{and} \quad \langle V, E_i\rangle = 0, \quad \forall\ i \geq 2.$$

Then, we arrive at x_0 that

$$\tau_i = \tau\gamma'\Lambda_i \tag{4.1}$$

and

$$\begin{aligned}\psi_{ii} =& -\frac{\tau_{ii}}{\tau} + \frac{(\tau_i)^2}{\tau^2} + \gamma''\Lambda_i^2 + \gamma'\Lambda_{ii} \\ =& -\frac{1}{\tau}\left(\sum_k (h_{iik} + \bar{R}_{0iik})\Lambda_k + f'h_{ii} - \tau\sum_k h_{ik}h_{ki}\right) \\ &+ \left((\gamma')^2 + \gamma''\right)\Lambda_i^2 + \gamma'(f' - \tau h_{ii})\end{aligned}$$

in view of

$$\tau_{ii} = \sum_k (h_{iik} + \bar{R}_{0iik})\langle V, E_k\rangle + f'h_{ii} - \tau\sum_k h_{ik}h_{ki}.$$

By (2.7), (2.8) and (4.1), we have at x_0

$$h_{11} = \tau\gamma', \quad h_{1i} = 0, \quad \forall\ i \geq 2. \tag{4.2}$$

Therefore, we can rotate the coordinate system such that $\{E_i\}_{i=1}^n$ are the principal curvature directions of the second fundamental form h_{ij}, i.e. $\kappa_i = h_{ii} = h_i^i = h_{ij}\delta^{ij}$. Since $\Lambda_1 = \langle V, E_1\rangle$, $\Lambda_i = \langle V, E_i\rangle$ for any $i \geq 2$. So, we can get

$$\begin{aligned}G^{ii}\psi_{ii} =& -\frac{f'}{\tau}G^{ii}h_{ii} - \frac{1}{\tau}G^{ii}(h_{ii1} + \bar{R}_{0ii1})\Lambda_1 + G^{ii}h_{ii}^2 \\ &+ \left((\gamma')^2 + \gamma''\right)G^{11}\Lambda_1^2 + \gamma' G^{ii}(f' - \tau h_{ii}).\end{aligned}$$

Noting that

$$G^{ij}h_{ij} = G - \sum_{l=0}^{k-2} t(k-l)\alpha_l G_l = \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t(k-l)\alpha_l G_l$$

and

$$G^{ij}h_{ij1} = \nabla_1\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t\nabla_1\alpha_l G_l,$$

we conclude

$$
\begin{aligned}
G^{ii}\psi_{ii} =& \frac{\Lambda_1}{\tau}\left(-\nabla_1\alpha_{k-1}(u,x,t)+\sum_{l=0}^{k-2} t\nabla_1\alpha_l G_l\right)\\
&+\frac{f'}{\tau}\left(-\alpha_{k-1}(u,x,t)+\sum_{l=0}^{k-2} t(k-l)\alpha_l G_l\right)+G^{ii}h_{ii}^2\\
&-\frac{1}{\tau}G^{ii}\bar{R}_{0ii1}\Lambda_1+\left((\gamma')^2+\gamma''\right)G^{11}\Lambda_1^2+\gamma' G^{ii}(f'-\tau h_{ii})\\
=&\frac{1}{\tau}\left(-\Lambda_1\nabla_1\alpha_{k-1}(u,x,t)-f'\alpha_{k-1}(u,x,t)\right)\\
&+\frac{1}{\tau}\sum_{l=0}^{k-2} tG_l\left(\Lambda_1\nabla_1\alpha_l+f'(k-l)\alpha_l\right)+G^{ii}h_{ii}^2\\
&-\frac{1}{\tau}G^{ii}\bar{R}_{0ii1}\Lambda_1+\left((\gamma')^2+\gamma''\right)G^{11}\Lambda_1^2+\gamma' G^{ii}(f'-\tau h_{ii}).
\end{aligned}
\tag{4.3}
$$

Since $\langle V, E_i\rangle = 0$ for $i = 2, \ldots, n$, we obtain

$$V = \langle V, E_1\rangle E_1 + \langle V, \nu\rangle \nu = \Lambda_1 E_1 + \tau\nu,$$

which results in

$$\Lambda_1\nabla_1\alpha_l(u,x)+(k-l)f'\alpha_l(u,x) = \bar{\nabla}_V\alpha_l(u,x)+(k-l)f'\alpha_l(u,x)-\tau\bar{\nabla}_\nu\alpha_l(u,x).$$

We know from the assumption (1.9) that

$$\left[(k-l)f'\alpha_l(u,x)+\bar{\nabla}_V\alpha_l(u,x)\right] = \left[(k-l)f'\alpha_l(u,x)+f\frac{\partial\alpha_l(u,x)}{\partial u}\right] \le 0.$$

Thus,

$$-\tau\bar{\nabla}_\nu\alpha_l(u,x) \ge \Lambda_1\nabla_1\alpha_l(u,x)+(k-l)f'\alpha_l(u,x) \tag{4.4}$$

and

$$
\begin{aligned}
&(1-t)\left(\varphi' f'+\varphi f''\right)\frac{\sigma_k(e)}{\sigma_{k-1}(e)}-\tau\bar{\nabla}_\nu\alpha_{k-1}(u,x,t)\\
&\qquad\ge \Lambda_1\nabla_1\alpha_{k-1}(u,x,t)+f'\alpha_{k-1}(u,x,t).
\end{aligned}
\tag{4.5}
$$

Taking (4.4) and (4.5) into (4.3), we have at x_0 that

(4.6)

$$\begin{aligned}
0 &\geq G^{ii}\psi_{ii} \\
&\geq G^{ii}h_{ii}^2 + \left((\gamma')^2+\gamma''\right)G^{11}\Lambda_1^2 + \gamma' G^{ii}(f'-\tau h_{ii}) - \frac{1}{\tau}G^{ii}\bar{R}_{0ii1}\Lambda_1 \\
&\quad - \frac{(1-t)}{\tau}\left(\varphi' f' + \varphi f''\right)\frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t\sum_{l=0}^{k-2} G_l \bar{\nabla}_\nu \alpha_l(u,x) + \bar{\nabla}_\nu \alpha_{k-1}(u,x,t) \\
&= G^{ii}\left(h_{ii} - \frac{1}{2}\gamma'\tau\right)^2 + \left((\gamma')^2+\gamma''\right)G^{11}\Lambda_1^2 + G^{ii}\left(\gamma' f' - \frac{1}{4}(\gamma')^2\tau^2\right) \\
&\quad - \frac{1}{\tau}G^{ii}\bar{R}_{0ii1}\Lambda_1 - \frac{(1-t)}{\tau}\left(\varphi' f' + \varphi f''\right)\frac{\sigma_k(e)}{\sigma_{k-1}(e)} \\
&\quad - t\sum_{l=0}^{k-2} G_l \bar{\nabla}_\nu \alpha_l(u,x) + \bar{\nabla}_\nu \alpha_{k-1}(u,x,t).
\end{aligned}$$

According to Remark 1.3, there always exists

$$\varphi' f' + \varphi f'' < 0.$$

Choosing

$$\gamma(t) = -\frac{\alpha}{t}$$

for sufficiently large positive constant α, we have

$$\gamma'(t) = \frac{\alpha}{t^2}, \quad \gamma''(t) = -\frac{2\alpha}{t^3}.$$

Therefore, (4.6) becomes

$$0 \geq G^{ii}\left(\gamma' f' - \frac{1}{4}(\gamma')^2\tau^2\right) - c_1\left(\sum_{l=0}^{k-2}|G_l| + 1\right) - \frac{1}{\tau}G^{ii}\bar{R}_{0ii1}\Lambda_1 \tag{4.7}$$

in view of

$$(\gamma')^2 + \gamma'' \geq 0,$$

where c_1 is a positive constant depending on $|\alpha_l|_{C^1}$. Since $V = \langle V, E_1\rangle E_1 + \langle V, \nu\rangle \nu$, we can find that $V \perp \mathrm{Span}(E_2, \ldots, E_n)$, i.e. V is orthogonal with the subspace spanned by $E_2, \ldots, E_n$. On the other hand, E_1, ν are orthogonal with $\mathrm{Span}(E_2, \ldots, E_n)$. It is possible to choose a suitable coordinate system such that $\bar{E}_1 \perp \mathrm{Span}(E_2, \ldots, E_n)$, which implies that the pairs $\{V, \bar{E}_1\}$ and $\{\nu, E_1\}$ lie in the same plane and

$$\mathrm{Span}(E_2, \ldots, E_n) = \mathrm{Span}(\bar{E}_2, \ldots, \bar{E}_n),$$

where of course $\{\bar{E}_0 = \bar{e}_0, \bar{E}_1, \ldots, \bar{E}_n\}$ is a local orthonormal frame field in $\bar{M}$. Therefore, we can choose $E_2 = \bar{E}_2, \ldots, E_n = \bar{E}_n$, and then vectors ν and

E_1 can be decomposed into

$$\nu = \langle \nu, \bar{e}_0 \rangle \bar{e}_0 + \langle \nu, \bar{E}_1 \rangle \bar{E}_1 = \frac{\tau}{f} \bar{e}_0 + \langle \nu, \bar{E}_1 \rangle \bar{E}_1,$$
$$E_1 = \langle E_1, \bar{e}_0 \rangle \bar{e}_0 + \langle E_1, \bar{E}_1 \rangle \bar{E}_1.$$

By (2.4) and the fact $V = \Lambda_1 E_1 + \tau\nu$, we can obtain

$$\begin{aligned}
\bar{R}_{0ii1} &= \bar{R}(\nu, E_i, E_i, E_1) \\
&= \frac{\tau}{f} \langle E_1, \bar{e}_0 \rangle \bar{R}(\bar{e}_0, \bar{E}_i, \bar{E}_i, \bar{e}_0) + \langle \nu, \bar{E}_1 \rangle \langle E_1, \bar{E}_1 \rangle \bar{R}(\bar{E}_1, \bar{E}_i, \bar{E}_i, \bar{E}_1) \\
&= \frac{\tau}{f} \langle E_1, \bar{e}_0 \rangle \bar{R}(\bar{e}_0, \bar{E}_i, \bar{E}_i, \bar{e}_0) - \tau \frac{\langle \nu, \bar{E}_1 \rangle^2}{\Lambda_1} \bar{R}(\bar{E}_1, \bar{E}_i, \bar{E}_i, \bar{E}_1) \\
&= \tau \left(\frac{1}{f} \langle E_1, \bar{e}_0 \rangle \bar{R}(\bar{e}_0, \bar{E}_i, \bar{E}_i, \bar{e}_0) - \frac{\langle \nu, \bar{E}_1 \rangle^2}{\Lambda_1} \bar{R}(\bar{E}_1, \bar{E}_i, \bar{E}_i, \bar{E}_1) \right),
\end{aligned} \tag{4.8}$$

where the third equality comes from $\langle V, \bar{E}_1 \rangle = 0$. Substituting (4.8) into (4.7) yields

$$0 \geq G^{ii} \left(\gamma' f' - \frac{1}{4} (\gamma')^2 \tau^2 \right) - c_1 \left(\sum_{l=0}^{k-2} |G_l| + 1 \right) - c_2 \sum_i G^{ii}, \tag{4.9}$$

where $c_2 > 0$ depends on the C^0 bound of f and the curvature tensor $\bar{R}$. To continue our proof, we need to estimate G_l for $0 \leq l \leq k-2$. Let $P \in \mathbb{R}$ be a fixed positive number.

(I) If $\frac{\sigma_k}{\sigma_{k-1}} \leq P$, then we get from $\alpha_l \geq c_l$ that

$$|G_l| = \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{1}{\alpha_l} \left(\frac{\sigma_k}{\sigma_{k-1}} + \alpha_l(u, x, t) \right) \leq c_3(P+1),$$

where the constant $c_3 > 0$ depends on c_l, $|\alpha_l|_{C^0}$.

(II) If $\frac{\sigma_k}{\sigma_{k-1}} > P$, then by Lemma 2.5, one has

$$|G_l| = \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{\sigma_l}{\sigma_{l+1}} \cdot \frac{\sigma_{l+1}}{\sigma_{l+2}} \cdot \dots \cdot \frac{\sigma_{k-2}}{\sigma_{k-1}} \leq c_4 \left(\frac{\sigma_{k-1}}{\sigma_k} \right)^{k-1-l} \leq P^{-(k-1-l)},$$

where the positive constant c_4 depends on k.

Hence, $|G_l|$ can be bounded for any $0 \leq l \leq k-2$. By the definition of operator G and a direct computation, we have $\Sigma_i G^{ii} \geq \frac{n-k+1}{k}$, and so we can choose sufficiently large α such that

$$0 \geq G^{ii} \left[\gamma' f' - (\gamma')^2 \tau^2 \right].$$

Thus,

$$\gamma' f' \leq (\gamma')^2 \tau^2,$$

which means

$$\tau \geq c_5$$

for some positive constant c_5 depending on n, k, c_l, $|\alpha_l|_{C^1}$, the C^0 bound of f and the curvature tensor $\bar{R}$. The conclusion of Lemma 4.1 follows directly. □

Remark 4.2. After several careful revisions to the manuscript of this paper, we prefer to number (by subscripts) nearly all the constants in the C^1 and C^2 estimates, and we believe that this way can reveal the relations among constants clearly to readers.

5. C^2 estimates

This section devotes to the C^2 estimates. However, before that, we need to make some preparations. First, we need the following fact.

Lemma 5.1. *Let $\mathcal{G} = \{(u(x), x)\,|x \in M^n\}$ be a $(k-1)$-admissible solution of* (3.1) *and assume that $\alpha_l(u, x) \geq 0$ for $0 \leq l \leq k-1$. Then, we have the following inequality*

$$G^{ij}h_{ijpp} \geq \nabla_p\nabla_p\alpha_{k-1}(u,x,t) + \sum_{l=0}^{k-2}\frac{1}{1+\frac{1}{k+1-l}}\frac{t(\nabla_p\alpha_l)^2}{\alpha_l}G_l - \sum_{l=0}^{k-2}t\nabla_p\nabla_p\alpha_l G_l.$$

Proof. Differentiating (3.1) once, we have

$$\nabla_p\alpha_{k-1}(u,x,t) = G^{ij}h_{ijp} + \sum_{l=0}^{k-2}t\nabla_p\alpha_l G_l.$$

Differentiating (3.1) twice, we obtain

$$\begin{aligned}
&\nabla_p\nabla_p\alpha_{k-1}(u,x,t)\\
&\quad = G^{ij,rs}h_{ijp}h_{rsp} + G^{ij}h_{ijpp} + 2\sum_{l=0}^{k-2}t\nabla_p\alpha_l G_l^{ij}h_{ijp} + \sum_{l=0}^{k-2}t\nabla_p\nabla_p\alpha_l G_l.
\end{aligned}$$

Moreover, since the operator $\left(\frac{\sigma_{k-1}}{\sigma_l}\right)^{\frac{1}{k-1-l}}$ is concave for $0 \leq l \leq k-2$, we have (see also (3.10) in [12])

$$G_l^{ij,rs}h_{ijp}h_{rsp} \leq \left(1+\frac{1}{k-1-l}\right)G_l^{-1}G_l^{ij}G_l^{rs}h_{ijp}h_{rsp}.$$

Thus, in view that G_k is concave in Γ_{k-1}, we have

$$\begin{aligned}
&\nabla_p\nabla_p\alpha_{k-1}(u,x,t)\\
&\leq \sum_{l=0}^{k-2}t\alpha_l G_l^{ij,rs}h_{ijp}h_{rsp} + G^{ij}h_{ijpp} + 2\sum_{l=0}^{k-2}t\nabla_p\alpha_l G_l^{ij}h_{ijp} + \sum_{l=0}^{k-2}t\nabla_p\nabla_p\alpha_l G_l\\
&\leq \sum_{l=0}^{k-2}t\alpha_l G_l^{-1}\left(1+\frac{1}{k-1-l}\right)(G_l^{ij}h_{ijp})^2 + G^{ij}h_{ijpp} + 2\sum_{l=0}^{k-2}t\nabla_p\alpha_l G_l^{ij}h_{ijp}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=0}^{k-2} t\nabla_p\nabla_p\alpha_l G_l \\
=&\frac{k-l}{k-1-l}\sum_{l=0}^{k-2} t\alpha_l G_l^{-1}\left(G_l^{ij}h_{ijp}+\frac{1}{1+\frac{1}{k-1-l}}\frac{\nabla_p\alpha_l}{\alpha_l}G_l\right)^2 \\
&-\sum_{l=0}^{k-2}\frac{1}{1+\frac{1}{k-1-l}}\frac{t(\nabla_p\alpha_l)^2}{\alpha_l}G_l+G^{ij}h_{ijpp}+\sum_{l=0}^{k-2} t\nabla_p\nabla_p\alpha_l G_l \\
\le&-\sum_{l=0}^{k-2}\frac{1}{1+\frac{1}{k-1-l}}\frac{t(\nabla_p\alpha_l)^2}{\alpha_l}G_l+G^{ij}h_{ijpp}+\sum_{l=0}^{k-2} t\nabla_p\nabla_p\alpha_l G_l,
\end{aligned}
$$

which completes the proof of Lemma 5.1. □

We also need the following truth.

LEMMA 5.2. *Let $\mathcal{G}=\{(u(x),x)\,|x\in M^n\}$ be a $(k-1)$-admissible solution of* (3.1) *with the position vector V in $\bar{M}$. We have the following equality*

$$
\begin{aligned}
&G^{ij}\tau_{ij}+\sum_k \tau G^{ij}h_{ik}h_{kj} \\
&\quad=\left(\nabla_p\alpha_{k-1}(u,x,t)-\sum_{l=0}^{k-2} t\nabla_p\alpha_l G_l+\sum_p G^{ij}\bar{R}_{0ijp}\right)\langle V,E_p\rangle \\
&\qquad+f'\left(\alpha_{k-1}(u,x,t)-\sum_{l=0}^{k-2}(k-l)t\alpha_l G_l\right).
\end{aligned}
$$

PROOF. By Lemma 2.4, we have

$$
\tau_{ij}=-\tau\sum_k h_{ik}h_{kj}+f'h_{ij}+\sum_k(h_{ijk}+\bar{R}_{0ijk})\langle V,E_k\rangle,
$$

which results in

$$
G^{ij}\tau_{ij}=-\tau G^{ij}\sum_k h_{ik}h_{kj}+f'G^{ij}h_{ij}+\sum_k G^{ij}(h_{ijk}+\bar{R}_{0ijk})\langle V,E_k\rangle.
$$

Note that

$$
G^{ij}h_{ij}=G-\sum_{l=0}^{k-2} t(k-l)\alpha_l G_l=\alpha_{k-1}(u,x,t)-\sum_{l=0}^{k-2} t(k-l)\alpha_l G_l
$$

and

$$
G^{ij}h_{ijp}=\nabla_p\alpha_{k-1}(u,x,t)-\sum_{l=0}^{k-2} t\nabla_p\alpha_l G_l.
$$

Thus,

$$G^{ij}\tau_{ij} = \sum_p \left(\nabla_p \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t\nabla_p \alpha_l G_l + G^{ij}\bar{R}_{0ijp} \right) \langle V, E_p \rangle \\ + f' \left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)t\alpha_l G_l \right) - \sum_k \tau G^{ij} h_{ik} h_{kj}.$$

Therefore, we complete the proof. □

Now we begin to estimate the second fundamental form.

Lemma 5.3 (C^2 estimates). *Assume that $k \geq 2$ and*

$$\alpha_l(u,x) \geq c_l > 0, \qquad \forall x \in M^n$$

for $0 \leq l \leq k-1$. If the k-admissible, closed graphic hypersurface $\mathcal{G} = \{(u(x),x) \,|\, x \in M^n\}$ satisfies (3.1) with the position vector V in $\bar{M}$, then there exists a constant C depending on n, k, c_l, $|\alpha_l|_{C^1}$, $|\alpha_l|_{C^2}$, $|\nabla u|_{C^0}$, the C^0, C^1 bounds of f and the curvature tensor $\bar{R}$ such that for $1 \leq i \leq n$, the principal curvatures of $\mathcal{G}$ at V satisfy

$$|\lambda_i(V)| \leq C, \qquad \forall x \in M^n.$$

Proof. Since $k \geq 2$, $\mathcal{G}$ is 2-admissible, for sufficiently large c_6, one has

$$|\lambda_i| \leq c_6 H,$$

where the positive constant c_6 depends on n, k. So, we only need to estimate the mean curvature H of $\mathcal{G}$. Taking the auxiliary function

$$W(x) = \log H - \log \tau.$$

Assume that x_0 is the maximum point of W. Then at x_0, one has

$$0 = W_i = \frac{H_i}{H} - \frac{\tau_i}{\tau} \tag{5.1}$$

and

$$0 \geq W_{ij}(x_0) = \frac{H_{ij}}{H} - \frac{\tau_{ij}}{\tau}. \tag{5.2}$$

Choosing a suitable coordinate system $\{x^1, x^2, \ldots, x^n\}$ in the neighborhood of $X_0 = (u(x_0), x_0) \in \mathcal{G}$ such that the matrix $(h_{ij})_{n\times n}$ is diagonal at X_0, i.e., $h_{ij} = h_{ii}\delta_{ij}$. This implies at x_0,

$$0 \geq G^{ij}W_{ij}(x_0) = \sum_{p=1}^{n} \frac{1}{H} G^{ii} h_{ppii} - \frac{G^{ii}\tau_{ii}}{\tau}. \tag{5.3}$$

By (2.3), we can obtain

$$h_{ppii} = h_{iipp} + h_{pp}^2 h_{ii} - h_{ii}^2 h_{pp} + \bar{R}_{0iip;p} + \bar{R}_{0pip;i} - 2h_{pp}\bar{R}_{piip} \\ + h_{ii}\bar{R}_{0i0i} + h_{pp}\bar{R}_{0ii0} + h_{ii}\bar{R}_{0p0p} + h_{ii}\bar{R}_{0ii0} - 2h_{ii}\bar{R}_{ipip}.$$

Note that

$$G^{ij}h_{ij} = G - \sum_{l=0}^{k-2}(k-l)\alpha_l G_l = \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)\alpha_l G_l.$$

So, we have

$$\begin{aligned}
&\sum_p G^{ii}h_{ppii}\\
&= \sum_p G^{ii}\left(h_{iipp} + \bar{R}_{0iip;p} + \bar{R}_{0pip;i}\right) - \sum_p h_{pp}G^{ii}\left(h_{ii}^2 + 2\bar{R}_{piip} - \bar{R}_{0ii0}\right)\\
&+ \sum_p G^{ii}h_{ii}\left(h_{pp}^2 - 2\bar{R}_{ipip} + \bar{R}_{0i0i} + \bar{R}_{0p0p} + \bar{R}_{0ii0}\right)\\
&\geq \sum_p G^{ii}h_{iipp} + \left(|A|^2 - c_8\right)\left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)\alpha_l G_l\right) - c_7\sum_i G^{ii}\\
&- HG^{ii}\left(h_{ii}^2 + c_9\right),
\end{aligned}$$

where the positive constant c_7 depends on the C^1 bound of the curvature tensor $\bar{R}$, the positive constants c_8, c_9 depend on the C^0 bound of the curvature tensor $\bar{R}$. Together with Lemma 5.1, we know that (5.3) becomes

$$\begin{aligned}
0 \geq &\frac{1}{H}\sum_{p=1}^n G^{ii}h_{iipp} + \frac{|A|^2 - c_8}{H}\left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)\alpha_l G_l\right) - \frac{G^{ii}\tau_{ii}}{\tau}\\
&- \frac{c_7}{H}\sum_i G^{ii} - G^{ii}(h_{ii}^2 + c_9)\\
\geq &\frac{1}{H}\sum_{p=1}^n\left(\nabla_p\nabla_p\alpha_{k-1}(u,x,t) + \sum_{l=0}^{k-2}\frac{1}{1+\frac{1}{k+1-l}}\frac{t(\nabla_p\alpha_l)^2}{\alpha_l}G_l - \sum_{l=0}^{k-2}t\nabla_p\nabla_p\alpha_l G_l\right)\\
&- \frac{G^{ii}\tau_{ii}}{\tau} + \frac{|A|^2 - c_8}{H}\left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)t\alpha_l G_l\right)\\
&- \frac{c_7}{H}\sum_i G^{ii} - G^{ii}(h_{ii}^2 + c_9).
\end{aligned}$$

By Lemma 5.2, the above inequality becomes

$$\begin{aligned}
0 \geq &\frac{1}{H}\sum_{p=1}^n\left(\nabla_p\nabla_p\alpha_{k-1}(u,x,t) + \sum_{l=0}^{k-2}\frac{1}{1+\frac{1}{k+1-l}}\frac{t(\nabla_p\alpha_l)^2}{\alpha_l}G_l - \sum_{l=0}^{k-2}t\nabla_p\nabla_p\alpha_l G_l\right)\\
&+ \frac{|A|^2 - c_8}{H}\left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)t\alpha_l G_l\right) - \frac{c_7}{H}\sum_i G^{ii} - G^{ii}(h_{ii}^2 + c_9)
\end{aligned}$$

$$- \frac{1}{\tau}\left(\nabla_p \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t\nabla_p \alpha_l G_l + \sum_p G^{ii}\bar{R}_{0iip}\right)\langle V, E_p\rangle$$
$$- \frac{f'}{\tau}\left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)t\alpha_l G_l\right) + G^{ii}h_{ii}^2.$$

Hence, we have

$$\begin{aligned}
0 \geq & \frac{|A|^2}{H}\left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)t\alpha_l G_l\right) - \left(\frac{c_7}{H} + c_9\right)\sum_i G^{ii} \\
& - \frac{\langle V, E_p\rangle}{\tau}\sum_p G^{ii}\bar{R}_{0iip} \\
& + \frac{1}{H}\sum_{p=1}^{n}\left(\nabla_p\nabla_p\alpha_{k-1}(u,x,t) + \sum_{l=0}^{k-2}\frac{1}{1+\frac{1}{k+1-l}}\frac{t(\nabla_p\alpha_l)^2}{\alpha_l}G_l\right. \\
& \qquad \left. - \sum_{l=0}^{k-2} t\nabla_p\nabla_p\alpha_l G_l\right) \\
& - \frac{\langle V, E_p\rangle}{\tau}\left(\nabla_p\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t\nabla_p\alpha_l G_l\right) \\
& - \left(\frac{c_8}{H} + \frac{f'}{\tau}\right)\left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)t\alpha_l G_l\right).
\end{aligned}$$

A direction calculation implies

$$|\nabla_p\alpha_{k-1}(u,x,t)| \leq c_{10}, \quad |\nabla_p\nabla_p\alpha_{k-1}(u,x,t)| \leq c_{11}(1+H), \tag{5.4}$$

where the positive constant c_{10} depends on $|\alpha_l|_{C^1}$, and the positive constant c_{11} depends on $|\alpha_l|_{C^2}$. So,

$$\begin{aligned}
& -\frac{1}{H}c_{12}\left(\sum_{l=0}^{k-2}|G_l| + 1\right)(H+1) - c_{13}\left(\sum_{l=0}^{k-2}|G_l| + 1\right) \\
\leq & \frac{1}{H}\sum_{p=1}^{n}\left(\nabla_p\nabla_p\alpha_{k-1}(u,x,t) + \sum_{l=0}^{k-2}\frac{1}{1+\frac{1}{k+1-l}}\frac{t(\nabla_p\alpha_l)^2}{\alpha_l}G_l\right. \\
& \qquad \left. - \sum_{l=0}^{k-2} t\nabla_p\nabla_p\alpha_l G_l\right) \\
& - \frac{\langle V, E_p\rangle}{\tau}\left(\nabla_p\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t\nabla_p\alpha_l G_l\right)
\end{aligned} \tag{5.5}$$

$$- \left(\frac{c_8}{H} + \frac{f'}{\tau}\right) \left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)t\alpha_l G_l\right)$$

holds, where the positive constant c_{12} depends on c_8, c_{10}, n, k, c_l, $|\alpha_l|_{C^1}$ and the C^1 bound of f, and the positive constant c_{13} depends on c_8, c_{11}, the C^1 bound of f. Then, together with the fact $|A|^2 \geq \frac{1}{n}H^2$, we have

$$\begin{aligned}
&\frac{1}{n}H\alpha_{k-1}(u,x,t) - \left(\frac{c_7}{H} + c_{14}\right)\sum_i G^{ii} \\
&\leq \frac{|A|^2}{H}\left(\alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2}(k-l)t\alpha_l G_l\right) \\
&\quad - \left(\frac{c_7}{H} + c_9\right)\sum_i G^{ii} - \frac{\langle V, E_p\rangle}{\tau}\sum_p G^{ii}\bar{R}_{0iip},
\end{aligned} \tag{5.6}$$

where the positive constant c_{14} depends on c_9, the C^0 bound of the curvature tensor $\bar{R}$. From [12, page 11-12], we have

$$\begin{aligned}
\sum_{i=1}^n G^{ii} &= (n-k+1) - (n-k+2)\frac{\sigma_k\sigma_{k-2}}{\sigma_{k-1}^2} + (n-k+2)\alpha_0\frac{\sigma_{k-2}}{\sigma_{k-1}^2} \\
&\quad + \sum_{l=1}^{k-2}\alpha_l\frac{(n-k+2)\sigma_l\sigma_{k-2} - (n-l+1)\sigma_{k-1}\sigma_{l-1}}{\sigma_{k-1}^2}.
\end{aligned}$$

Since graphic hypersurface $\mathcal{G}$ is k-admissible and $\alpha_l > 0$ for all $0 \leq l \leq k-1$, thus

$$\begin{aligned}
\sum_{i=1}^n G^{ii} &\leq (n-k+1) + (n-k+2)\alpha_0\frac{\sigma_{k-2}}{\sigma_{k-1}^2} + \sum_{l=1}^{k-2}\alpha_l\frac{(n-k+2)\sigma_l\sigma_{k-2}}{\sigma_{k-1}^2} \\
&= (n-k+1) + \sum_{l=0}^{k-2}\alpha_l\frac{(n-k+2)\sigma_l\sigma_{k-2}}{\sigma_{k-1}^2}.
\end{aligned}$$

Together with Lemma 4.1, we have

$$\begin{aligned}
\Sigma_i G^{ii} &\leq (n-k+1) + (n-k+2)(k-1)\sum_{l=0}^{k-2}\alpha_l|G_l||G_{k-2}| \\
&\leq (n-k+1) + (n-k+2)(k-1)\sup|\alpha_l|\sup|G_l|^2.
\end{aligned}$$

Combining inequalities (5.5) and (5.6) with the fact that $\Sigma_i G^{ii}$ has positive upper bound estimate, we have

$$0 \geq \frac{1}{n} H\alpha_{k-1}(u,x,t) - \left(\frac{c_7}{H} + c_{14}\right) - \frac{1}{H} c_{12} \left(\sum_{l=0}^{k-2} |G_l| + 1\right)(H+1) - c_{13} \left(\sum_{l=0}^{k-2} |G_l| + 1\right).$$

Let us divide the rest of the proof into two cases.

Case I. If $\frac{\sigma_k}{\sigma_{k-1}} \leq H^{\frac{1}{k}}$, then we get from $\alpha_l \geq c_l$ that

$$|G_l| = \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{1}{\alpha_l}\left(\frac{\sigma_k}{\sigma_{k-1}} + \alpha_l(u,x,t)\right) \leq c_{15}(H^{\frac{1}{k}} + 1),$$

where the positive constant c_{15} depends on c_l, $|\alpha_l|_{C^0}$. Thus, we have a contradiction when H is large enough, which implies $H \leq C$.

Case II. If $\frac{\sigma_k}{\sigma_{k-1}} > H^{\frac{1}{k}}$, then by Lemma 2.5, one has

$$|G_l| = \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{\sigma_l}{\sigma_{l+1}} \cdot \frac{\sigma_{l+1}}{\sigma_{l+2}} \cdots \frac{\sigma_{k-2}}{\sigma_{k-1}} \leq c_{16}\left(\frac{\sigma_{k-1}}{\sigma_k}\right)^{k-1-l} \leq H^{-\frac{k-1-l}{k}},$$

where the constant $c_{16} > 0$ depends on k. In this case, we can also derive $H \leq C$ easily.

In sum, the conclusion of Lemma 5.3 follows directly by using the fact $|\lambda_i| \leq c_6 H$. □

6. Existence

In this section, we use the degree theory for nonlinear elliptic equation developed in [16] to prove Theorem 1.2.

After establishing a priori estimates (see Lemmas 3.1, 4.1 and 5.3), we know that (3.1) is uniformly elliptic. By [5], [14] and Schauder estimates, we have

$$|u|_{C^{4,\alpha}(M^n)} \leq C \tag{6.1}$$

for any k-convex solution $\mathcal{G}$ to the equation (3.1). Define

$$C_0^{4,\alpha}(M^n) = \{u \in C^{4,\alpha}(M^n) : \mathcal{G} = \{(u(x), x)\,|x \in M^n\} \text{ is } k\text{−convex}\}.$$

Let us consider the function

$$F(\cdot;t) : C_0^{4,\alpha}(M^n) \to C^{2,\alpha}(M^n),$$

which is defined by

$$F(u,x,t) = \frac{\sigma_k(\kappa(V))}{\sigma_{k-1}(\kappa(V))} - \sum_{l=0}^{k-2} t\alpha_l(u,x)\frac{\sigma_l(\kappa(V))}{\sigma_{k-1}(\kappa(V))} - \alpha_{k-1}(u,x,t).$$

Set

$$\mathcal{O}_R = \{u \in C_0^{4,\alpha}(M^n) : |u|_{C^{4,\alpha}(M^n)} < R\},$$

which clearly is an open set in $C_0^{4,\alpha}(M^n)$. Moreover, if R is sufficiently large, $F(u, x, t) = 0$ does not have solution on $\partial\mathcal{O}_R$ by the priori estimate established in (6.1). Therefore, the degree $\deg(F(\cdot;t), \mathcal{O}_R, 0)$ is well-defined for $0 \le t \le 1$. Using the homotopic invariance of the degree, we have

$$\deg(F(\cdot;1), \mathcal{O}_R, 0) = \deg(F(\cdot;0), \mathcal{O}_R, 0).$$

Lemma 3.2 shows that $u = u_0$ is the unique solution to the above equation for $t = 0$. By direct calculation, one has

$$F(su_0, x; 0) = [1 - \varphi(su_0)]\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'(su_0)}{f(su_0)}.$$

Using the fact $\varphi(u_0) = 1$, we have

$$\begin{aligned}
\delta_{u_0} F(u_0, x; 0) &= \frac{d}{ds}\bigg|_{s=1} F(su_0, x; 0) \\
&= -\varphi'(u_0)\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'(u_0)}{f(u_0)}u_0 \\
&\quad + [1 - \varphi(u_0)]\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f''(u_0)f(u_0) - (f'(u_0))^2}{f(u_0)}u_0 \\
&= -\varphi'(u_0)\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'(u_0)}{f(u_0)}u_0 > 0,
\end{aligned}$$

where $\delta F(u_0, x; 0)$ is the linearized operator of F at u_0. Clearly, $\delta F(u_0, x; 0)$ has the form

$$\begin{aligned}
\delta_\omega F(u_0, x; 0) &= \frac{d}{ds}\bigg|_{s=0} F(u_0 + sw, x; 0) \\
&= -a^{ij}\omega_{ij} + b^i\omega_i - \varphi'(u_0)\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'(u_0)}{f(u_0)}\omega,
\end{aligned}$$

where $(a^{ij})_{n\times n}$ is a positive definite matrix. Since $-\varphi'(u_0)\frac{\sigma_k(e)}{\sigma_{k-1}(e)}\frac{f'(u_0)}{f(u_0)} > 0$, then $\delta F(u_0, x; 0)$ is an invertible operator. Therefore,

$$\deg(F(\cdot;1), \mathcal{O}_R, 0) = \deg(F(\cdot;0), \mathcal{O}_R, 0) = \pm 1,$$

which implies that we can obtain a solution at $t = 1$. This finishes the proof of Theorem 1.2.

Acknowledgements.

This work is partially supported by the NSF of China (Grant Nos. 11801496 and 11926352), the Fok Ying-Tung Education Foundation (China), Hubei Key Laboratory of Applied Mathematics (Hubei University), and Key Laboratory of Intelligent Sensing System and Security (Hubei University), Ministry of Education. The authors sincerely thank Mr. Agen Shang and Prof. Qiang Tu for sending the digital version of the reference [28] to them. The authors are grateful to the anonymous referee for careful reading and valuable comments such that the paper appears as its present version.

References

[1] F. Andrade, J. Barbosa and J. de Lira, *Closed Weingarten hypersurfaces in warped product manifolds*, Indiana Univ. Math. J. **58** (2009), 1691–1718.

[2] L. Chen, A. G. Shang and Q. Tu, *A class of prescribed Weingarten curvature equations in Euclidean space*, Comm. Partial Differential Equations **46** (2021), 1326–1343.

[3] D. G. Chen, H. Z. Li and Z. Z. Wang, *Starshaped compact hypersurfaces with prescribed Weingarten curvature in warped product manifolds*, Calc. Var. Partial Differential Equations **57** (2018), Paper No. 42.

[4] X. J. Chen, Q. Tu and N. Xiang, *k-convex hypersurfaces with prescribed Weingarten curvature in warped product manifold*, Nonlinear Anal. **249** (2024), Article No. 113640.

[5] L. C. Evans, *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math. **35** (1982), 333–363.

[6] P. Freitas, J. Mao and I. Salavessa, *Spherical symmetrization and the first eigenvalue of geodesic disks on manifolds*, Calc. Var. Partial Differential Equations **51** (2014), 701–724.

[7] J. X. Fu and S. T. Yau, *A Monge-Ampère type equation motivated by string theory*, Comm. Anal. Geom. **15** (2007), 29–75.

[8] J. X. Fu and S. T. Yau, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation*, J. Differential Geom. **78** (2008), 369–428.

[9] P. F. Guan and J. Li, *A mean curvature type flow in space forms*, Int. Math. Res. Not. IMRN 2015 (2015), 4716–4740.

[10] P. F. Guan and X. N. Ma, *The Christoffel-Minkowski problem. I. Convexity of solutions of a Hessian equation*, Invent. Math. **151** (2003), 553–577.

[11] P. F. Guan, C. Y. Ren and Z. Z. Wang, *Global C^2-estimates for convex solutions of curvature equations*, Comm. Pure Appl. Math. **68** (2015), 1287–1325.

[12] P. F. Guan and X. W. Zhang, *A class of curvature type equations*, Pure Appl. Math. Q. **17** (2021), 865–907.

[13] N. N. Katz and K. Kondo, *Generalized space forms*, Trans. Amer. Math. Soc. **354** (2002), 2279–2284.

[14] N. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations in a domain*, Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), 75–108.

[15] N. Krylov, *On the general notion of fully nonlinear second order elliptic equation*, Trans. Amer. Math. Soc. **347** (1995), 857–895.

[16] Y. Y. Li, *Degree theory for second order nonlinear elliptic operators and its applications*, Comm. Partial Differential Equations **14** (1989), 1541–1578.

[17] M. Lin and N. S. Trudinger, *On some inequalities for elementary symmetric functions*, Bull. Aust. Math. Soc. **50** (1994), 317–326.

[18] W. Lu, J. Mao, C. X. Wu and L. Z. Zeng, *Eigenvalue estimates for the drifting Laplacian and the p-Laplacian on submanifolds of warped products*, Appl. Anal. **100** (2021), 2275–2300.

[19] J. Mao, *Eigenvalue inequalities for the p-Laplacian on a Riemannian manifold and estimates for the heat kernel*, J. Math. Pures Appl. (9) **101** (2014), 372–393.

[20] J. Mao, *Volume comparisons for manifolds with radial curvature bounded*, Czech. Math. J. **66(141)** (2016), 71–86.

[21] J. Mao, F. Du and C. X. Wu, Eigenvalue problems on manifolds, Science Press, Beijing, 2017.
[22] J. Mao, *Geometry and topology of manifolds with integral radial curvature bounds*, Differ. Geom. Appl. **91** (2023), Article No. 102064.
[23] B. O'Neill, Semi-Riemannian geometry. With applications to relativity, Academic Press, New York, 1983.
[24] P. Petersen, Riemannian geometry, Springer, New York, 2006.
[25] D. H. Phong, S. Picard and X. W. Zhang, *The Fu-Yau equation with negative slope parameter*, Invent. Math. **209** (2017), 541–576.
[26] D. H. Phong, S. Picard and X. W. Zhang, *On estimates for the Fu-Yau generalization of a Strominger system*, J. Reine Angew. Math. **751** (2019), 243–274.
[27] D. H. Phong, S. Picard and X. W. Zhang, *Fu-Yau Hessian equations*, J. Differential Geom. **118** (2021), 147–187.
[28] A. G. Shang and Q. Tu, *A class of fully nonlinear equations with linear combination of σ_k-curvature in hyperbolic space*, preprint.
[29] N. S. Trudinger, *The Dirichlet problem for the prescribed curvature equations*, Arch. Ration. Mech. Anal. **111** (1990), 153–179.
[30] Y. Zhao, C. X. Wu, J. Mao and F. Du, *Eigenvalue comparisons in Steklov eigenvalue problem and some other eigenvalue estimates*, Rev. Mat. Complut. **33** (2020), 389–414.

Y. Gao
School of Mathematical Science and Academy for Multidisciplinary Studies
Capital Normal University
Beijing 100048, China
Faculty of Mathematics and Statistics
Key Laboratory of Applied Mathematics of Hubei Province
Hubei University
Wuhan 430062, China
E-mail: `Echo-gaoya@outlook.com`

C. Liu
Faculty of Mathematics and Statistics
Key Laboratory of Applied Mathematics of Hubei Province
Hubei University
Wuhan 430062, China
E-mail: `1109452431@qq.com`

J. Mao
Faculty of Mathematics and Statistics
Key Laboratory of Applied Mathematics of Hubei Province
Hubei University
Wuhan 430062, China
Key Laboratory of Intelligent Sensing System and Security (Hubei University)
Ministry of Education
China
E-mail: `jiner120@163.com`

Received: 15.10.2024.
Revised: 17.6.2025.

ZADANE JEDNADŽBE WEINGARTENOVE ZAKRIVLJENOSTI U PRODUKTNIM MNOGOSTRUKOSTIMA ZAKRIVLJENE METRIKE

Y. Gao, C. Liu and J. Mao

Sažetak. U ovom radu, pod odgovarajućim uvjetima, možemo dokazati egzistenciju rješenja za određenu klasu zadanih jednadžbi Weingartenove zakrivljenosti u produktnim mnogostrukostima zakrivljene metrike posebnog tipa, koristeći standardnu teoriju stupnja temeljenu na apriornim ocjenama za rješenja. Drugim riječima, u danoj produktnoj mnogostrukosti zakrivljene metrike posebnog tipa, može se osigurati postojanje zatvorene hiperplohe (koja je graf u odnosu na baznu mnogostrukost i čija k-Weingartenova zakrivljenost zadovoljava određene uvjete).