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PERMUTATION TEST OF INDEPENDENCE IN TAILS FOR DEPENDENT PROCESSES

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Abstract. In this article, we propose a permutation test for independence in the tails of two strongly mixing and strictly stationary sequences. We establish the asymptotic validity of the test by demonstrating that both the test statistic and its permutation distribution are asymptotically normal. These results build upon and generalize findings from Basrak and Brborović [1]. Additionally, we conduct a simulation study to evaluate the size and power properties of the proposed test.

1. Introduction

In this article, we present a permutation test of independence in tails for a sequence of bivariate random vectors $X_i = (Y_i, Z_i)$, $i \in \mathbb{N}$, which is strictly stationary and strongly mixing. We assume Y_i and Z_i are non-negative for all $i \in \mathbb{N}$. Our test statistic is based on the number of joint upcrossings of the sequences (Y_i) and (Z_i) over high thresholds, i.e.

$$\sum_{i=1}^{n} I_{\{Y_i > u'_n, Z_i > v'_n\}}, \tag{1.1}$$

for some suitable increasing sequences (u'_n) and (v'_n).

The permutation test of independence for i.i.d. (independent and identically distributed) X_i and based on the sample correlation statistic is a well-known test that was generalized by DiCiccio and Romano [8]. In [8], the authors extended the validity of the testing procedure under the null hypothesis of uncorrelated samples by applying suitable studentization to the test statistic. More details on the studentization procedure can be found in Chung

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and Romano [5] and the references therein. Subsequently, Romano and Tirlea [21] further generalized the permutation testing procedure for strictly stationary and strongly mixing sequences under the null hypothesis that first-order or higher-order autocorrelations are equal to zero. Additional results on permutation testing for time series can be found in Tirlea [23].

Despite the usefulness of correlations as a measure of dependence between two distributions, it has been noted in various references (see, for example, Embrechts, McNeil, Straumann [9] or de la Pena, Ibragimov, Sharakhmetov [7]) that correlation is an insufficient measure of dependence when dealing with distributions that are not elliptic. Furthermore, correlation may not be applicable to some heavy-tailed distributions, as their moments, even the first moments, may be unbounded. On the other hand, many time series exhibit heavy-tailed behavior (see Cont [6] for a discussion related to financial applications).

Inspired by such observations, the authors in [1] propose a permutation test of independence for the case where X_i are i.i.d. and based on a test statistic that focuses on the tails of the distributions of Y_i and Z_i, as in (1.1). By applying similar ideas to those in [8], but in the context of triangular arrays, the authors in [1] extend the proposed permutation test procedure to test for tail dependence by using appropriate studentization of the test statistic. An illustration of the application of the proposed permutation test to financial data can also be found in [1].

In this article, we present a generalization of the permutation test of independence given in [1] (see Remark 2 in [1]) to the case where the sequence (X_i) is strictly stationary and strongly mixing. Precise definitions, along with further details are provided in the next section. We note that a generalization of the test of independence for m-dependent processes Y_i and Z_i is presented in Brborović [2]. The proofs of the main results in [2] were rather technical and involved lengthy arguments, while the results presented here, especially Theorem 2.5, are more in the spirit of the proof of the main Theorem 1 in [1]. Please note that we assume the marginal distributions of the sequences of random variables (Y_i) and (Z_i) are known; thus, we do not incorporate the estimation of these marginal distributions into our testing procedure.

We will work under the null hypothesis

$$H_0 : Y_i \textit{ and } Z_j \textit{ are independent above some threshold level } u_0 > 0,\ i, j \in \mathbb{N}.$$

Clearly, if sequences (Y_i) and (Z_i) are independent the null hypothesis H_0 is valid. Therefore, the permutation test we are proposing may be used as a test of independence. In order to justify the application of the proposed permutation test, we prove that under the null hypothesis H_0 both the test statistic T_n given in (2.10) and its permutation distribution $\hat{R}_n$ asymptotically follow the standard normal distribution. To define the permutation distribution of the test statistic, let $X^n = (X_1, \ldots, X_n)$, $Y^n = (Y_1, \ldots, Y_n)$, $Z^n = (Z_1, \ldots, Z_n)$,

$n \in \mathbb{N}$. Denote the finite group of permutations of the set $\{1, 2, \ldots, n\}$ by $\mathbf{G}_n$. The group action of $\mathbf{G}_n$ on $\mathbb{R}^{2n}$ is defined by the action of an element $\pi \in \mathbf{G}_n$ as

$$\pi((y^1, z^1), \ldots (y^n, z^n)) = ((y^1, z^{\pi(1)}), \ldots (y^n, z^{\pi(n)})), \tag{1.2}$$

where $((y^1, z^1), \ldots (y^n, z^n)) \in \mathbb{R}^{2n}$. The permutation distribution of the statistic T_n is defined as

$$\hat{R}_n(t) = \frac{1}{n!} \sum_{\pi \in \mathbf{G}_n} I_{\{T_n(Y^n, Z^n_\pi) \leq t\}}, \ t \in \mathbb{R}, \tag{1.3}$$

where $Z^n_\pi = (Z_{\pi(1)}, \ldots, Z_{\pi(n)})$. Its $(1-\alpha)$ quantile is defined as

$$\hat{r}(1-\alpha) = \hat{R}_n^{-1}(1-\alpha) = \inf\{t : \hat{R}_n(t) \geq 1-\alpha\}.$$

The permutation test rejects the null hypothesis if the value of the statistic T_n is greater than $\hat{r}(1-\alpha)$.

The asymptotic results we prove in this article allow us to construct a permutation test that uses the permutation distribution as the null distribution. To perform the test, we calculate the test statistic T_n for multiple permutations of the vector $(Z_1, \ldots, Z_n)$ and then reject the null hypothesis whenever the original test statistic exceeds a predetermined quantile of the empirical permutation distribution. A standard reference for permutation tests is the book by Lehmann and Romano [17], Section 15.2. For a brief overview of permutation tests, you can also refer to the Appendix in [1].

The article is organized into four sections. In the next section, we present our main theoretical results supporting the construction of the permutation test of independence. Section 3 presents a simulation study concerning the suggested test's power and in Section 4 we present proofs of the theoretical results from Section 2.

2. Main results

Let $X_i = (Y_i, Z_i)$, $i \in \mathbb{N}$, be a sequence of strictly stationary, strongly mixing, and non-negative bivariate random vectors defined on a probability space $(\Omega, \mathcal{F}, P)$. We use Rosenblatt's α-mixing coefficient to define the strong mixing property. For $n \in \mathbb{N}$ and sequence $(X_i)_{i\in\mathbb{N}}$ we define σ-algebras

$$\mathcal{E}_n = \sigma(X_k \ : \ k \leq n), \quad \mathcal{F}_n = \sigma(X_k \ : \ k \geq n).$$

The α-mixing coefficient is given by

$$\alpha_X(n) = \sup_{k \in \mathbb{N}} \sup_{A \in \mathcal{E}_k, B \in \mathcal{F}_{k+n}} |P(A \cap B) - P(A)P(B)|, \ n \in \mathbb{N}.$$

The sequence $(X_i)_{i\in\mathbb{N}}$ is said to be strongly mixing if $\alpha_X(n) \to 0$, $n \to \infty$. We will further assume:

$$\sum_{n=1}^{\infty} \alpha_X(n) < \infty. \tag{2.1}$$

REMARK 2.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ (or $\mathbb{R}$) be a Borel measurable function and let $V_i = f(X_i) = f(Y_i, Z_i)$. Then, the sequence $(V_i)_{i\in\mathbb{N}}$ is also strongly mixing, with

$$\alpha_V(n) \le \alpha_X(n),\ n \in \mathbb{N}.$$

To understand why, see Remarks 1.4 (III), 1.8, and 1.9 in Bradley [4]. The argument is based on the fact that, for example, $\sigma(V_k : k \le n) \subset \sigma(X_k : k \le n)$, $n \in \mathbb{N}$. Consequently, the sequences $(Y_i)_{i\in\mathbb{N}}$ and $(Z_i)_{i\in\mathbb{N}}$ are also strongly mixing, with

$$\alpha_Y(n) \le \alpha_X(n), \quad \alpha_Z(n) \le \alpha_X(n), \quad n \in \mathbb{N}.$$

Thus, for them the analogue of (2.1) also holds. Similarly, for $g : \mathbb{R} \to \mathbb{R}$ Borel measurable and $W_i = g(Y_i)$, it follows that the sequence $(W_i)_{i\in\mathbb{N}}$ is also strongly mixing, with $\alpha_W(n) \le \alpha_Y(n)$, $n \in \mathbb{N}$. □

Due to strict stationarity, all Y_i, $i \in \mathbb{N}$, and all Z_i, $i \in \mathbb{N}$, have the same distribution. Let $(m_n)_{n\in\mathbb{N}}$ be an intermediate sequence of integers such that $m_n \to \infty$ as $n \to \infty$ and

$$m_n = O\big(n^{1-\tau}\big), \tag{2.2}$$

for some $0 < \tau < 1$. Condition (2.2) implies that $m_n/n \to 0$, as $n \to \infty$.

Denote by F_Y and F_Z the distribution functions of Y_1 and Z_1, respectively. Suppose that there exist two sequences (u_n) and (v_n) of positive real numbers such that $u_n \to \sup\{x : F_Y(x) < 1\}$, $v_n \to \sup\{x : F_Z(x) < 1\}$ and

$$nP(Y_1 > u_n) \to 1, \quad nP(Z_1 > v_n) \to 1,\ n \to \infty. \tag{2.3}$$

Note that the existence of such sequences (u_n) and (v_n) is immediate for continuous random variables (see p. 430 in [17]). The same is true for regularly varying sequences (see Theorem 3.6. in Resnick [20])

Let $I_{Y,i} = I_{\{Y_i > u_{\sqrt{m_n}}\}}$ and $I_{Z,i} = I_{\{Z_i > v_{\sqrt{m_n}}\}}$. Note that $\sqrt{m_n}$ may not be an integer, but in that case, we use $\lfloor\sqrt{m_n}\rfloor$ as an index in $u_{\sqrt{m_n}}$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$ (floor). For notational simplicity, we omit the floor notation in the rest of the text.

Additionally, let $p_Y = P(Y_i > u_{\sqrt{m_n}})$ and $p_Z = P(Z_i > v_{\sqrt{m_n}})$. Note that $I_{Y,i}$, $I_{Z,i}$, p_Y and p_Z depend on n. From (2.3) we have

$$p_Y = p_Y(n) \sim \frac{1}{\sqrt{m_n}}, \quad p_Z = p_Z(n) \sim \frac{1}{\sqrt{m_n}}. \tag{2.4}$$

where $\sim$ denotes asymptotic equivalence.

REMARK 2.2. Our null hypothesis H_0, presented in Introduction, requires that all random events $\{Y_i > u\}$ and $\{Z_j > v\}$ are independent for $u, v > u_0$ and for all $i, j \in \mathbb{N}$. Therefore, the sequences (u_n) and (v_n) should be chosen such that $u_n, v_n > u_0$, $n \in \mathbb{N}$. This condition may be restrictive in some cases under assumption (2.3), but it is always satisfied when (Y_i) and (Z_i) are independent. We emphasize that the term 'independence in tails' refers to the specific context of the null hypothesis and the clarification above. It should not be confused with the concept of tail independence, as defined in [1]. □

We proceed with the following lemma, which is an immediate consequence of assumptions (2.1), (2.2) and (2.3). Its proof is given in Section 4.

LEMMA 2.3. *Let (Y_i, Z_i), $i \in \mathbb{N}$, be a sequence of strictly stationary, strong mixing and non-negative bivariate random vectors such that (2.1) holds. Suppose that a sequence of integers m_n and a sequence of thresholds u_n and v_n are chosen such that (2.2) holds for some $0 < \tau < 1$ and (2.3) holds as $n \to \infty$. Then*

$$\frac{\sqrt{m_n}}{n} \sum_{i=1}^{n} (I_{Y,i} - p_Y) \xrightarrow{P} 0, \; n \to \infty.$$

An analogous result holds for the sequences (Z_n) and (v_n).

Let G_n be a random element on Ω with uniform distribution on the permutation group $\mathbf{G}_n$. We assume that G_n and X^n are independent throughout the rest of the document. Let P_{G_n} be the probability on $\mathbf{G}_n$ induced by G_n. Clearly $P_{G_n}(\{\pi\}) = 1/n!$, for $\pi \in \mathbf{G}_n$. Consider the following auxiliary statistic

$$S_n(X^n) = \frac{m_n}{n} \sum_{i=1}^{n} I_{\{Y_i > u_{\sqrt{m_n}}\}} I_{\{Z_i > v_{\sqrt{m_n}}\}} = \frac{m_n}{n} \sum_{i=1}^{n} I_{Y,i} I_{Z,i}.$$

and define the permuted sum $S_n^{G_n}$ as

$$S_n^{G_n} = S_n(G_n X^n) := \frac{m_n}{n} \sum_{i=1}^{n} I_{\{Y_i > u_{\sqrt{m_n}}\}} I_{\{Z_{G_n(i)} > v_{\sqrt{m_n}}\}} = \frac{m_n}{n} \sum_{i=1}^{n} I_{Y,i} I_{Z,G_n(i)}.$$

Because of the independence between X^n and G_n, by Theorem 6.4 in Kallenberg [15], we obtain

$$\begin{aligned} E(S_n^{G_n} \mid X^n) &= \int_{\mathbf{G}_n} \frac{m_n}{n} \sum_{i=1}^{n} I_{Y,i} I_{Z,\pi(i)} P_{G_n}(d\pi) \quad \text{(a.s.)} \\ &= \frac{1}{n!} \sum_{\pi \in \mathbf{G}_n} \frac{m_n}{n} \sum_{i=1}^{n} I_{Y,i} I_{Z,\pi(i)} \quad \text{(a.s.)}. \end{aligned} \tag{2.5}$$

Similarly, by the same theorem, we conclude that almost surely

$$\begin{aligned}
&P\Big(S_n^{G_n} - E(S_n^{G_n} \mid X^n) \le t\sqrt{\mathrm{Var}\,(S_n^{G_n} \mid X^n)} \mid X^n\Big) \\
&= \int_{\mathbf{G}_n} I_{\{\frac{m_n}{n}\sum_{i=1}^n I_{Y,i}I_{Z,\pi(i)} - E(S_n^{G_n}|X^n) \le t\sqrt{\mathrm{Var}\,(S_n^{G_n}|X^n)}\}} P_{G_n}(d\pi) \\
&= \frac{1}{n!}\sum_{\pi\in\mathbf{G}_n} I_{\{\frac{m_n}{n}\sum_{i=1}^n I_{Y,i}I_{Z,\pi(i)} - E(S_n^{G_n}|X^n) \le t\sqrt{\mathrm{Var}\,(S_n^{G_n}|X^n)}\}},
\end{aligned} \tag{2.6}$$

for $t \in \mathbb{R}$. A more detailed explanation of relations (2.5) and (2.6) is provided in [1]. Further details can be found in Section 1.3. in [2].

Let

$$\bar{I}_Y = \frac{1}{n}\sum_{i=1}^n I_{Y,i} \quad \text{and } \bar{I}_Z = \frac{1}{n}\sum_{i=1}^n I_{Z,i}.$$

Then we have the following lemma, whose proof is given in Section 4.

Lemma 2.4. *With the same notation as above, we have*

$$E(S_n^{G_n} \mid X^n) = m_n \bar{I}_Y \bar{I}_Z \quad (a.s.), \tag{2.7}$$

and

$$\mathrm{Var}\,(S_n^{G_n} \mid X^n) = \frac{1}{n-1}\frac{m_n^2}{n^2}\sum_{i=1}^n\sum_{j=1}^n \Big(I_{Y,i} - \bar{I}_Y\Big)^2\Big(I_{Z,j} - \bar{I}_Z\Big)^2 \ (a.s.). \tag{2.8}$$

Our main asymptotic result is summarized in the following theorem.

Theorem 2.5. *Let $X_i = (Y_i, Z_i)$, $i \in \mathbb{N}$, be a sequence of strictly stationary, strongly mixing, and non-negative bivariate random vectors for which (2.1) holds. Suppose that a sequence of integers m_n and a sequence of thresholds u_n and v_n are chosen such that (2.2) holds for some $0 < \tau < 1$ and (2.3) holds as $n \to \infty$. Then, for $t \in \mathbb{R}$, we have the following convergence in probability*

$$(P)\lim_{n\to\infty} P\big(S_n^{G_n} - E(S_n^{G_n} \mid X^n) \le t\sqrt{\mathrm{Var}\,(S_n^{G_n} \mid X^n)} \mid X^n\big) = \Phi(t), \tag{2.9}$$

where Φ is the standard normal cumulative distribution function.

Define the statistic T_n as

$$T_n(X^n) := \sqrt{n-1}\,\frac{\sum_{i=1}^n I_{Y,i}I_{Z,i} - n\bar{I}_Y\bar{I}_Z}{\sqrt{\sum_{i=1}^n \big(I_{Y,i} - \bar{I}_Y\big)^2}\sqrt{\sum_{j=1}^n \big(I_{Z,j} - \bar{I}_Z\big)^2}}. \tag{2.10}$$

Then we have

$$T_n(G_nX^n) = \sqrt{n-1}\,\frac{\sum_{i=1}^n I_{Y,i}I_{Z,G_n(i)} - n\bar{I}_Y\bar{I}_Z}{\sqrt{\sum_{i=1}^n \big(I_{Y,i} - \bar{I}_Y\big)^2}\sqrt{\sum_{j=1}^n \big(I_{Z,j} - \bar{I}_Z\big)^2}}.$$

Use the expressions in (2.7) and (2.8) and rearrange slightly the terms of $(S_n^{G_n} - E(S_n^{G_n} \,|\, X^n))/\sqrt{\mathrm{Var}\,(S_n^{G_n} \,|\, X^n)}$ to see that

$$T_n(G_n X^n) = \frac{S_n^{G_n} - E(S_n^{G_n} \,|\, X^n)}{\sqrt{\mathrm{Var}\,(S_n^{G_n} \,|\, X^n)}}. \tag{2.11}$$

Recall the definition of the permutation distribution $\hat{R}_n(t)$ of the statistic T_n given in (1.3). By using the expression in (2.6), we conclude from (2.11) that Theorem 2.5 states that the permutation distribution $\hat{R}_n(t)$ of the statistic T_n converges in probability to the standard normal distribution function Φ. In other words, we have shown

$$(P) \lim_{n\to\infty} \hat{R}_n(t) = \Phi(t), \; t \in \mathbb{R}, \quad \text{as } n \to \infty. \tag{2.12}$$

Observe that when Y and Z are independent, X^n and $G_n X^n$ have the same distribution for any permutation G_n. Then, by definition, the randomization hypothesis holds for X^n (see [17], Definition 15.2.1.). Note that the permutation distribution $\hat{R}_n(t)$ is a function of the random variables $I_{Y,i}$ and $I_{Z,i}$, which are independent under the null hypothesis. Since the randomization hypothesis holds for these variables, and due to the linearity of expectation, it follows from the definition of the permutation distribution that

$$E(\hat{R}_n(t)) = P(T_n \leq t), \quad t \in \mathbb{R}.$$

Given the convergence in probability in (2.12), and the fact that $\hat{R}_n(t)$ is uniformly bounded by 1 for all $n \in \mathbb{N}$, Theorem 25.12 in Billingsley [3] implies that

$$\lim_{n\to\infty} E(\hat{R}_n(t)) = E(\Phi(t)) = \Phi(t), \; t \in \mathbb{R}.$$

From the above, we deduce that

$$\lim_{n\to\infty} P(T_n \leq t) = \Phi(t), \; t \in \mathbb{R}.$$

Thus, we conclude that the distribution of the test statistic T_n also converges to the standard normal distribution. Hence, the permutation distribution $\hat{R}_n(t)$ asymptotically approximates the true sampling distribution of the statistic T_n, enabling the construction of the permutation test. Note that the convergence of the quantiles of the permutation distribution of the test statistic T_n to the quantiles of the standard normal distribution follows from Lemma 11.2.1 in [17].

We summarize the above considerations in the following theorem. Its proof is given in Section 4.

THEOREM 2.6. *Under the same assumptions as in Theorem 2.5, and under the null hypothesis H_0, the permutation distribution $\hat{R}_n$ of the statistic T_n*

satisfies

$$\sup_{t\in\mathbb{R}} \left|\hat{R}_n(t) - \Phi(t)\right| \to 0,\ n \to \infty, \tag{2.13}$$

in probability. It also holds that

$$\sup_{t\in\mathbb{R}} |R_n(t) - \Phi(t)| \to 0,\ n \to \infty, \tag{2.14}$$

where R_n denotes the distribution of the statistic T_n.

REMARK 2.7. In Remark 2.2, we noted that independent processes (Y_i) and (Z_i) are also independent in tails. One may naturally ask whether there are examples of processes that are independent in tails, yet not fully independent. Such an example can be constructed as follows: Let $(B_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernoulli random variables with parameter $\eta \in (0,1)$. Define the bivariate sequence

$$(Y_i, Z_i) = (1 - B_i)(U_i^1, V_i^1) + B_i(U_i^2, V_i^2), \quad i \in \mathbb{N},$$

where: (U_i^1, V_i^1) are dependent random vectors supported on $[0, u_0]^2$, and U_i^2 and V_i^2 are independent Pareto random variables with distribution function $F(x) = (1 - u_0/x)I_{\{x>u_0\}}$. We assume B_i are independent of U_i^2 and V_i^2.

In this construction, dependence between U_i^1 and V_i^1 may be introduced in various ways. For instance, one can take (U_i^0, V_i^0) to follow a dependent copula C on $[0,1]^2$, and then define $U_i^1 = u_0 U_i^0$ and $V_i^1 = u_0 V_i^0$. In this setup, (Y_i, Z_i) are dependent overall, but independent above the threshold u_0, since joint exceedances beyond u_0 only arise from the independent Pareto random variables. □

3. SIMULATIONS

In this section, we investigate the behaviour of the test statistic T_n defined in (2.10) through a simulation study. We denote the simulated data by $(Y_1, Z_1), \dots, (Y_n, Z_n)$, where $n \in \mathbb{N}$. The threshold levels used to calculate the value of the statistic T_n are determined by the empirical upper quantiles of the given data. After calculating the value of T_n, the following permuted values of T_n are computed:

$$T_n(Y^n, Z_\pi^n) := \sqrt{n-1}\frac{\sum_{i=1}^n I_{Y,i} I_{Z,\pi(i)} - n\bar{I}_Y \bar{I}_Z}{\sqrt{\sum_{i=1}^n \left(I_{Y,i} - \bar{I}_Y\right)^2}\sqrt{\sum_{j=1}^n \left(I_{Z,j} - \bar{I}_Z\right)^2}}.$$

For a given significance level α, we calculate $k = n - \lfloor n\alpha \rfloor$ and compare the value of the statistic with the k-th largest value among the permuted values of T_n. The permutation test rejects the null hypothesis when the value of the test statistic exceeds the k-th largest value of the permuted statistics. In the case where the value equals the k-th largest value, the test randomizes (see test ϕ in [17], Section 15). This procedure is repeated to obtain simulated rejection probabilities, referred to as empirical rejection probabilities.

3.1. *Independent sequences* (Y_i) *and* (Z_i). A sequence $\{Y_i, i \in \mathbb{N}\}$ is called M-dependent, $M \in \mathbb{N}_0$, if for all $j \in \mathbb{N}$, the vector $(Y_1, \ldots, Y_j)$ is independent of $(Y_{j+k}, Y_{j+k+1}, \ldots)$ whenever $k > M$. An i.i.d. sequence is zero-dependent (i.e., $M = 0$).

It is natural to investigate the proposed permutation testing procedure for two independent sequences (Y_i) and (Z_i) that are 1-dependent, 2-dependent or 3-dependent. Where applicable, we simulate Pareto-type random variables. Specifically, to simulate Pareto-distributed random variables, we first generate i.i.d. random variables U_i from a continuous uniform distribution $U(0,1)$, and then define the Pareto-distributed random variables $X_i = 1/U_i$. Below, we list various combinations of independent sequences (Y_i) and (Z_i) for which we provide simulation results.

1) Let U_i and V_i, $i \in \mathbb{N}$, be two independent sequences of i.i.d. random variables from standard normal distribution. Define two independent sequences as $Y_i = U_i \cdot U_{i+1}$ and $Z_i = V_i \cdot V_{i+1}$. This type of 1-dependent sequences is more thoroughly analysed in Example 2.1. in [21].
2) Let U_i and V_i, $i \in \mathbb{N}$, be two independent sequences of i.i.d. Pareto-distributed random variables. Define two independent sequences as $Y_i = U_i \cdot U_{i+1}$ and $Z_i = V_i \cdot V_{i+1}$. (Y_i) and (Z_i) are 1-dependent sequences.
3) Let U_i and V_i, $i \in \mathbb{N}$, be two independent sequences of i.i.d. Pareto-distributed random variables. Define two independent sequences as $Y_i = U_i + U_{i+1}$ and $Z_i = V_i + V_{i+1}$. (Y_i) and (Z_i) are 1-dependent sequences.
4) Let U_i and V_i, $i \in \mathbb{N}$, be two independent sequences of i.i.d. Pareto-distributed random variables. Define two independent sequences as $Y_i = U_i + U_{i+1}$ and $Z_i = V_i + V_{i+1} + V_{i+2} + V_{i+3}$. Here, (Y_i) is a 1-dependent sequence, while (Z_i) is a 3-dependent sequence.
5) Let U_i and V_i, $i \in \mathbb{N}$, be two independent sequences of i.i.d. Pareto-distributed random variables. Define two independent sequences $Y_i = U_i + U_{i+1}$ and $Z_i = V_i + V_{i+2}$. (Y_i) is 1-dependent sequences while (Z_i) is 2-dependent.

A generalized autoregressive conditional heteroscedastic (GARCH) process $(X_t)_{t\in\mathbb{Z}}$ with parameters p and q, denoted as GARCH(p,q), and volatility $(\sigma_t)_{t\in\mathbb{Z}}$, is a solution to the equations:

$$X_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \quad t \in \mathbb{Z},$$

where (ϵ_i) is a sequence of i.i.d. random variables. A GARCH(p, q) admits strictly stationary solution if

$$E(\epsilon_0^2)\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j < 1.$$

More on this topic can be found in Lindner [18] (see Theorem 3).

In models 6) and 7), we utilize independent GARCH processes to generate the sequences (Y_i) and (Z_i), where ϵ_i are standard normal random variables, with the following parameter settings:

6) (Y_i) and (Z_i) are GARCH(1, 1) processes with parameters $\alpha_0 = 0.5$, $\alpha_1 = 0.2$ and $\beta_1 = 0.7$.
7) (Y_i) and (Z_i) are GARCH(2, 2) processes with parameters $\alpha_0 = 0.5$, $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\beta_1 = 0.4$ and $\beta_2 = 0.2$.

Below, we provide empirical rejection probabilities for the above-listed cases at various threshold levels (30%, 20%, 10%). Those threshold levels refer to upper-tail quantiles. For example, the threshold level of 20% indicates that the upper 20% quantile of the simulated data was used (i.e., the 80% quantile of the data). In other words, in this case, we are using the most extreme 20% of the simulated data points. In all simulations, the significance level of the test is set at 5%, the sample size is 1,000, and the number of permutations and repetitions is 2,000. Additionally, simulations were conducted with varying significance levels, sample sizes, and numbers of permutations and repetitions, yielding results comparable to those presented here.

TABLE 3.1. Empirical rejection probabilities for various cases of dependence in independent sequences (Y_i) and (Z_i). The significance level of the test is set at 5%, the sample size is 1,000, and the number of permutations and repetitions is 2,000.

Model for Y_i and Z_i	30% threshold	20% threshold	10% threshold
Case 1)	0.0490	0.0515	0.0560
Case 2)	0.0675	0.0635	0.0655
Case 3)	0.0740	0.0710	0.0760
Case 4)	0.0905	0.0880	0.0875
Case 5)	0.0500	0.0505	0.0465
Case 6)	0.0515	0.0530	0.0605
Case 7)	0.0575	0.0570	0.0550

Permutation tests are exact for independent data (see Section 15 in [17]). Therefore, for the significance level set at 5%, the empirical rejection probabilities presented in Table 3.1 should be close to 5%. In most of the cases presented in Table 3.1, this is indeed the case. However, rows 3 and 4 indicate that the dependence caused by the addition of i.i.d. Pareto-type random variables increases the empirical rejection probabilities above the desired level. This issue becomes more pronounced as the dependence strengthens. It is worth noting, though we do not present the simulation results here, that this problem was less severe when U_i and V_i were drawn from other distributions, such as the standard normal or uniform distribution.

Pareto-type random variables typically show many small values punctuated by occasional large spikes. The dependence models in cases 2), 3) and 4) combine these spikes with neighbouring elements of Y_i and Z_i, forming clusters of large values. Please note the difference in case 5), where Z_i is not defined as the sum of two consecutive members of the sequence (V_i), thereby partially preventing the clustering of large spikes mentioned above.

To investigate whether other types of clustering result in similar deviations from expected rejection probabilities, we added models 6) and 7), based on GARCH processes, which are known for modelling clusters of extremes, such as volatility clustering in financial markets. As shown in Table 3.1, the empirical rejection probabilities for the GARCH models closely match the expected value of 5%. Similar results were observed when simulations were performed using different GARCH parameters. Additionally, we note that GARCH processes can attain both positive and negative values. Therefore, a 30% upper quantile reported in Table 3.1 refers to 60% of extreme data points in the upper tail, which may be excessive. However, when simulations were run with a 5% threshold level (i.e., using 10% of data in the upper quantile) we obtained similar results: the empirical rejection probability in case 6) was 0.049, while in case 7) it was 0.047.

To gain further insight into the impact of dependence on the empirical rejection probabilities in cases 3) and 4), we conducted additional simulations by modifying the process (Y_i) to $Y_i = U_i + b \cdot U_{i+1}$, where $b \in [0, 1]$. The parameter b controls the strength of dependence between adjacent observations, allowing us to quantify how varying levels of autocorrelation within sequences impact the performance of the test. The results for selected values of b are shown in the table below, with a threshold level set to 20%.

Table 3.2. Empirical rejection probabilities for different values of b. The significance level of the test is set at 5%, the sample size is $1,000$, and the number of permutations and repetitions is $2,000$.

Model for Y_i and Z_i	$b = 0.8$	$b = 0.6$	$b = 0.4$	$b = 0.2$	$b = 0$
Case 3), $Y_i = U_i + b \cdot U_{i+1}$	0.0750	0.0665	0.0650	0.0605	0.0575
Case 4), $Y_i = U_i + b \cdot U_{i+1}$	0.0865	0.0815	0.0730	0.0685	0.0455

As expected, as the dependence within sequences decreases (i.e., as b decreases), the empirical rejection probabilities approach 5%.

Overall, we conclude that the proposed permutation test demonstrates solid size characteristics. The somewhat inflated rejection rates observed in cases 3) and 4) are likely a result of the strong dependence caused by the addition of i.i.d. Pareto-type random variables. These issues could potentially be addressed by using block permutation tests, such as those analyzed in [2]. Note that the results presented in Section 3 of [2] are valid under the assumption of independence in tails, as described by the null hypothesis of this article.

3.2. *Dependent sequences* (Y_i) *and* (Z_i). Let (U_i^1), (U_i^2), (U_i^3), (U_i^4) and (U_i^5) be five independent i.i.d. sequences of random variables distributed as $U(0,1)$. We define the corresponding sequences of independent Pareto-distributed random variables as $X_i^1 = 1/U_i^1$, $X_i^2 = 1/U_i^2$, $X_i^3 = 1/U_i^3$, $X_i^4 = 1/U_i^4$ and $X_i^5 = 1/U_i^5$, $i = 1, 2, \ldots$. We then analyse the following models of linear dependence:

a) Let $Y_i = X_i^1 + a \cdot X_i^2$ and $Z_i = X_i^3 + a \cdot X_i^2$, where $a \in [0,1]$. This type of dependence is more thoroughly analysed in Section 3.2 in [1].
b) Let $Y_i = X_i^1 + X_i^2 + a \cdot X_i^3$ and $Z_i = X_i^4 + X_i^5 + a \cdot X_i^3$, where $a \in [0,1]$.
c) Let $Y_i = X_i^1 + X_{i+1}^1 + a \cdot X_i^2$ and $Z_i = X_i^3 + X_{i+1}^3 + a \cdot X_i^2$, where $a \in [0,1]$.

Figure 3.1 shows simulation results for these models, with the threshold level set at 20%, the significance level of the test set at 5% and the number of permutation and repeats set to $2,000$.

As the parameter a increases, the dependence between the sequences (Y_i) and (Z_i) increases in all three models. The rejection probabilities converge toward 1 most rapidly in model a), which is expected since the dependence structure in this model depends directly on the parameter a. In models b) and c), the increase in rejection probabilities is comparatively slower, as the dependence on a is diluted across additional terms. However, the rise remains satisfactorily fast, especially considering the behaviour of Pareto-type processes, which exhibit many small values with occasional large spikes. Similar results were obtained when different threshold levels (e.g., 10% or 5%) were used.

Next, we present simulation results for dependent GARCH processes. Specifically, we used GARCH$(1,1)$ and GARCH$(2,2)$ models with the same parameters as described in cases 6) and 7) above. To introduce dependence between the sequences (Y_i) and (Z_i), we simulated correlated innovations ϵ_i for both sequences. These innovations were drawn from a bivariate normal distribution with mean zero, unit variance, and correlation parameter ρ. The relationship between the strength of dependence between the sequences (Y_i) and (Z_i), measured by ρ, and the empirical rejection probabilities of the

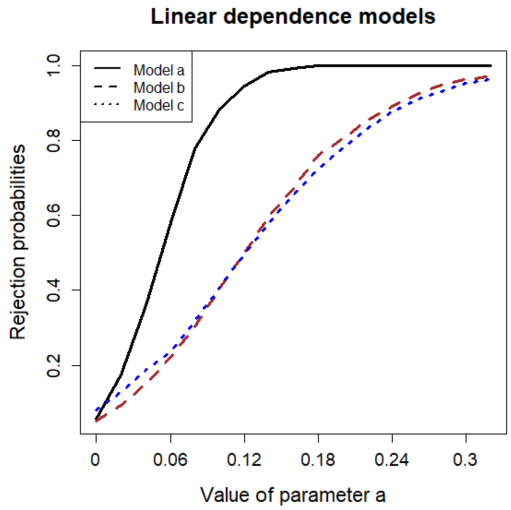

FIGURE 3.1. Rejection probabilities for models a)-c). The threshold level is set at 20%, the significance level of the test is 5% and the number of permutation and repeats is 2,000.

proposed permutation test is presented in Table 3.3. For illustration, we chose threshold levels of 10% and 5%. Due to the possibility of both positive and negative values in the simulated GARCH processes, these threshold levels correspond to the 20% and 10% upper-tail quantiles of the simulated data, respectively.

TABLE 3.3. Empirical rejection probabilities for dependent GARCH sequences with different values of ρ. The significance level of the test is set at 5%, the sample size is 1,000, and the number of permutations and repetitions is 2,000. Parameter ρ denotes the correlation between innovations of two GARCH processes we simulate.

Model for Y_i and Z_i	$\rho=0.0$	$\rho=0.1$	$\rho=0.2$	$\rho=0.3$	$\rho=0.4$
GARCH$(1,1)$, thresh. 10%	0.0455	0.2910	0.6780	0.9390	0.9980
GARCH$(1,1)$, thresh. 5%	0.0535	0.1770	0.4065	0.7040	0.8975
GARCH$(2,2)$, thresh. 10%	0.0420	0.2960	0.7140	0.9570	0.9985
GARCH$(2,2)$, thresh. 5%	0.0495	0.1925	0.4590	0.7500	0.9270

As we can see from the Table 3.3, there are notable differences in rejection probabilities for different threshold levels. A similar effect was observed in simulation results from [1] (see Section 3.4 in [1]).

Although not explicitly shown here, we tested our statistic T_n on various models of i.i.d. data $(Y_i, Z_i)_{i\in\mathbb{N}}$. Our simulation results were consistent with those obtained for the studentized statistic $\hat{T}_n$ in all cases reported in [1], including those based on the Gumbel-Hougaard, Morgenstern and Normal copula. Additionally, we conducted simulations using the GARCH$(1,1)$ model with innovations generated from the Gumbel-Hougaard copula, and the results were comparable to those shown in Figure 2 in [1].

Overall, our simulation studies suggest that the proposed test demonstrates considerable power against alternatives.

All the simulations and analysis were done in *R* [19] using the publicly available packages `permute` [22], `rugarch` [10] and `copula` [12, 13, 16, 24].

4. Proofs

Proof of Lemma 2.3. Let $n \in \mathbb{N}$ and $\epsilon > 0$ be arbitrarily chosen. By the Chebyshev's inequality, we have

$$P\Big(\frac{\sqrt{m_n}}{n}\Big|\sum_{i=1}^{n}(I_{Y,i}-p_Y)\Big| > \epsilon\Big) \le \frac{m_n}{n^2\epsilon^2}\mathrm{Var}\,\Big(\sum_{i=1}^{n}(I_{Y,i}-p_Y)\Big). \tag{4.1}$$

Due to the stationarity of the sequence (Y_i), we have

$$\begin{aligned}\mathrm{Var}\,\Big(\sum_{i=1}^{n}(I_{Y,i}-p_Y)\Big) &= n\mathrm{Var}\,(I_{Y,1}-p_Y)\\ &\quad + 2\sum_{i=2}^{n}(n-i+1)\mathrm{Cov}\,(I_{Y,1}-p_Y, I_{Y,i}-p_Y).\end{aligned} \tag{4.2}$$

Since $E(I_{Y,i}-p_Y)=0$, we immediately get

$$\mathrm{Var}\,(I_{Y,1}-p_Y) = E(I_{Y,1}-p_Y)^2 = p_Y - p_Y^2 = p_Y(1-p_Y).$$

To bound the covariance term, we use the fact that $0 \le I_{Y,i} \le 1$ (a.s.), $i = 1, 2, \dots n$, and apply an inequality for covariances of strongly mixing stationary sequences due to Ibragimov [14]. For $i \in \{2, \dots, n\}$ we have

$$\begin{aligned}|\mathrm{Cov}\,(I_{Y,1}-p_Y, I_{Y,i}-p_Y)| &= |\mathrm{Cov}\,(I_{Y,1}, I_{Y,i})|\\ &\le 2\alpha_{I_Y}(i)\|I_{Y,1}\|_\infty\|I_{Y,i}\|_\infty \le 2\alpha_Y(i).\end{aligned}$$

Note that α_{I_Y} is the α-mixing coefficient of the stationary sequence $(I_{Y,i})_{i\in\mathbb{N}}$. The last inequality above follows from the argument given in Remark 2.1.

Taking the absolute value in (4.2), using the triangle inequality, and applying the last inequality above, we conclude that the right-hand side in (4.1) can be bounded by

$$\frac{m_n}{n^2\epsilon^2}np_Y(1-p_Y)+4\frac{m_n}{n^2\epsilon^2}\sum_{i=2}^{n}(n-i+1)\alpha_Y(i)\leq\frac{m_n}{n\epsilon^2}p_Y+4\frac{m_n}{n\epsilon^2}\sum_{i=1}^{\infty}\alpha_Y(i).$$

Note that $\sum_{i=1}^{\infty}\alpha_Y(i)<\infty$ follows from assumption (2.1) and Remark 2.1. Also, $p_Y\sim m_n^{-1/2}$ by assumption (2.3). Then, by assumption (2.2) it follows that the right-hand side in (4.1) converges to zero as $n\to\infty$, and thus the claim of the Lemma follows. ∎

PROOF OF LEMMA 2.4. The proofs of this lemma and Theorem 2.5 are based on results proven in Hoeffding [11], with the main result given by the Combinatorial central limit theorem (Theorem 4 in [11]). The setup of the problem analyzed in [11] is the following: assume that for each $n\in\mathbb{N}$ we are given $2n$ real numbers $a_n(i), b_n(i)$, $i=1,2,\ldots,n$, such that neither all instances of $a_n(i)$ nor those of $b_n(i)$ are equal. Let

$$S_n=\sum_{i=1}^{n}a_n(i)b_n(G_n(i)). \tag{4.3}$$

The mean and variance of S_n can be expressed explicitly, as is shown in Theorem 2 in [11], to get:

$$ES_n=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}a_n(i)b_n(j), \tag{4.4}$$

$$\mathrm{Var}\,(S_n)=\frac{1}{n-1}\sum_{i=1}^{n}\sum_{j=1}^{n}d_n^2(i,j), \tag{4.5}$$

where

$$\begin{aligned}d_n(i,j)&=a_n(i)b_n(j)-\frac{1}{n}\sum_{g=1}^{n}a_n(g)b_n(j)\\&\quad-\frac{1}{n}\sum_{h=1}^{n}a_n(i)b_n(h)+\frac{1}{n^2}\sum_{g=1}^{n}\sum_{h=1}^{n}a_n(g)b_n(h).\end{aligned} \tag{4.6}$$

To apply the above setup, we define two triangular arrays of random variables, $a_n(i)$ and $b_n(i)$, $i\in\{1,\ldots,n\}$, as

$$a_n(i)=\sqrt{\frac{m_n}{n}}I_{Y,i}\quad\text{ and }\quad b_n(i)=\sqrt{\frac{m_n}{n}}I_{Z,i}.$$

Next, we define

$$\bar{a}_n:=\frac{1}{n}\sum_{i=1}^{n}a_n(i)=\sqrt{\frac{m_n}{n}}\frac{1}{n}\sum_{i=1}^{n}I_{Y,i}$$

and

$$\bar{b}_n := \frac{1}{n}\sum_{i=1}^{n} b_n(i) = \sqrt{\frac{m_n}{n}}\frac{1}{n}\sum_{i=1}^{n} I_{Z,i}.$$

On the right-hand side of (2.5) we recognize the expectation of S_n, relative to the probability measure P_{G_n}. From (4.4) we conclude that $E(S_n^{G_n} \mid X^n)$ is almost surely equal to

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_n(i)b_n(j) = \frac{1}{n}\frac{m_n}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} I_{Y,i}I_{Z,j}$$

and so (2.7) follows. To prove (2.8) first observe that by relation (4.6) we have

$$\begin{aligned} d_n(i,j) &= \frac{m_n}{n} I_{Y,i}I_{Z,j} - \frac{1}{n}\frac{m_n}{n}\sum_{k=1}^{n} I_{Y,k}I_{Z,j} \\ &\quad - \frac{1}{n}\frac{m_n}{n}\sum_{l=1}^{n} I_{Y,i}I_{Z,l} + \frac{1}{n^2}\frac{m_n}{n}\sum_{k=1}^{n}\sum_{l=1}^{n} I_{Y,k}I_{Z,l} \\ &= \frac{m_n}{n} I_{Z,j}\Big(I_{Y,i} - \frac{1}{n}\sum_{k=1}^{n} I_{Y,k}\Big) - \frac{m_n}{n}\frac{1}{n}\sum_{l=1}^{n} I_{Z,l}\Big(I_{Y,i} - \frac{1}{n}\sum_{k=1}^{n} I_{Y,k}\Big) \\ &= \frac{m_n}{n}\Big(I_{Y,i} - \bar{I}_Y\Big)\Big(I_{Z,j} - \bar{I}_Z\Big). \end{aligned} \tag{4.7}$$

Use similar arguments as those employed in (2.5), together with (2.7), to obtain

$$\operatorname{Var}(S_n^{G_n} \mid X^n) = \frac{1}{n!}\sum_{\pi\in\mathbf{G}_n}\Big(\frac{m_n}{n}\sum_{i=1}^{n} I_{Y,i}I_{Z,\pi(i)} - m_n\bar{I}_Y\bar{I}_Z\Big)^2 \quad \text{(a.s.)}.$$

Note that the right-hand side of the above expression is the variance of S_n and then apply (4.5), along with the expression for $d_n(i,j)$ in (4.7), to conclude that (2.8) holds. □

In the proofs of the next two theorems, we will repeatedly use Theorem 20.5 from [3], which provides a characterization of convergence in probability: a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in probability to a random variable X if and only if every subsequence $\{n_j : j \in \mathbb{N}\} \subset \mathbb{N}$ contains a further subsequence $\{n_{j_k} : k \in \mathbb{N}\} \subset \{n_j : j \in \mathbb{N}\}$ such that

$$X_{n_{j_k}} \to X,\ k \to \infty\ (a.s.).$$

Proof of Theorem 2.5. The proof of this theorem relies on the Combinatorial Central Limit Theorem due to Hoeffding (Theorem 4 in [11]). Following the notation used in the proof of Lemma 2.4 we need to verify the condition from Hoeffding's CLT

$$\lim_{n\to\infty} n \frac{\max_{1\le i\le n}(a_n(i)-\bar{a}_n)^2}{\sum_{i=1}^n (a_n(i)-\bar{a}_n)^2} \frac{\max_{1\le i\le n}(b_n(i)-\bar{b}_n)^2}{\sum_{i=1}^n (b_n(i)-\bar{b}_n)^2} = 0. \tag{4.8}$$

Since $a_n(i)$ and $b_n(i)$ are random variables, we need the almost sure convergence in the above expression to hold. The idea behind the proof of this theorem is to use the characterization of convergence in probability presented before this proof.

For simplicity, we focus separately on the numerator and the denominator of the expression in (4.8). The numerator in (4.8) is equal to

$$n \max_{1\le i\le n}\left(\sqrt{\frac{m_n}{n}} I_{Y,i} - \sqrt{\frac{m_n}{n}} \frac{1}{n}\sum_{j=1}^n I_{Y,j}\right)^2 \max_{1\le i\le n}\left(\sqrt{\frac{m_n}{n}} I_{Z,i} - \sqrt{\frac{m_n}{n}} \frac{1}{n}\sum_{j=1}^n I_{Z,j}\right)^2,$$

or, equivalently,

$$\frac{m_n^2}{n} \max_{1\le i\le n}\left(I_{Y,i} - \bar{I}_Y\right)^2 \max_{1\le i\le n}\left(I_{Z,i} - \bar{I}_Z\right)^2.$$

Both maxima in the numerator are almost surely bounded by 1 since this is true for each $I_{Y,i}$ and $I_{Z,i}$. Therefore, the numerator is almost surely bounded by m_n^2/n. Consequently, the expression under the limit in (4.8) is almost surely bounded by

$$\frac{1}{\frac{n}{m_n^2}\sum_{i=1}^n (a_n(i)-\bar{a}_n)^2 \sum_{i=1}^n (b_n(i)-\bar{b}_n)^2}. \tag{4.9}$$

By Lemma 2.3, we know that

$$\frac{\sqrt{m_n}}{n}\sum_{i=1}^n (I_{Y,i} - p_Y) \xrightarrow{P} 0, \quad \text{and} \quad \frac{\sqrt{m_n}}{n}\sum_{i=1}^n (I_{Z,i} - p_Z) \xrightarrow{P} 0, \text{ as } n\to\infty.$$

Let $\{n_j : j \in \mathbb{N}\}$ be an arbitrary subsequence of natural numbers. Using Theorem 20.5 from [3], we conclude that there exists a further subsequence $\{n_{j_k} : k\in\mathbb{N}\} \subset \{n_j : j\in\mathbb{N}\}$ such that

$$\frac{\sqrt{m_{n_{j_k}}}}{n_{j_k}}\sum_{i=1}^{n_{j_k}} (I_{Y,i} - p_Y) \to 0, \ k\to\infty \ (a.s.) \tag{4.10}$$

and yet another subsequence $\{n_{j_{k_l}} : l\in\mathbb{N}\} \subset \{n_{j_k} : k\in\mathbb{N}\}$ such that

$$\frac{\sqrt{m_{n_{j_{k_l}}}}}{n_{j_{k_l}}}\sum_{i=1}^{n_{j_{k_l}}} (I_{Z,i} - p_Z) \to 0, \ l\to\infty \ (a.s.). \tag{4.11}$$

Clearly, (4.10) also holds with $n_{j_{k_l}}$ instead of n_{j_k}. We conclude that there exists a subsequence of $\{n_j : j\in\mathbb{N}\}$ such that both (4.10) and (4.11) are true.

For notational simplicity we will denote that subsequence by $\{n_{j_k}\}$ in the rest of the proof. Note that now

$$I_{Y,i} = I_{\{Y_i > \sqrt{u_{m_{n_{j_k}}}}\}}$$

with an analogous relation for the sequence Z. We will show that

$$\lim_{k\to\infty} n_{j_k} \frac{\max_{1\le i\le n_{j_k}} (a_{n_{j_k}}(i) - \bar{a}_{n_{j_k}})^2}{\sum_{i=1}^{n_{j_k}} (a_{n_{j_k}}(i) - \bar{a}_{n_{j_k}})^2} \frac{\max_{1\le i\le n_{j_k}} (b_{n_{j_k}}(i) - \bar{b}_{n_{j_k}})^2}{\sum_{i=1}^{n_{j_k}} (b_{n_{j_k}}(i) - \bar{b}_{n_{j_k}})^2} = 0 \tag{4.12}$$

holds on the set A, i.e. almost surely. Using the calculations from the beginning of the proof of this theorem, we conclude that the expression under the limit in (4.12) is almost surely bounded by

$$\frac{1}{\frac{n_{j_k}}{m^2_{n_{j_k}}} \sum_{i=1}^{n_{j_k}} (a_{n_{j_k}}(i) - \bar{a}_{n_{j_k}})^2 \sum_{i=1}^{n_{j_k}} (b_{n_{j_k}}(i) - \bar{b}_{n_{j_k}})^2}. \tag{4.13}$$

We will show that the denominator in (4.13) tends to $+\infty$ almost surely. Both sums in the denominator of (4.13) can be treated analogously, so we focus on the first sum. We have:

$$\begin{aligned}
\sum_{i=1}^{n_{j_k}} (a_{n_{j_k}}(i) - \bar{a}_{n_{j_k}})^2 &= \sum_{i=1}^{n_{j_k}} \Big(\sqrt{\frac{m_{n_{j_k}}}{n_{j_k}}} I_{Y,i} - \frac{1}{n_{j_k}} \sum_{j=1}^{n_{j_k}} \sqrt{\frac{m_{n_{j_k}}}{n_{j_k}}} I_{Y,j}\Big)^2 \\
&= \frac{m_{n_{j_k}}}{n_{j_k}} \left(\sum_{i=1}^{n_{j_k}} I^2_{Y,i} - 2 \sum_{i=1}^{n_{j_k}} I_{Y,i} \frac{1}{n_{j_k}} \sum_{j=1}^{n_{j_k}} I_{Y,j} + \sum_{i=1}^{n_{j_k}} \Big(\frac{1}{n_{j_k}} \sum_{j=1}^{n_{j_k}} I_{Y,j} \Big)^2 \right) \\
&= \frac{m_{n_{j_k}}}{n_{j_k}} \sum_{i=1}^{n_{j_k}} I_{Y,i} - m_{n_{j_k}} \Big(\frac{1}{n_{j_k}} \sum_{j=1}^{n_{j_k}} I_{Y,j} \Big)^2 = m_{n_{j_k}} \bar{I}_Y (1 - \bar{I}_Y)
\end{aligned}$$

and analogously

$$\sum_{i=1}^{n_{j_k}} (b_{n_{j_k}}(i) - \bar{b}_{n_{j_k}})^2 = m_{n_{j_k}} \bar{I}_Z (1 - \bar{I}_Z).$$

Note, now we have

$$\bar{I}_Y = \frac{1}{n_{j_k}} \sum_{i=1}^{n_{j_k}} I_{Y,i}.$$

We conclude that the denominator in (4.13) can be written as

$$\frac{\sqrt{n_{j_k}}}{\sqrt{m_{n_{j_k}}}} \sqrt{m_{n_{j_k}}}\ \bar{I}_Y (1 - \bar{I}_Y) \frac{\sqrt{n_{j_k}}}{\sqrt{m_{n_{j_k}}}} \sqrt{m_{n_{j_k}}}\ \bar{I}_Z (1 - \bar{I}_Z).$$

Consider the term $\sqrt{m_{n_{j_k}}} \bar{I}_Y$. We have

$$\sqrt{m_{n_{j_k}}} \bar{I}_Y = \frac{\sqrt{m_{n_{j_k}}}}{n_{j_k}} \sum_{i=1}^{n_{j_k}} (I_{Y,i} - p_Y + p_Y) = \frac{\sqrt{m_{n_{j_k}}}}{n_{j_k}} \sum_{i=1}^{n_{j_k}} (I_{Y,i} - p_Y) + \sqrt{m_{n_{j_k}}} p_Y,$$

We know, by the assumption (2.3), that $\sqrt{m_n} p_Y \to 1$, as $n \to \infty$. This is also true for the subsequence $(m_{n_{j_k}})$ of (m_n). By using Lemma 2.3 and the choice of the subsequence (n_{j_k}), we conclude that $\sqrt{m_{n_{j_k}}}\bar{I}_Y$ converges to 1 almost surely. Therefore, $\bar{I}_Y \to 0$ almost surely and the same is then true for $\sqrt{m_{n_{j_k}}}\bar{I}_Y\bar{I}_Y$. Thus, $\sqrt{m_{n_{j_k}}}\bar{I}_Y(1-\bar{I}_Y) \to 1$ almost surely as $k \to \infty$.

We conclude that both $\sqrt{m_{n_{j_k}}}\bar{I}_Y(1-\bar{I}_Y)$ and $\sqrt{m_{n_{j_k}}}\bar{I}_Z(1-\bar{I}_Z)$ almost surely converge to 1. Then, the whole expression in (4.13) converges to zero because of the term $\sqrt{n_{j_k}}/\sqrt{m_{n_{j_k}}}$ in the denominator, which tends to infinity (recall (2.2)).

Note that, after taking into account the definition of the probability P_{G_n}, we can explicitly write the statement of the Combinatorial central limit theorem as

$$\lim_{n\to\infty} \frac{1}{n!} \sum_{\pi \in \mathbf{G}_n} I_{\{(S_n^\pi - ES_n)/\sqrt{\operatorname{Var} S_n} \le x\}} = \Phi(x), \text{ (a.s.)} \tag{4.14}$$

where we sum over all permutations π of the set $\{1, 2, \ldots, n\}$ and

$$S_n^\pi = \sum_{i=1}^n a_n(i) b_n(\pi(i)).$$

Compare (4.14) with relation (2.6) to conclude that almost surely

(4.15)

$$\lim_{k\to\infty} P\Big(S_{n_{j_k}}^{G_{n_{j_k}}} - E(S_{n_{j_k}}^{G_{n_{j_k}}} \mid X^{n_{j_k}}) \le t\sqrt{\operatorname{Var}(S_{n_{j_k}}^{G_{n_{j_k}}} \mid X^{n_{j_k}}) \mid X^{n_{j_k}}}\Big) = \Phi(t).$$

We conclude that for any arbitrary subsequence of natural numbers $\{n_j\}$, there exists a further subsequence $\{n_{j_k}\}$ of $\{n_j\}$ such that (4.15) holds. Then, again by Theorem 20.5. from [3], it follows that (2.9) holds. $\square$

PROOF OF THEOREM 2.6. The relation in (2.13) follows by a subsequence argument, specifically by the repeated use of Theorem 20.5 from [3].

Suppose that (2.12) holds, and choose an arbitrary subsequence (n_k) in $\mathbb{N}$ and a dense countable set $D = \{t_1, t_2, \ldots\}$ in $\mathbb{R}$. Then, for $t_1 \in D$, there exists a further subsequence $(n_{1,i})$ such that $\hat{R}_{n_{1,i}}(t_1) \to \Phi(t_1)$ holds almost surely. Furthermore, for this sequence, there exists yet another subsequence, say $(n_{2,i})$, such that $\hat{R}_{n_{2,i}}(t_2) \to \Phi(t_2)$ holds almost surely. We continue this process for each $t_j \in D$, creating a sequence of subsequences $(n_{j,i})$, where each $(n_{j+1,i})$ is a subsequence of $(n_{j,i})$.

To handle convergence for all points in D, we apply the diagonal argument. Define a new sequence l_i by taking $l_i = n_{i,i}$. The sequence (l_i) is constructed such that, for each $t \in D$, $\hat{R}_{l_i}(t) \to \Phi(t)$ almost surely. Due to the continuity of Φ and right continuity of R_{l_i}, this convergence also holds for all $t \in \mathbb{R}$.

By Polya's theorem (see Theorem 11.2.9 in [17]), which states that if a sequence of distribution functions converges to a continuous distribution function at all continuity points, then the convergence is uniform, we obtain that

$$\sup_{t\in\mathbb{R}}\left|\hat{R}_{l_i}(t)-\Phi(t)\right|\to 0,\qquad \text{as}\quad i\to\infty.$$

Since the original subsequence (n_k) was arbitrary, we have shown that for every subsequence, there exists a further subsequence along which the convergence is almost surely uniform. By Theorem 20.5 from [3], this implies that (2.13) holds. The relation in (2.14) follows directly from Polya's theorem as well. □

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References

[1] B. Basrak and D. Brborović, *Permutation test of tail dependence*, Stat. Methods Appl. **33** (2024), 89–129.

[2] D. Brborović, Statistical analysis of the tail behaviour of dependent sequences, PhD thesis, University of Zagreb, Faculty of Science, Department of Mathematics, 2022.

[3] P. Billingsley, Probability and measure, John Wiley & Sons, Inc., New York, 1995.

[4] R. C. Bradley, Introduction to strong mixing conditions, Kendrick Press, Heber City, 2009.

[5] E. Chung and J. P. Romano, *Exact and asymptotically robust permutation tests*, Ann. Statist **41** (2013), 484–507.

[6] R. Cont, *Empirical properties of asset returns: stylized facts and statistical issues*, Quantitative Finance **1** (2001), 223–236.

[7] V. H. de la Pena, R. Ibragimov and S. Sharakhmetov, *Characterizations of joint distributions, copulas, information, dependence and decoupling, with applications to time series*, Institute of Mathematical Statistics, Beachwood, 2006, 183–209.

[8] C. J. DiCiccio and J. P. Romano *Robust permutation tests for correlation and regression coefficients*, J. Amer. Statist. Assoc. **112** (2017), 1211–1220.

[9] P. Embrechts, A. McNeil and D. Straumann *Correlation and dependence in risk management: properties and pitfalls*, In Risk Management: Value at Risk and Beyond. Cambridge University Press, Cambridge, 2002, 176–223.

[10] A. Galanos, rugarch: Univariate GARCH models. R package version 1.5-3., 2024.

[11] V. Hoeffding, *A combinatorial central limit theorem*, Ann. Math. Statistics **22** (1951), 558–566.

[12] M. Hofert, I. Kojadinovic, M. Maechler and J. Yan, copula: Multivariate dependence with copulas. R package version 1.1-6, 2024, `https://CRAN.R-project.org/package=copula`.

[13] M. Hofert and Ma. Maechler, Nested Archimedean copulas meet R: The nacopula package. Journal of Statistical Software **39** (2011), 1-20. `https://www.jstatsoft.org/v39/i09/`.

[14] I. A. Ibragimov, *Some limit theorems for stationary processes*, Teor. Verojatnost. i Primenen. **7** (1962), 361–392.

[15] O. Kallenberg, Foundations of modern probability, Springer-Verlag, New York, 2002.

[16] I. Kojadinovic and J. Yan, *Modeling multivariate distributions with continuous margins using the copula R package*, Journal of Statistical Software **34** (2010), 1–20. `https://www.jstatsoft.org/v34/i09/`.

[17] E. L. Lehman, and J. P. Romano, Testing statistical hypotheses, Springer, New York, 2005.

[18] A. M. Lindner, *Stationarity, mixing, distributional properties and moments of GARCH(p, q)–processes.* In Handbook of financial time series, Springer, Berlin, Heidelberg, 2009.

[19] R Core Team (2024). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. `https://www.R-project.org/`

[20] S. I. Resnick, Heavy tail phenomena, probabilistic and statistical modeling, Springer New York, 2007.

[21] J. P. Romano and M. A. Tirlea, *Permutation testing for dependence in time series*, J. Time Series Anal. **43** (2022), 781–807.

[22] G. Simpson, permute: Functions for generating restricted permutations of data. R package version 0.9-7, 2022, `https://CRAN.R-project.org/package=permute`.

[23] M. A. Tirlea, Permutation-based inference in time series analysis, PhD Thesis, Stanford University, 2023.

[24] J. Yan, Enjoy the joy of copulas: with a package copula, Journal of Statistical Software **21** (2007), 1–21. `https://www.jstatsoft.org/v21/i04/`.

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PERMUTACIJSKI TEST REPNE NEZAVISNOSTI ZA ZAVISNE PROCESE

D. Brborović

Sažetak. U članku prezentiramo permutacijski test repne nezavisnosti za stacionarne procese koji zadovoljavaju tzv. uvjet jakog miješanja. Asimptotska ispravnost testa je opravdana činjenicom da su i testna statistika i njena permutacijska distribucija normalno distribuirane. Rezultati u ovom članku se u velikoj mjeri oslanjaju na rezulate iz članka Basrak and Brborović [1]. Karakteristike predloženog permutacijskog testa su dodatno ilustrirane simulacijskom studijom.