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COMPUTABLE SUBCONTINUA OF CIRCULARLY CHAINABLE CONTINUA

David Tarandek

Abstract. This paper explores, in computable metric spaces, circularly chainable continua which are not chainable. Given such a continuum K, if we endow it with semicomputability, its computability follows. Conditions under which semicomputability implies computability, typically topological, are extensively studied in the literature. When these conditions are not satisfied, it is natural to explore approximate approaches. In this article we investigate specific computable subcontinua of K. The main result establishes that, given two points on a semicomputable, circularly chainable, but non-chainable continuum K, one can approximate them by computable points such that there exists a computable subcontinuum connecting these approximations. As a consequence, given disjoint computably enumerable open sets U and V intersected by K, the intersection of K with the complement of their union necessarily contains a computable point, provided that this intersection is totally disconnected.

1. Introduction

A compact set $K \subseteq \mathbb{R}$ is said to be *computable* if it can be effectively approximated by finitely many rational points with any given precision. A compact set $K \subseteq \mathbb{R}$ is said to be *semicomputable* if there is a computable function $f : \mathbb{R} \to \mathbb{R}$ such that the set of its zeros equals K.

In $\mathbb{R}$ the following implication holds:

$$K \text{ computable} \implies K \text{ semicomputable}.$$

This also holds in more general ambient spaces such as $\mathbb{R}^n$, computable metric spaces and computable topological spaces.

The converse,

$$K \text{ semicomputable} \implies K \text{ computable},$$

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does not hold. A well known counterexample from Miller [12] goes like this: Let $\gamma \in [0,1]$ be a left-computable, but not computable, real number. Then the segment $[\gamma, 1]$ is semicomputable, but not computable.

But, for each $\epsilon > 0$ there exists $\gamma_\epsilon \in \langle \gamma, \gamma + \epsilon \rangle$ such that $[\gamma_\epsilon, 1]$ is computable. In that way we inner approximate the semicomputable set $[\gamma, 1]$ with the computable sets $[\gamma_\epsilon, 1]$.

This example illustrates two common approaches in computable analysis.

1. Which topological conditions render a semicomputable set K computable?

2. If this conditions aren't met, under which conditions can K be inner approximated by a desired class of computable sets?

Example 1.1.

- A semicomputable set homeomorphic to the circle $S^1 \subseteq \mathbb{R}^2$ is computable [12].
- Specker [15] shows that there exists a computable function $f : \mathbb{R} \to \mathbb{R}$ which has zeros, but none of them are computable. It follows that $K = f^{-1}(\{0\})$ is a semicomputable set that contains no computable points. In particular, K has no nonempty computable subsets.
- It is shown that K from the previous example is homeomorphic to the Cantor set. It is compact, but not connected. However, $(K \times [0,1]) \cup ([0,1] \times K) \subseteq \mathbb{R}^2$ is a compact and connected set, i.e., a *continuum*, which is semicomputable and has no nonempty computable subsets.
- There exists a contractible, locally contractible semicomputable curve in $\mathbb{R}^2$ which cannot be inner approximated by computable continua [11].

Topological properties have a significant impact on the behavior of sets with respect to approaches **1.** and **2.** Among other objects, arcs and topological circles are often studied in the literature, as well as their respective generalizations, *chainable* and *circularly chainable* continua. Here are some important results concerning approach **1.**

[6] : In a computable metric space, every semicomputable circularly chainable continuum that is not chainable is computable.

[5] : The same holds in a computable topological space. Also, every semicomputable continuum K that is chainable from a to b, where a and b are computable points, is computable.

[10] : A semicomputable manifold with a semicomputable boundary is computable.

We now present several important results regarding approach **2.**

[9] : If A is a semicomputable arc in a computable metric space with endpoints a and b, then for every $\epsilon > 0$ there exist computable points a' and b' such that $d(a, a') < \epsilon$, $d(b, b') < \epsilon$, and a computable arc A' whose endpoints are a' and b' such that $A' \subseteq A$.

[3] : The same holds in a computable topological space. The same conclusion holds for semicomputable chainable continua under the additional assumption of decomposability.

[8] : The same holds even without the decomposability assumption, if K is a semicomputable continuum chainable from a to b, where a is a computable point.

The first goal is to prove that, given a circularly chainable, but not chainable, continuum K in a computable metric space and two distinct points $a, b \in K$ we can find computable points $a', b' \in K$ arbitrarily close to a and b and a computable subcontinuum L of K chainable from a' to b'. Some similarities and differences are highlighted when contrasting this result with the approach **2.**

The second goal is to generalize a well known result stated in Pour-El, Richards [14]:

THEOREM 1.2. ***Computable Intermediate Value Theorem.*** *A computable function $f : [0,1] \to \mathbb{R}$ such that $f(0) < 0$, $f(1) > 0$ has a computable zero.*

Some generalizations have been studied in [9], with the most prominent result being:

THEOREM 1.3. *Let (X, d, α) be a computable metric space and let U and V be disjoint c.e. open sets in X. Let $S = X \setminus (U \cup V)$. Suppose K is a continuum in X chainable from a to b, where $a \in U$ and $b \in V$. Suppose K is a computable set and $K \cap S$ is totally disconnected. Then $K \cap S$ contains a computable point.*

We plan to show that the modification of Theorem 1.3 holds, where K is a circularly chainable, but not chainable, continuum which intersects both U and V.

Our choice of setting is a computable metric space. The result [1, Theorem 3.4], shows that every semicomputable set S in a computable topological space can be effectively embedded into the Hilbert cube and therefore computable topological spaces do not lead to a more general result.

2. Preliminaries

Here we state some basic definitions and facts about computable metric spaces. See [6, 9, 10, 16, 17].

Let $k \in \mathbb{N} \setminus \{0\}$. A function $f : \mathbb{N}^k \to \mathbb{Q}$ is said to be *computable* if there exist computable (i.e., recursive) functions $a, b, c : \mathbb{N}^k \to \mathbb{N}$ such that

$$f(x) = (-1)^{c(x)} \cdot \frac{a(x)}{b(x) + 1}$$

for each $x \in \mathbb{N}^k$. A function $f : \mathbb{N}^k \to \mathbb{R}$ is said to be *computable* if there exists a computable function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that

$$|f(x) - F(x, i)| < 2^{-i}$$

for all $x \in \mathbb{N}^k$, $i \in \mathbb{N}$.

Let (X, d) be a metric space, and let α be a sequence in X such that $\alpha(\mathbb{N})$ is a dense set in (X, d). We say that (X, d, α) is a *computable metric space* if the function

$$(i, j) \mapsto d(\alpha_i, \alpha_j) : \mathbb{N}^2 \to \mathbb{R}$$

is computable.

Let (X, d, α) be a computable metric space. A point $x \in X$ is said to be *computable* in (X, d, α) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$d\left(x, \alpha_{f(k)}\right) < 2^{-k}$$

for all $k \in \mathbb{N}$.

We fix computable functions

$$(j, i) \mapsto (j)_i : \mathbb{N}^2 \to \mathbb{N} \quad \text{and} \quad j \mapsto \overline{j} : \mathbb{N} \to \mathbb{N}$$

such that

$$\{((j)_0, (j)_1, \dots, (j)_{\overline{j}}) \mid j \in \mathbb{N}\}$$

is the set of all nonempty finite sequences in $\mathbb{N}$.

For $j \in \mathbb{N}$, let

$$[j] := \{(j)_0, \dots, (j)_{\overline{j}}\}.$$

Then each nonempty finite subset of $\mathbb{N}$ is equal to $[j]$ for some $j \in \mathbb{N}$.

If (X, d, α) is a computable metric space, $i \in \mathbb{N}$, and $q \in \mathbb{Q}$ with $q > 0$, we say that $B(\alpha_i, q)$ is a *rational open ball* in this space. Here, for $x \in X$ and $r > 0$, we denote by $B(x, r) = \{y \in X : d(x, y) < r\}$ the open ball of radius r centered at x. Let $\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$ be fixed computable functions such that

$$\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2,$$

and let $q : \mathbb{N} \to \mathbb{Q}$ be a fixed computable function whose image is the set of all positive rational numbers. Let $(\lambda_i)_{i \in \mathbb{N}}$ be the sequence of points in X defined by $\lambda_i = \alpha_{\tau_1(i)}$, and let $(\rho_i)_{i \in \mathbb{N}}$ be the sequence of rational numbers defined by $\rho_i = q_{\tau_2(i)}$. For $i \in \mathbb{N}$, we define

$$I_i = B(\lambda_i, \rho_i), \quad \hat{I}_i = \overline{B}(\lambda_i, \rho_i).$$

Note that $\{I_i \mid i \in \mathbb{N}\}$ is the set of all rational open balls in (X, d, α). Therefore, the sequence $(I_i)_{i \in \mathbb{N}}$ represents an effective enumeration of all rational open balls.

Let (X, d, α) be a computable metric space. Any finite union of rational open balls in this space is said to be a *rational open set*. For $j \in \mathbb{N}$ we define

$$J_j = \bigcup_{i\in[j]} I_i, \quad \hat{J}_j = \bigcup_{i\in[j]} \hat{I}_i.$$

Then (J_j) is an effective enumeration of all rational open sets in (X, d, α).

Let (X, d, α) be a computable metric space. For every $j \in \mathbb{N}$, we define

$$\operatorname{fdiam}(j) := \operatorname{diam}\{\lambda_u \mid u \in [j]\} + 2\max\{\rho_u \mid u \in [j]\},$$

and call it the *formal diameter* of J_j. This is formally a $\mathbb{N} \to \mathbb{R}$ function of j, not of J_j. We define a function fmesh : $\mathbb{N} \to \mathbb{R}$ by

$$\operatorname{fmesh}(l) = \max_{0\le p\le \bar{l}} \operatorname{fdiam}\big((l)_p\big).$$

It is straightforward to conclude that the functions fdiam and fmesh are computable [6, Proposition 2, Proposition 13].

Let (X, d) be a metric space, $A, B \subseteq X$, and $\varepsilon > 0$. We say that A and B are ε-close, and write $A \approx_\varepsilon B$, if

$$(\forall a \in A)(\exists b \in B)\,(d(a, b) < \varepsilon) \quad \text{and} \quad (\forall b \in B)(\exists a \in A)\,(d(a, b) < \varepsilon)\,.$$

If A and B are nonempty compact sets in (X, d), the number

$$\inf\{\varepsilon > 0 \mid A \approx_\varepsilon B\}$$

is called the *Hausdorff distance* from A to B, and it is denoted by $d_H(A, B)$.

It is not hard to check that, for $\varepsilon > 0$, we have $d_H(A, B) < \varepsilon$ if and only if $A \approx_\varepsilon B$.

Definition 2.1. *Let (X, d, α) be a computable metric space. We say that a compact set $S \subseteq X$ is* computable *in (X, d, α) if either $S = \emptyset$ or there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$S \approx_{2^{-k}} \{\alpha_i \mid i \in [f(k)]\}, \quad \textit{for all } k \in \mathbb{N}.$$

Definition 2.2. *Let (X, d, α) be a computable metric space.*

(i) A closed set $S \subseteq X$ is said to be computably enumerable (c.e.) *in (X, d, α) if the set*

$$\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$$

is computably enumerable.

(ii) A compact set $S \subseteq X$ is said to be semicomputable *in (X, d, α) if the set*

$$\{j \in \mathbb{N} \mid S \subseteq J_j\}$$

is computably enumerable.

(iii) An open set $U \subseteq X$ is said to be computably enumerable open in (X, d, α) if

$$U = \bigcup_{i\in A} I_i,$$

for some c.e. subset A of $\mathbb{N}$.

It is not hard to see that these definitions do not depend on the particular choices of the functions q, τ_1, τ_2, and $j \mapsto [j]$.

For compact sets K in (X, d, α) we have the following important equivalence from [7, Proposition 2.6]:

$$K \text{ is computable} \iff K \text{ is semicomputable and c.e.}$$

We state the following basic facts about computable functions of type $\mathbb{N}^k \to \mathbb{N}^n$ and $\mathbb{N}^k \to \mathbb{R}$:

Proposition 2.3.

(i) ***(Projection theorem)*** *Let $T \subseteq \mathbb{N}^{k+n}$ be a computably enumerable set. Then the set*

$$S = \{x \in \mathbb{N}^k \mid \exists y \in \mathbb{N}^n : (x, y) \in T\}$$

is computably enumerable.

(ii) ***(Single-valuedness theorem)*** *Suppose $T \subseteq \mathbb{N}^{k+n}$, $S_1 \subseteq \mathbb{N}^k$, and $S_2 \subseteq \mathbb{N}^n$ are computably enumerable sets such that for each $x \in S_1$ there exists $y \in S_2$ with $(x, y) \in T$. Then there exists a partial computable (partial recursive) function $f : S_1 \to \mathbb{N}^n$ such that $f(S_1) \subseteq S_2$ and*

$$(x, f(x)) \in T \quad \text{for each } x \in S_1.$$

(iii) If $S \subseteq \mathbb{N}^n$ is a computably enumerable set and $f : \mathbb{N}^k \to \mathbb{N}^n$ is a computable function, then the set $f^{-1}(S)$ is computably enumerable.

Proposition 2.4.

(i) If $f, g : \mathbb{N}^k \to \mathbb{R}$ are computable functions, then $f + g$ and $f - g$ are computable.

(ii) If $f, g : \mathbb{N}^k \to \mathbb{R}$ are computable functions, then the set

$$\{x \in \mathbb{N}^k \mid f(x) > g(x)\}$$

is computably enumerable.

For $m \in \mathbb{N}$, let $\mathbb{N}_m = \{0, \dots, m\}$. For $n \geq 1$, let

$$\mathbb{N}_m^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{N}_m\}.$$

We say that a function $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ is *effectively finitely valued* or *e.f.v.* if the function $\overline{\Phi} : \mathbb{N}^{k+n} \to \mathbb{N}$ defined by

$$\overline{\Phi}(x, y) = \chi_{\Phi(x)}(y), \quad x \in \mathbb{N}^k, \, y \in \mathbb{N}^n,$$

is computable (where $\chi_S : \mathbb{N}^n \to \mathbb{N}$ denotes the characteristic function of $S \subseteq \mathbb{N}^n$) and if there exists a computable function $\varphi : \mathbb{N}^k \to \mathbb{N}$ such that

$$\Phi(x) \subseteq \mathbb{N}_{\varphi(x)}^n, \quad \forall x \in \mathbb{N}^k.$$

In the following proposition, we state some elementary facts about e.f.v. functions.

PROPOSITION 2.5.

(i) *Let* $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ *and* $\Psi : \mathbb{N}^n \to \mathcal{P}(\mathbb{N}^m)$ *be e.f.v. functions. Let* $\Lambda : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^m)$ *be defined by:*

$$\Lambda(x) = \bigcup_{z \in \Phi(x)} \Psi(z), \quad \forall x \in \mathbb{N}^k.$$

Then Λ *is an e.f.v. function.*

(ii) *Let* $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ *be e.f.v. and let* $T \subseteq \mathbb{N}^n$ *be c.e. Then the set*

$$S = \{x \in \mathbb{N}^k \mid \Phi(x) \subseteq T\}$$

is c.e.

(iii) *If* $\Phi, \Psi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ *are e.f.v. functions, then the sets*

$$\{x \in \mathbb{N}^k \mid \Phi(x) = \Psi(x)\} \quad \textit{and} \quad \{x \in \mathbb{N}^k \mid \Phi(x) \subseteq \Psi(x)\}$$

are computable.

3. Chains and circular chains

DEFINITION 3.1. *Let* X *be a set. Let* $\mathcal{C} = (C_0, \dots, C_m)$ *be a finite sequence of subsets of* X*. Then* $\mathcal{C}$ *is said to be a* chain *in* X *if for all* $i, j \in \{0, \dots m\}$

$$|i - j| > 1 \iff C_i \cap C_j = \emptyset.$$

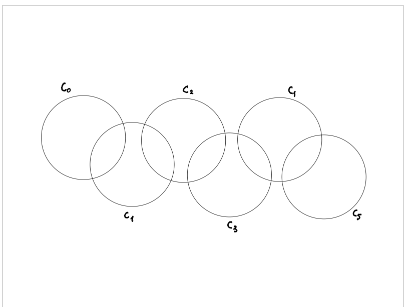
The finite sequence $\mathcal{C}$ *is said to be a* circular chain *in* X *if for all* $i, j \in \{0, \dots m\}$

$$1 < |i - j| < m \iff C_i \cap C_j = \emptyset.$$

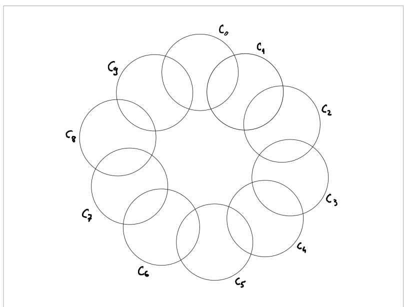
If for all $i, j \in \{0, \dots m\}$

$$|i - j| > 1 \implies C_i \cap C_j = \emptyset,$$

we call $\mathcal{C}$ *a* quasi-chain.



(A) A chain in (X, d)



(B) A circular chain in (X, d)

For a (circular) chain $\mathcal{C} = (C_0, \ldots, C_m)$ *and a natural number* $i \in \{0, \ldots, m\}$ *we call* C_i *a link of* $\mathcal{C}$. *If* $\mathcal{C}$ *is a chain,* C_0 *and* C_m *are called* end links.

If $\mathcal{A} = (A_0, \ldots, A_n)$ *is a finite sequence of bounded nonempty subsets of* (X, d)*, we define*

$$\operatorname{mesh}(\mathcal{A}) = \max_{0 \leq i \leq n} \operatorname{diam}(A_i).$$

Let $\mathcal{C}$ *be a (circular) chain and* $\epsilon > 0$. *If* $\operatorname{mesh}(\mathcal{C}) < \epsilon$*, we say that* $\mathcal{C}$ *is an* ϵ*-(circular) chain.*

A (circular) chain in some metric space (X, d) *is said to be an open (circular) chain in (X, d) if each of its links is an open set in* (X, d). *Similarly, a (circular) chain* (X, d) *is said to be a compact (circular) chain in* (X, d) *if each of its links is a compact set in* (X, d).

Remark 3.2. Let X be a set. We note the following:

- Let $\mathcal{C} = (C_0, \ldots, C_m)$ be a circular chain. Each link of $\mathcal{C}$ is a nonempty set by definition.
- If C_0 is a nonempty set in X, the sequence (C_0) is both a chain and a circular chain. The same holds for the sequence (C_0, C_0).
- Let $m \geq 2$ and $\mathcal{C} = (C_0, \ldots, C_m)$ be a chain in X. We claim: if $i \neq j$, then $C_i \neq C_j$. Namely, if $|i - j| > 1$, then C_i and C_j are disjont. Since they are nonempty, we have $C_i \neq C_j$. If $|i - j| = 1$, we can without loss of generality assume $i < j$. Then $i \in \{0, \ldots m - 1\}, j = i + 1$. Then we suppose the opposite, $C_i = C_{i+1}$. If $i < m - 1$, then the link C_{i+2} intersects the link $C_{i+1} = C_i$, so we have derived a contradiction. If $i = m - 1$, then the link C_{m-2} intersects $C_{m-1} = C_m$, again a contradiction.
- For nonempty intersecting sets C_0 and C_1 in X, the sequence (C_0, C_1) is both a chain and a circular chain. It easily follows that collections of two-link chains and two-link circular chains coincide.
- Let $\mathcal{C} = (C_0, \ldots, C_m)$ be a sequence of sets in X with $m \geq 2$. Then $\mathcal{C}$ can not be both a chain and a circular chain. If $\mathcal{C}$ is a chain, then C_0 and C_m are disjoint, so $\mathcal{C}$ is not a circular chain.
- Let $\mathcal{C} = (C_0, \ldots, C_m)$ be a chain in X. If $m < 2$, then we say that $\mathcal{C}$ is *trivial.* If $m \geq 2$, we say that $\mathcal{C}$ is *non-trivial.* To summarize: non-trivial chains cannot be circular, and their links are mutually distinct sets.

Definition 3.3. *Suppose* X *is a set,* $S \subseteq X$ *and* $\mathcal{C} = (C_0, \ldots, C_m)$ *a finite sequence of subsets of* X. *We say that* $\mathcal{C}$ *covers* S *if* $S \subseteq C_0 \cup \cdots \cup C_m$. *If* $a, b \in X$*, we say that* $\mathcal{C}$ *covers* S *from* a *to* b *if* $\mathcal{C}$ *covers* S *and* $a \in C_0, b \in C_m$.

Let (K, d) *be a continuum. We say that* (K, d) *is a (circularly) chainable continuum if for each* $\epsilon > 0$ *there exists an open* ϵ*-(circular) chain*

$(C_0, \dots, C_m)$ in (K, d) which covers K. If $a, b \in K$, we say that (K, d) is a continuum chainable from a to b if for each $\epsilon > 0$ there exists an open ϵ-chain $(C_0, \dots, C_m)$ in (K, d) which covers K from a to b.

We state a well-known fact from [9]:

PROPOSITION 3.4 ([9, Proposition 3.3, Proposition 6.11]). *Let (X, d) be a continuum. Then (X, d) is (circularly) chainable if and only if for each $\epsilon > 0$ there exists a compact (circular) ϵ-chain in (X, d) which covers X.*

REMARK 3.5. Let (X, d) be a continuum. If X is not chainable, then there exists $\epsilon_0 > 0$ such that no compact ϵ_0-chain covers X. Using the Lebesgue number lemma, it can easily be seen that then no open ϵ_0-chain covers X. If this is the case, we will say that X is not ϵ_0-chainable.

EXAMPLE 3.6. Here we list some well-known examples (see [2, 4, 6, 13]):

- Closed interval $[0, 1]$ is chainable from 0 to 1, but not circularly chainable.
- An *arc* is a topological space homeomorphic to $[0, 1]$. If A is an arc and $f : [0, 1] \to A$ a homeomorphism, then $a = f(0)$ and $b = f(1)$ are called its *endpoints*. A is then chainable from a to b.
- The unit circle S^1 in $\mathbb{R}^2$ is circularly chainable, but not chainable.
- Topological circles (homeomorphic images of S^1) are also circularly chainable, but not chainable.
- *Closed topological sine curve.* Let
$$K = \left\{ (x, \sin\left(\frac{1}{x}\right)) : 0 < x \leq 1 \right\} \cup \{(0, y) : y \in [-1, 1]\}.$$
Let $a = (0, -1)$, $b = (0, 0)$, $c = (1, \sin 1)$. K is a continuum chainable from a to c. K is not chainable from b to c.

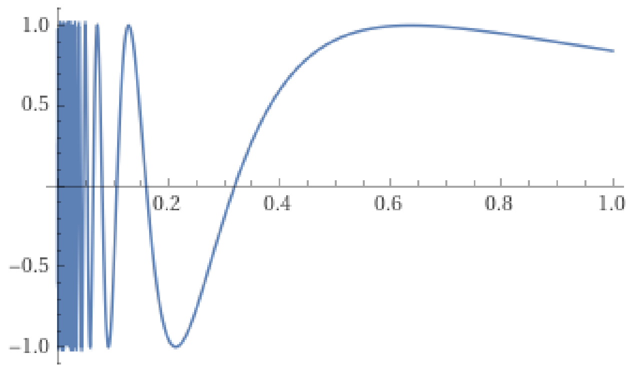


FIGURE 2. Closed topological sine curve

- The *Warsaw circle*. Let K be the curve from the previous example. We define $W = K \cup (\{0\} \times [-2,-1]) \cup ([0,1] \times \{-2\}) \cup (\{1\} \times [-2, \sin 1])$. W is a circularly chainable continuum.

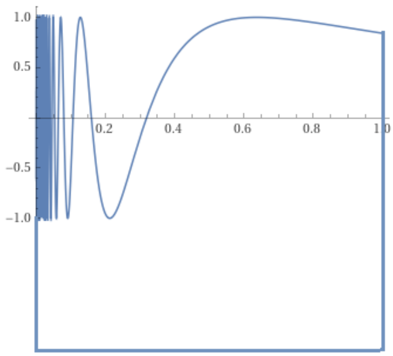


FIGURE 3. The Warsaw circle

- Let us consider the *double topological sine curve*:

$$D = \left\{ (x, \sin\left(\left|\frac{1}{x}\right|\right)) \mid x \in [-1,1] \setminus \{0\} \right\} \cup \{(0,y) : y \in [-1,1]\},$$

and now we add some line segments to construct a closed curve

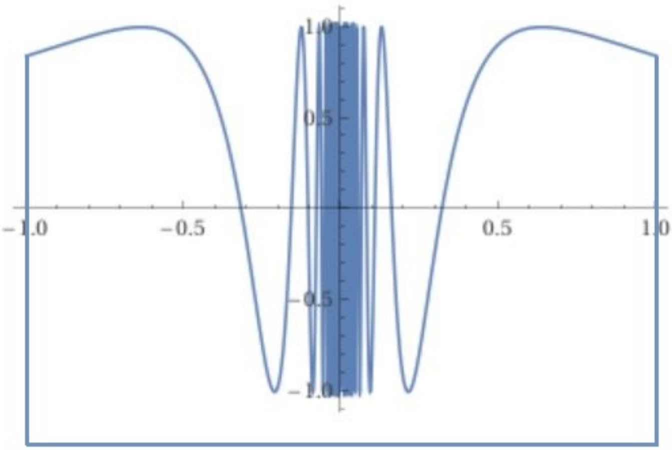


FIGURE 4. Double topological sine curve

$$D' = D \cup (\{-1\} \times [-2, \sin 1]) \cup ([-1, 1] \times \{-2\}) \\ \cup (\{1\} \times [-2, \sin 1]).$$

Then D' is a circularly chainable continuum, but it is not chainable. Notice that if $a = (0, 0)$ and $b = (1, \sin 1)$ there doesn't exist a subcontinuum of D' chainable from a to b. Furthermore, if subcontinuum L of D' is chainable from $a = (0, 0)$ to some $b \in D'$, then $b \in \{0\} \times [-1, 1]$.

- A continuum is called *decomposable* if $K = K_1 \cup K_2$, where K_1 and K_2 are proper subcontinua of K. A continuum is called *indecomposable* if it is not decomposable. A decomposable continuum K is said to be 2-*indecomposable* if there exist no subcontinua K_1, K_2 and K_3 of K such that $K = K_1 \cup K_2 \cup K_3$ and $K_1 \not\subseteq K_2 \cup K_3$, $K_2 \not\subseteq K_1 \cup K_3$ and $K_3 \not\subseteq K_1 \cup K_2$.

  A continuum can be both chainable and circularly chainable. Then it is either indecomposable or 2-indecomposable. In this article, we focus on circularly chainable, but not chainable, continua.

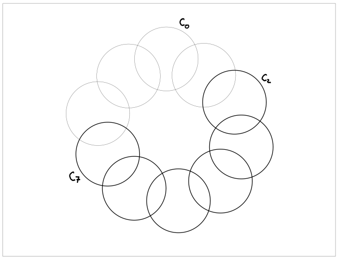
REMARK 3.7. Nadler [13] also uses the following terminology for continua:

- chainable: *arc-like*,
- circularly chainable: *circle-like*,
- circularly chainable but not chainable: *proper circle-like.*

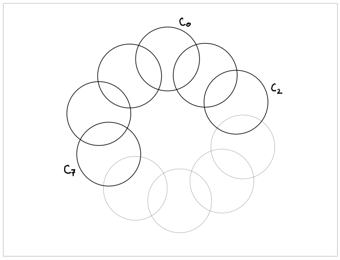
DEFINITION 3.8 (subchain). *Let $\mathcal{C} = (C_0, \dots C_m)$ be a circular chain. For $k, l \in \{0, \dots m\}$ we define a following sequence. If $k \leq l$ let $\mathcal{C}_{k,l} = (C_k, \dots, C_l)$. If $k > l$ let $\mathcal{C}_{k,l} = (C_k, \dots, C_m, C_0, \dots, C_l)$. If $l \neq k - 1$ and $(k, l) \neq (0, m)$ we say that $\mathcal{C}_{k,l}$ is a* subchain *of $\mathcal{C}$, and we write $\mathcal{C}_{k,l} \leq \mathcal{C}$.*

REMARK 3.9.

- Note that for $l = k - 1$, sequence $\mathcal{C}_{k,l}$ is a circular chain, and therefore not a chain in general. Same holds for $(k, l) = (0, m)$.
- If $l \neq k - 1$ and $(k, l) \neq (0, m)$, then $\mathcal{C}_{k,l}$ is a chain.
- Also note that for $l \neq k$ and $m \geq 3$ we have $\mathcal{C}_{k,l} \neq \mathcal{C}_{l,k}$.
- We can obtain a subchain of $\mathcal{C}$ by omitting one of its links. For example, if $1 \leq k \leq m-1$, then by omitting C_k we get $(C_{k+1}, C_{k+2}, \dots, C_{k-1}) = \mathcal{C}_{k+1,k-1}$.
- Next we note that for nonempty subset C_0 of X, sequence (C_0, C_0, C_0) is a circular chain but not a chain.
- Let $m \geq 3$ and let $\mathcal{C} = (C_0, \dots, C_m)$ be a circular chain in X. If $i \neq j$ then $C_i \neq C_j$. For $i \in \{0, \dots, m\}$ we omit the link C_i to obtain a chain whose all links are pairwise distinct (by Remark 3.2). From this it follows that all links of $\mathcal{C}$ are pairwise distinct.
- Let $\mathcal{C} = (C_0, \dots, C_m)$ be a circular chain in X. If $m < 3$, we will call such $\mathcal{C}$ *trivial*, and if $m \geq 3$ we will call it *non-trivial.*
- By omitting a link of a non-trivial circular chain, we obtain a non-trivial chain.



(A) $\mathcal{C}_{2,7}$



(B) $\mathcal{C}_{7,2}$

Definition 3.10. *Suppose X is a set, let $\mathcal{C} = (C_0, \ldots, C_m)$ and $\mathcal{D} = (D_0, \ldots, D_n)$ be sequences of sets in X. If for each $i \in \{0, \ldots, n\}$ there exists $j \in \{0, \ldots, m\}$ such that $D_i \subseteq C_j$ we say that $\mathcal{D}$* refines *$\mathcal{C}$. If $\mathcal{D}$ refines $\mathcal{C}$ and both $D_0 \subseteq C_0, D_n \subseteq C_m$, we say that $\mathcal{D}$* strictly refines *$\mathcal{C}$.*

Definition 3.11. *Let $(C_0, \ldots, C_m)$ be a sequence of sets in X. We define the* counter-sequence *of $\mathcal{C}$:*

$$-\mathcal{C} = (C_m, C_{m-1}, \ldots, C_0)$$

.

We note:

- If $\mathcal{C}$ is a (circular) chain, then $-\mathcal{C}$ is also a (circular) chain.
- For a chain $\mathcal{C}'$ and a circular chain $\mathcal{C}$ we have: If $\mathcal{C}' \leq \mathcal{C}$, then $-\mathcal{C}' \leq -\mathcal{C}$.
- If $\mathcal{C}$ is sequence of sets, $-(-\mathcal{C})) = \mathcal{C}$.

Let X be a set and let $\mathcal{C} = (C_0, \ldots, C_m)$ be a sequence of sets in X. Let $\text{links}(\mathcal{C}) = \{C_0, \ldots, C_m\}$, the set of all links of $\mathcal{C}$. Obviously, $\text{links}(\mathcal{C}) = \text{links}(-\mathcal{C})$.

Lemma 3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be non-trivial chains such that $\text{links}(\mathcal{C}) = \text{links}(\mathcal{D})$. Then $\mathcal{C} = \mathcal{D}$ or $\mathcal{C} = -\mathcal{D}$.*

Proof. Let $\mathcal{C} = (C_0, \ldots, C_m)$, $\mathcal{D} = (D_0, \ldots, D_n)$. From Remark 3.2 we conclude that $|\,\text{links}(\mathcal{C})| = m + 1$ and $|\,\text{links}(\mathcal{D})| = n + 1$. From this we conclude that $m = n$. We have $C_0 = D_i$, for $0 \leq i \leq n$. We observe that the only sets that C_0 intersects are itself and C_1. Every link D_i such that $0 < i < n$ intersects itself, D_{i-1} and D_{i+1}. Since $D_{i-1} \neq D_{i+1}$, we conclude that $i = 0$ or $i = n$.

If $i = 0$, we observe that C_1 intersects $C_0 = D_0$ but is itself not equal to it. The only link of $\mathcal{D}$ that intersects D_0 other than itself is D_1. Therefore, $C_1 = D_1$. Similarly, we conclude $C_i = D_i$ for all $0 \leq i \leq n$, so $\mathcal{C} = \mathcal{D}$.

If $i = n$, we observe that C_1 intersects $C_0 = D_n$ but is not equal to it. The only link of $\mathcal{D}$ that intersects D_n other than itself is D_{n-1}. Therefore,

$C_1 = D_{n-1}$. Similarly, we conclude $C_i = D_{n-i}$ for all $0 \le i \le n$, so $\mathcal{C} = -\mathcal{D}$. ∎

Definition 3.13. *Let X be a set and $\mathcal{C} = (C_0, \dots, C_m)$ a circular chain in X. For $0 \le k \le m$ we define*

$$\mathcal{C}_{+k} = (C_k, C_{k+1}, \dots, C_m, C_0, \dots, C_{k-1}).$$

In addition, we define $\mathcal{C}_{+(m+1)} = \mathcal{C}_{+0}$. Informally, the operation $\mathcal{C} \mapsto \mathcal{C}_{+k}$ will be referred to as a *rotation.* The operation $\mathcal{C} \mapsto -\mathcal{C}$ will be referred to as a *reflection.* We observe:

- $\mathcal{C}_{+0} = \mathcal{C}$;
- for each $0 \le k \le m$, $\mathcal{C}_{+k}$ is a circular chain and $\text{links}(\mathcal{C}) = \text{links}(\mathcal{C}_{+k})$;
- for a chain $\mathcal{C}'$, if $\mathcal{C}' \le \mathcal{C}$, then $\mathcal{C}' \le \mathcal{C}_{+k}$, for all $0 \le k \le m$;
- rotating and then reflecting a circular chain can be obtain by first reflecting and then rotating it, $-\mathcal{C}_{+k} = (-C)_{+(m-k+1)}$.

Lemma 3.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be non-trivial circular chains such that* $\text{links}(\mathcal{C}) = \text{links}(\mathcal{D})$. *Then there exists $k = 0, \dots, m$ such that $\mathcal{C} = \mathcal{D}_{+k}$ or $\mathcal{C} = -\mathcal{D}_{+k}$.*

Proof. Let $\mathcal{C} = (C_0, \dots, C_m)$, $\mathcal{D} = (D_0, \dots, D_n)$. Next we define $\mathcal{C}' = (C_0, \dots, C_{m-1})$. This sequence is obtained from $\mathcal{C}$ by omitting its last link C_m. From Remark 3.9 we know that such sequence is a non-trivial chain. Then $C_m = D_i$, for some $i \in \{0, \dots, n\}$. We define $\mathcal{D}'$ by omitting D_i from $\mathcal{D}$, i.e. $\mathcal{D}' = (D_{i+1}, D_{i+2}, \dots, D_n, D_0, \dots D_{i-1})$. Since $\text{links}(\mathcal{C}') = \text{links}(\mathcal{D}')$, either $\mathcal{C}' = \mathcal{D}'$ or $\mathcal{C}' = -\mathcal{D}'$.

If we look $\mathcal{C}' = \mathcal{D}'$ componentwise, we get $(C_0, \dots, C_{m-1}) = (D_{i+1}, D_{i+2}, \dots, D_{i-1})$. By appending $C_m = D_i$ to the end of the sequences, we get $(C_0, \dots, C_m) = (D_{i+1}, D_{i+2}, \dots, D_i)$, i. e. $\mathcal{C} = \mathcal{D}_{+(i+1)}$.

If $\mathcal{C}' = -\mathcal{D}'$, then we have

$$(C_0, \dots, C_{m-1}) = (D_{i-1}, D_{i-2}, \dots, D_0, D_n, \dots, D_{i+1}).$$

By appending $C_m = D_i$ to the end of the sequences, we get $(C_0, \dots, C_m) = (D_{i-1}, D_{i-1}, \dots, D_0, D_n, \dots, D_i)$, i. e. $\mathcal{C} = -\mathcal{D}_{+i}$. ∎

Remark 3.15. Let $\mathcal{C} = (C_0, \dots, C_m)$ be a nontrivial circular chain. Now we can describe the family of all circular chains $\mathcal{D}$ such that $\text{links}(\mathcal{D}) = \text{links}(\mathcal{C})$. It consists of $m+1$ circular chains of the form $C_{+k}, k \in \{0, \dots, m\}$ (which are said to have the *same orientation* as $\mathcal{C}$) and $m+1$ circular chains of the form $(-\mathcal{C})_{+k}, k \in \{0, \dots, m\}$ (which are said to have the *opposite orientation* with regard to $\mathcal{C}$).

Proposition 3.16. *Let (X, d) be a metric space and $S \subseteq X$ a circularly chainable continuum. Then for each open circular chain $\mathcal{C}$ that covers S and each $\epsilon > 0$ there exists a compact ϵ-circular chain $\mathcal{D}$ in S which covers S and refines $\mathcal{C}$.*

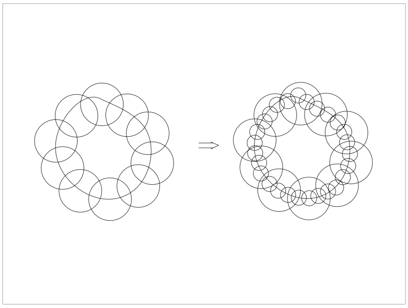


FIGURE 6. Refinement

PROOF. Let $\mathcal{C} = (C_0, \dots, C_m)$ be a circular chain that covers S and let $\epsilon > 0$. Then the set $\{C_0 \cap S, \dots, C_m \cap S\}$ is an open cover of S. Since S is compact, there exists $\lambda > 0$, a Lebesgue number of this cover. This means that for each $A \subseteq S$ such that $\operatorname{diam}(A) < \lambda$ there exists a member of this cover which contains A. By Proposition 3.4, there exists a compact $\min\{\epsilon, \lambda\}$-circular chain $\mathcal{D} = (D_0, \dots, D_n)$ in S which covers S. For each $i \in \{0, \dots, n\}$, since $\operatorname{diam} D_i < \lambda$, there exists $j \in \{0, \dots, m\}$ such that $D_i \subseteq C_j \cap S \subseteq C_j$, so $\mathcal{D}$ refines $\mathcal{C}$. ◻

Two main results of this section follow, Lemma 3.17 and Theorem 3.19.

LEMMA 3.17. *Let X be a set, $\mathcal{C} = (C_0, \dots, C_m)$ a circular chain in X and $k \in \{1, \dots, m-1\}$. Let $\mathcal{D} = (D_0, \dots, D_n)$ be a compact circular chain in X which refines $\mathcal{C}$. Suppose there is a subchain $\mathcal{D}' \leq \mathcal{D}$ whose one end link, D_a is contained in C_0, and other end link is D_i, $\mathcal{D}'$ refines $(C_0, \dots, C_{k-1})$. Furthermore suppose there is a subchain $\mathcal{D}'' \leq \mathcal{D}$ whose one end link, D_b is contained in C_k, and other end link is D_j, $\mathcal{D}''$ refines $(C_1, \dots, C_k)$. Let D_i and D_j intersect. Then there exists a subchain of $\mathcal{D}$ which strictly refines $(C_0, \dots C_k)$ or $-(C_0, \dots C_k)$.*

PROOF. We start by observing that either $\mathcal{D}' = \mathcal{D}_{a,i}$ or $\mathcal{D}' = \mathcal{D}_{i,a}$. Next, we consider the simple **Case I.** where $\mathcal{D}' = \mathcal{D}_{0,i}$. Then we consider **Case II.**, with $\mathcal{D}' = \mathcal{D}_{a,i}$, and **Case III.** $\mathcal{D}' = \mathcal{D}_{i,a}$. We reduce Case II. and Case III. to Case I.

Case I. Let $\mathcal{D}' = \mathcal{D}_{0,i}$. Subchain $\mathcal{D}''$ can then be obtained in two ways: **a)** $\mathcal{D}'' = \mathcal{D}_{j,b}$ and **b)** $\mathcal{D}'' = \mathcal{D}_{b,j}$.

First, **a)** $\mathcal{D}'' = \mathcal{D}_{j,b}$. We will now examine subcases dependent on the position of index b: **1.** $j < b < n$, **2.** $b \leq j < n$, **3.** $j \leq b = n$ and **4.** $b < j = n$.

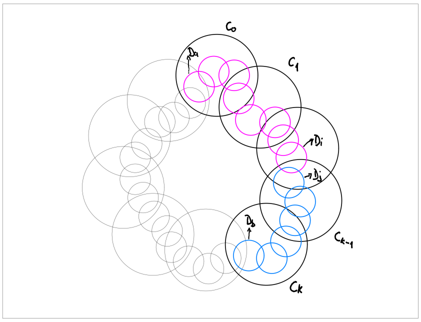


FIGURE 7. Subchain addition

1. If $j < b < n$, we consider $\mathcal{D}_{0,b} = D_0, \dots, D_b$. In general we can imagine this sequence as either

$$\begin{aligned} &\text{A) } D_0, \dots, D_j, D_i, \dots, D_b, \\ &\text{B) } D_0, \dots, D_i = D_j, \dots, D_b, \\ \text{or } &\text{C) } D_0, \dots, D_i, D_j, \dots, D_b, \end{aligned}$$

three possibilities depending on the placement of intersecting links D_i and D_j. Notice that in such cases each link in $\mathcal{D}_{0,b}$ is contained in $\mathcal{D}'$ or in $\mathcal{D}''$. Since $\mathcal{D}'$ refines $\mathcal{C}_{0,k-1}$ and $\mathcal{D}''$ refines $\mathcal{C}_{1,k}$, it follows that $\mathcal{D}_{0,b}$ refines $\mathcal{C}_{0,k}$. Strict refinement follows from the fact that $D_0 \subseteq C_0$ and $D_b \subseteq C_k$. Since $b < n$, $\mathcal{D}_{0,b}$ is a subchain of $\mathcal{D}$.

2. If $b \le j < n$, then we consider if $b \le i$. If that is the case, then $\mathcal{D}_{0,b}$ is a subsequence of $\mathcal{D}_{0,i} = \mathcal{D}'$.

If $b > i$, then $\mathcal{D}_{0,b} = (D_0, \dots, D_i, D_b = D_j)$. Each link of $\mathcal{D}_{0,b}$ is in $\mathcal{D}'$ except $D_b = D_j$ which is in $\mathcal{D}'' = \mathcal{D}_{j,b}$. Again it holds that each link in $\mathcal{D}_{0,b}$ is contained in $\mathcal{D}'$ or in $\mathcal{D}''$. We follow the arguments from the above paragraph to prove that $\mathcal{D}_{0,b}$ is a subchain of $\mathcal{D}$ that strictly refines $\mathcal{C}_{0,k}$.

3. If $b = n$, we observe that $\mathcal{D}_{0,b}$ is now not a subchain of $\mathcal{D}$. We have now $D_{b=n} \subseteq C_k$ and $D_0 \subseteq C_0$. Since D_n and D_0 intersect, so do C_k and C_0. It follows $k \in \{0, 1, m\}$, but by the condition of the lemma only $k = 1$ is possible. Since $\mathcal{D}'$ refines $(C_0, \dots, C_{k-1} = C_0)$ it follows that $D_i \subseteq C_0$. Similarly, $\mathcal{D}''$ refines $(C_1, \dots, C_k = C_1)$, so $D_j \subseteq C_1$. If D_i is contained in C_1, then one-link chain (D_i) refines both C_0 and C_1 and therefore strictly refines $(C_0, \dots C_{k=1})$. If $D_i \nsubseteq C_1$, then D_i is not a link of $\mathcal{D}'' = D_j, \dots, D_b$, and then $i < j$ holds. If $j = n$ and $i = 0$, then (D_j, D_i) is a subchain of $\mathcal{D}$ which strictly refines $-(C_0, \dots, C_k)$. Otherwise, (D_i, D_j) is a subchain of $\mathcal{D}$ which strictly refines $(C_0, \dots, C_k)$.

4. If $b < j = n$, then D_i and $D_{n=j}$ intersect. Then $i \in \{n-1, n, 0\}$ follows. If $i = n-1$ then the circular chain $\mathcal{D}$ is equal to $(D_0, \dots, D_b, \dots, D_{i=n-1}, D_{j=n})$. Subsequence $\mathcal{D}_{0,b}$ is a subchain, and

$$\text{links}(\mathcal{D}_{0,b}) \subseteq \text{links}(\mathcal{D}'),$$

so again it follows that $\mathcal{D}_{0,b}$ is the solution. If $i = n$ then $\mathcal{D}' = \mathcal{D}_{0,i=n} = \mathcal{D}$. But $\mathcal{D}$ is a circular chain, which contradicts condition of lemma which states that $\mathcal{D}'$ is a subchain of $\mathcal{D}$. If $i = 0$ then $\mathcal{D}'' = (D_{j=n}, D_{i=0}, \dots, D_b)$. We can see that $\mathcal{D}_{0,b}$ is a chain and that $\text{links}(\mathcal{D}_{0,b}) \subseteq \text{links}(\mathcal{D}'')$.

b) Now, $\mathcal{D}'' = \mathcal{D}_{b,j}$. We consider the following subcases: **1.** $b \le j < n$, **2.** $j < b < n$, **3.** $b = n$, **4.** $b < j = n$.

1. $b \le j < n$. If $b \le i$, then $\mathcal{D}_{0,b}$ is a subsequence of $\mathcal{D}_{0,i} = \mathcal{D}'$.

If $b > i$, then $\mathcal{D}_{0,b} = (D_0, \dots, D_i, D_b = D_j)$.

In both cases each link in $\mathcal{D}_{0,b}$ is contained in $\mathcal{D}'$ or in $\mathcal{D}''$. Following the arguments from previous cases, we conclude that $\mathcal{D}_{0,b}$ is subchain of $\mathcal{D}$ which strictly refines $\mathcal{C}_{0,k}$.

2. If $j < b < n$, first we observe that $b \ne 1$. If $b = 1$, then $j = 0$ follows. But then the sequence $\mathcal{D}'' = \mathcal{D}_{b,j} = \mathcal{D}_{1,0}$ is not a subchain of $\mathcal{D}$. Next, we note that $\mathcal{D}_{b,0}$ is a subsequence of $\mathcal{D}_{b,j} = \mathcal{D}''$. The fact that $b \ne 1$ implies that $\mathcal{D}_{b,0}$ is a subchain of $\mathcal{D}$. In addition, $\mathcal{D}_{b,0}$ strictly refines $-(C_0, \dots, C_k)$.

3. If $b = n$, we again conclude that the two-link chain (D_n, D_0) strictly refines $-\mathcal{C}_{0,k}$.

4. If $b < j = n$, we consider the sequence $\mathcal{D}_{b,0} = (D_b, \dots D_{j=n}, D_0)$. The links $D_b, \dots D_{j=n}$ are contained in $\mathcal{D}''$, and the link D_0 is contained in $\mathcal{D}'$. So, if $\mathcal{D}_{b,0}$ is a chain, then it is the desired subchain of $\mathcal{D}$. If $\mathcal{D}_{b,0}$ is not a chain, then $b = 1$. Now we have the two-link chain $(D_0, D_{b=1})$ which strictly refines $\mathcal{C}_{0,k}$.

Case II. If $\mathcal{D}' = \mathcal{D}_{a,i}$ then we consider the circular chain $\mathcal{D}_{+a} = (D_a, D_{a+1}, \dots, D_{a-1})$.

- Subchains of $\mathcal{D}$ and $\mathcal{D}_{+a}$ coincide. Therefore $\mathcal{D}'$ and $\mathcal{D}''$ are subchains of $\mathcal{D}_{+a}$.
- Furthermore, $\mathcal{D}' = (\mathcal{D}_{+a})_{0,i'}$. We have in fact changed the enumeration of circular chain $\mathcal{D}$ so that starting link of subchain $\mathcal{D}'$ has index 0.
- We apply the method from **Case I.** to obtain subchain $\mathcal{E}$ of $\mathcal{D}_{+a}$ which strictly refines $\mathcal{C}_{0,k}$ or $-\mathcal{C}_{0,k}$. $\mathcal{E}$ is the desired solution, since it is a subchain of $\mathcal{D}$.

Case III. If $\mathcal{D}' = \mathcal{D}_{i,a}$ first we rotate: $\mathcal{D}_{+(a+1)} = (D_{a+1}, D_{a+2}, \dots, D_n, D_0, \dots, D_a)$. Next we reflect: $-\mathcal{D}_{+(a+1)} = (D_a, D_{a-1}, \dots, D_0, D_n, \dots, D_{a+1})$.

We note:

- If $\mathcal{E} \leq \mathcal{D}$, then $-\mathcal{E} \leq -\mathcal{D}_{+(a+1)}$. This entails that both $-\mathcal{D}'$ and $-\mathcal{D}''$ are subchains of $-\mathcal{D}_{+(a+1)}$.
- Moreover, $-\mathcal{D}' = (-\mathcal{D}_{+(a+1)})_{0,i'}$, i.e. we have rotated and reflected circular chain $\mathcal{D}$, so that $\mathcal{D}'$ again starts with link 0.
- Therefore, by applying method from **Case I.**, we obtain a subchain $\mathcal{E}$ of $-\mathcal{D}_{+(a+1)}$ such that it strictly refines $\mathcal{C}_{0,k}$ or $-\mathcal{C}_{0,k}$.
- If $\mathcal{E}$ strictly refines $\mathcal{C}_{0,k}$, then $-\mathcal{E}$ strictly refines $-\mathcal{C}_{0,k}$. Therefore $-\mathcal{E}$ is a subchain of $\mathcal{D}$ that strictly refines $\mathcal{C}_{0,k}$ or $-\mathcal{C}_{0,k}$.

□

REMARK 3.18. Notice that the previous lemma holds even without the compactness of $\mathcal{D}$.

THEOREM 3.19. *Let (X, d) be a metric space, $S \subseteq X$ a circularly chainable, but not chainable continuum. Let $\epsilon_0 > 0$ be such that no compact ϵ_0-chain covers S. Let $\mathcal{C} = (C_0, \dots, C_m)$ be an open circular ϵ_0-chain that covers S. For each compact circular chain $\mathcal{D} = (D_0, \dots, D_n)$ which covers S and $\mathcal{D} \leq \mathcal{C}$ and for each $k \in \{2, \dots, m-1\}$, there exists a subchain $\mathcal{D}'$ of $\mathcal{D}$ such that $\mathcal{D}'$ strictly refines $(C_0, \dots, C_k)$ or $-(C_0, \dots, C_k)$.*

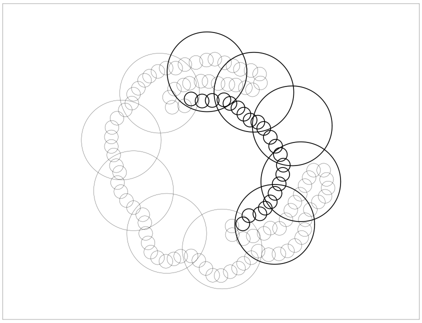


FIGURE 8. Strict refinement

PROOF. First we fix arbitrary $\mathcal{D}$ which covers S and refines $\mathcal{C}$, then we fix arbitrary $k \in \{2, \dots, m-1\}$. Let us suppose the following:

(*) no subchain of $\mathcal{D}$ strictly refines $(C_0, \dots, C_k)$ or $-(C_0, \dots, C_k)$,

and derive a contradiction.

Let us consider the set A of all $i \in \{0, \dots n\}$ that are connected to C_0, i.e. such that there exists a subchain $\mathcal{D}'$ of $\mathcal{D}$ in which one end link is D_i, and the other is contained in C_0 and $\mathcal{D}'$ refines $(C_0, \dots, C_{k-1})$. Similarly, let B be a set of all $i \in \{0, \dots n\}$ that are connected to C_k, i.e. such that there exists a subchain $\mathcal{D}''$ of $\mathcal{D}$ in which one end link is D_i, and the other is contained in C_k and $\mathcal{D}''$ refines $(C_1, \ldots, C_k)$.

Now let

$$C_1' = \bigcup_{\substack{i \in B \\ D_i \subseteq C_1}} D_i$$

$$\vdots$$

$$C_{k-1}' = \bigcup_{\substack{i \in B \\ D_i \subseteq C_{k-1}}} D_i$$

$$C_k' = \bigcup_{D_i \subseteq C_k} D_i$$

$$\vdots$$

$$C_m' = \bigcup_{D_i \subseteq C_m} D_i$$

$$C_0' = \bigcup_{D_i \subseteq C_0} D_i$$

$$C_1'' = \bigcup_{\substack{i \in A \\ D_i \subseteq C_1}} D_i$$

$$\vdots$$

$$C_{k-1}'' = \bigcup_{\substack{i \in A \\ D_i \subseteq C_{k-1}}} D_i$$

Let us consider the sequence of sets

$$\mathcal{C}' = (C_1', \ldots, C_k', \ldots, C_m', C_0', C_1'', \ldots, C_{k-1}'').$$

After noting that some of this sets might be empty, we claim that the sequence a) covers S, b) is a quasi-chain.

a) We will prove that $\mathcal{D}$ refines $\mathcal{C}'$. Then, since $\mathcal{D}$ covers S, so does $\mathcal{C}'$. Let us choose arbitrary D_i, for $i \in \{0, \ldots, m\}$. Since $\mathcal{D}$ refines $\mathcal{C}$, $D_i \subseteq C_0 \cup \cdots \cup C_m$. If D_i is contained in one of the sets $C_k, \ldots, C_m, C_0$, then by definition D_i is contained in one of the sets $C_k', \ldots, C_m', C_0'$.

If D_i is not contained in one of the sets $C_k, \ldots, C_m, C_0$, then it must be contained in one of the sets $C_1, \ldots, C_{k-1}$. In this case it will follow that $i \in A$ or $i \in B$. Note that there exists a link of $\mathcal{D}$ which isn't contained in $C_1 \cup \cdots \cup C_{k-1}$. Otherwise $\mathcal{D}$ refines $(C_1, \ldots, C_{k-1})$. Since $\mathcal{D}$ covers S, so does $(C_1, \ldots, C_{k-1})$. But then S is covered by an open ϵ_0-chain. Using

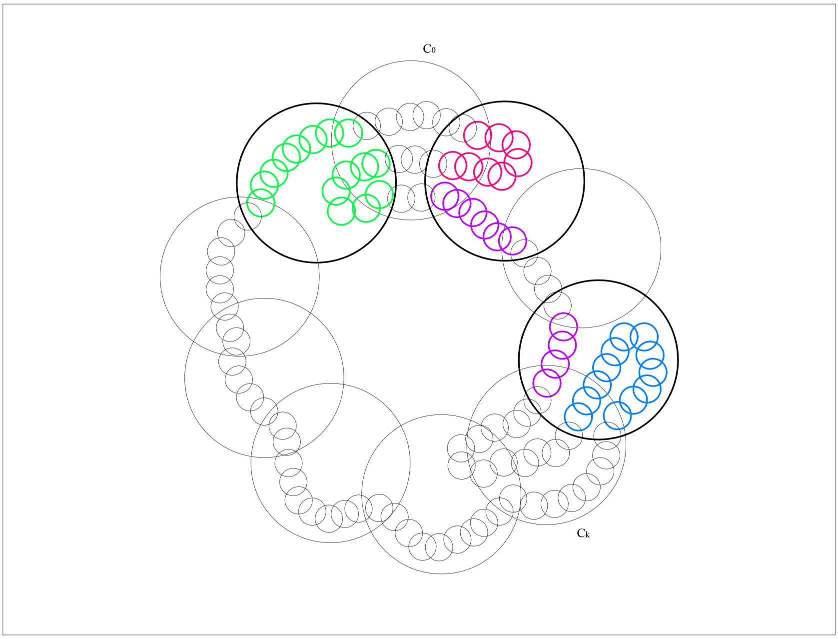


FIGURE 9. Links of $\mathcal{C}'$

the Lebesgue number lemma, we can easily construct a compact ϵ_0-chain that covers S. This is in contradiction with assumption that S can not be covered by a compact ϵ_0-chain. Notice that if D_j is contained in one of the sets $C_1, \dots, C_{k-1}$, then D_{j+1} is either also contained in one of the sets $C_1, \dots, C_{k-1}$, or $D_{j+1} \subseteq C_0$ or $D_{j+1} \subseteq C_k$. Therefore there exists $l \in \{0, \dots, m\}$ such that $(D_i, \dots D_{l-1})$ refines $(C_1, \dots C_{k-1})$ and $D_l \subseteq C_0$ or $D_l \subseteq C_k$.

- If $D_l \subseteq C_0$, then $(D_i, \dots, D_{l-1}, D_l)$ refines $(C_0, \dots, C_{k-1})$, one end link is D_i, and the other end link, D_l, is contained in C_0. Therefore $i \in A$. It follows that D_i is contained in one of the sets $C''_1, \dots, C''_{k-1}$.
- If $D_l \subseteq C_k$, then $(D_i, \dots, D_{l-1}, D_l)$ refines $(C_1, \dots, C_k)$, one end link is D_i, and the other end link, D_l, is contained in C_k. Therefore $i \in B$. It follows that D_i is contained in one of the sets $C'_1, \dots, C'_{k-1}$.

b) Now let us see that $(C'_1, \dots, C'_k, \dots, C'_m, C'_0, C''_1, \dots, C''_{k-1})$ is a quasi-chain. Let $\mathcal{B} = \{C'_1, \dots, C'_{k-1}\}$, $\mathcal{E} = \{C'_k, \dots, C'_m, C'_0\}$, $\mathcal{A} = \{C''_1, \dots, C''_{k-1}\}$. First we choose two non-neighbouring links from same family, for instance $C'_i, C'_j \in \mathcal{B}$. Since by their definition $C'_i \subseteq C_i$, $C'_j \subseteq C_j$, and C_i and C_j being disjoint, C'_i and C'_j must also be disjoint. Similarly, one concludes that two non-neighbouring links are disjoint if they are both in $\mathcal{E}$ or both in $\mathcal{A}$.

Now let us observe two non-neighbouring links where one is in $\mathcal{B}$ and the other in $\mathcal{E}$. We conclude that the only nontrivial possibility are sets C'_1 and C'_0. Now, let us suppose that they aren't disjoint i.e. there exists $x \in C'_1 \cap C'_0$. That means that there exist $i \in B$ and $j \in \{0, \dots, m\}$ such that $x \in D_i \subseteq C_1$ and $x \in D_j \subseteq C_0$. So, to summarize, one-link chain (D_j) refines $\mathcal{C}_{0,k-1}$, there exists a subchain of $\mathcal{D}$ which refines $\mathcal{C}_{1,k}$, whose one end link is in C_k, and the other end link D_i intersects D_j. By applying Lemma 3.17, it follows that there is a subchain of $\mathcal{D}$ which strictly refines $(C_0, \dots, C_k)$ or $-(C_0, \dots, C_k)$, so we have derived a contradiction with (*). So C'_1 and C'_0 are disjoint.

If we observe two non-neighbouring links where one is in $\mathcal{E}$ and the other in $\mathcal{A}$, we can see that only nontrivial possibilities are C'_k and C''_{k-1}. We proceed as in the previous paragraph.

If we take two non-neighbouring links where one is in $\mathcal{B}$ and the other in $\mathcal{A}$, we can see that only nontrivial possibilities are C'_u, C''_v such that C_u and C_v intersect. Let us suppose that C'_u and C''_v intersect, so there exist $i \in B$, $j \in A$ and $x \in X$ such that $x \in D_i \subseteq C_u$ and $x \in D_j \subseteq C_v$. So there is a subchain of $\mathcal{D}$ which refines $\mathcal{C}_{0,k-1}$, whose one end link is in C_0 and the other is D_j, there also exists a subchain of $\mathcal{D}$ which refines $\mathcal{C}_{1,k}$, whose one end link is in C_k, and the other end link D_i intersects D_j. By applying Lemma 3.17, it follows again that there is a subchain of $\mathcal{D}$ which strictly refines $(C_0, \dots, C_k)$ or $-(C_0, \dots, C_k)$, so we have derived a contradiction. So C'_u and C''_v are disjoint.

So by now we have proven that $\mathcal{C}'$ is a quasi-chain which covers S. By construction, $\mathcal{C}'$ refines $\mathcal{C}$, so $\mathcal{C}'$ is an ϵ_0-quasi-chain. By removing empty links, we construct a compact ϵ_0-chain which covers S. But this is in contradiction with our assumption that no compact ϵ_0-chain covers S. Therefore we must reject (*), so there exists subchain $\mathcal{D}'$ of $\mathcal{D}$ such that $\mathcal{D}'$ strictly refines $\mathcal{C}_{0,k}$ or $-\mathcal{C}_{0,k}$. ∎

4. Formal circular chains

Let (X, d, α) be a computable metric space and let $a \in X$ be a computable point in this space. Using Proposition 2.4 it is easy to conclude that the set $\{i \in \mathbb{N} \mid a \in I_i\}$ is c.e. From this it follows that the set $\{j \in \mathbb{N} \mid a \in J_j\}$ is also c.e. Consequently we have the following proposition:

Proposition 4.1. *Let (X, d, α) be a computable metric space and let $a \in X$ be a computable point in this space. Then*

$$\{l \in \mathbb{N} \mid a \in J_{(l)_0}\}$$

is a c.e. set.

Definition 4.2. *Let (X, d, α) be a computable metric space and let $v, w \in \mathbb{N}$. We say that I_v is formally contained in I_w and we write $I_v \subseteq_F I_w$ if*

$$d(\lambda_v, \lambda_w) + \rho_v < \rho_w.$$

Let $a, b \in \mathbb{N}$. We say that J_a is formally contained in J_b and we write $J_a \subseteq_F J_b$ if

$$\text{for each } u \in [a] \text{ there exists } v \in [b] \text{ such that } I_u \subseteq_F I_v.$$

Remark 4.3. Note that $I_u \subseteq_F I_v$ denotes a relation between the indices u and v, not between the sets I_u and I_v. The same holds for $J_a \subseteq_F J_b$.

If $I_u \subseteq_F I_v$, then $\hat{I}_u \subseteq I_v$. Therefore $J_a \subseteq_F J_b$ implies $\hat{J}_a \subseteq J_b$.

Proposition 4.4. *Let (X, d, α) be a computable metric space. Then*

$$A = \{(u, w) \in \mathbb{N}^2 \mid I_u \subseteq I_v\}$$

is a c.e. set.

Proof.

$$(u, v) \in A \iff d(\lambda_u, \lambda_v) + \rho_u < \rho_v,$$

and now we apply Proposition 2.4. ∎

Definition 4.5. *Let (X, d, α) be a computable metric space and let $i, j \in \mathbb{N}$. We say that I_i and I_j are formally disjoint, and we write $I_i \diamond I_j$, if*

$$d(\lambda_i, \lambda_j) > \rho_i + \rho_j.$$

Let $a, b \in \mathbb{N}$. We say that J_a and J_b are formally disjoint, and we write $J_a \diamond J_b$, if

$$I_i \diamond I_j, \quad \text{for every } i \in [a] \text{ and } j \in [b].$$

Remark 4.6. Formal disjointness is not a relation between the sets I_i and I_j, but between the indices i and j. The same holds for $J_a \diamond J_b$.

If $I_i \diamond I_j$, then $\hat{I}_i \cap \hat{I}_j = \emptyset$. Consequently, $J_a \diamond J_b$ implies $\hat{J}_a \cap \hat{J}_b = \emptyset$.

The following proposition easily follows from Remark 4.3, Remark 4.6 and the fact that in a computable metric space $\mathrm{Cl}(J_j) \subseteq \hat{J}_j$ holds.

Proposition 4.7. *Let (X, d, α) be a computable metric space and let $a, b \in \mathbb{N}$. Then the following implications hold:*

$$J_a \subseteq_F J_b \implies \mathrm{Cl}(J_a) \subseteq J_b,$$
$$J_a \diamond J_b \implies \mathrm{Cl}(J_a) \cap \mathrm{Cl}(J_b) = \emptyset.$$

Definition 4.8. *Let (X, d, α) be a computable metric space. Let $l \in \mathbb{N}$. We define*

$$\mathcal{H}_l = (J_{(l)_0}, \dots, J_{(l)_{\bar{l}}}).$$

We say that l represents a formal circular chain or, shortly, that $\mathcal{H}_l$ is a formal circular chain if for all $i, j \in \{0, \dots \bar{l}\}$

$$1 < |i - j| < \bar{l} \implies J_{(l)_i} \diamond J_{(l)_j}.$$

We define

$$\bigcup \mathcal{H}_l = \bigcup_{k \in [l]} J_k.$$

Using Propositions 2.3, 2.4 and 2.5, it is not hard to prove the following proposition. For details, see [6, Proposition 8] and [7, Proposition 5.4.].

PROPOSITION 4.9. *Let (X, d, α) be a computable metric space. The following sets are c.e.:*

$$\{(i, j) \in \mathbb{N}^2 \mid I_i \diamond I_j\},$$
$$\{(a, b) \in \mathbb{N}^2 \mid J_a \diamond J_b\},$$
$$\{(a, b) \in \mathbb{N}^2 \mid J_a \subseteq_F J_b\},$$
$$\{l \in \mathbb{N} \mid \mathcal{H}_l \textit{ is a formal circular chain}\}.$$

By [7, Proposition 5.3.] we have:

PROPOSITION 4.10. *Let K be a semicomputable set in a computable metric space (X, d, α). The set*

$$\{l \in \mathbb{N} \mid K \subseteq \bigcup \mathcal{H}_l\}$$

is c.e.

REMARK 4.11. Let $\epsilon_0 > 0$ and let K be a circularly chainable, but not ϵ_0-chainable, continuum in a computable metric space (X, d, α). Let $l \in \mathbb{N}$ be such that $\mathcal{H}_l$ is a formal circular chain which covers K and such that $\text{fmesh}(l) < \epsilon_0$. Then $(J_{(l_y)_0} \cap K, \dots, J_{(l_y)_{\overline{l_y}}} \cap K)$ is a circular chain in K. Namely, non-neighbouring links do not intersect because $\mathcal{H}_l$ is a formal circular chain. Neighbouring links intersect because otherwise there would exist an ϵ_0 chain in K covering K, which contradicts the fact that K is not ϵ_0-chainable.

DEFINITION 4.12. *Let (X, d, α) be a computable metric space. Let $l, n \in \mathbb{N}$. We say that $\mathcal{H}_l$ formally refines $\mathcal{H}_n$ and we write*

$$\mathcal{H}_l \leq \mathcal{H}_n,$$

if for all $i \in [l]$ there exists $j \in [n]$ such that $J_i \subseteq_F J_j$.

Similarly, we say that $(J_{a_0}, \dots, J_{a_k})$ formally refines $(J_{b_0}, \dots, J_{b_l})$ if for each $i \in \{0, \dots, k\}$ there exists $j \in \{0, \dots, l\}$ such that $J_{a_i} \subseteq_F J_{b_j}$.

DEFINITION 4.13. *Let (X, d, α) be a computable metric space. Let $l, l', k, k' \in \mathbb{N}$. We write*

$$\mathcal{H}_{l'}^{k'} \leq \mathcal{H}_l^k$$

if the following holds:

- $\mathcal{H}_{l'} \leq \mathcal{H}_l$,
- $(J_{(l')_0}, \ldots, J_{(l')_{k'}})$ *formally refines* $(J_{(l)_0}, \ldots, J_{(l)_k})$,
- $J_{(l')_0} \subseteq_F J_{(l)_0}$, $J_{(l')_{k'}} \subseteq_F J_{(l)_k}$.

Definition 4.14. *Let (X, d, α) be a computable metric space and let $A \subseteq X$, $j \in \mathbb{N}$ and $r > 0$. We are going to write*

$$A \subseteq_r J_j$$

if the following holds:

$$\begin{aligned} &A \subseteq J_j, \\ &I_i \cap A \neq \emptyset, \quad \textit{for each } i \in [j], \\ &\rho_i < r, \quad \textit{for each } i \in [j]. \end{aligned}$$

Definition 4.15. *Let (X, d, α) be a computable metric space, let $A, B \subseteq X$ and $r > 0$. We say that the number r is an (A, B)- separator if for all $i, j \in \mathbb{N}$ the following implication holds:*

$$(A \subseteq_r J_i \textit{ and } B \subseteq_r J_j) \implies J_i \diamond J_j.$$

Definition 4.16. *Let (X, d, α) be a computable metric space, let K be a compact set in (X, d), suppose $r > 0$ and let $a \in \mathbb{N}$. We say that the number r is an (K, a)- augmentator if for each $j \in \mathbb{N}$ the following implication holds:*

$$K \subseteq_r J_j \implies J_j \subseteq_F J_a.$$

Remark 4.17. Note if A, B are compact sets in (X, d), r is an (A, B)-separator and $r' \in \langle 0, r\rangle$, then r' is also an (A, B)- separator. Similarly, if r is a (K, a)-augmentator and $r' \in \langle 0, r\rangle$, then r' is also an (K, a)- augmentator.

Here we state four useful results: Lemma 4.18, Proposition 4.19, Lemma 4.20 and Proposition 4.21. The proofs can be found in [9, Lemma 4.8. - Proposition 4.13.].

Lemma 4.18. *Let (X, d, α) be a computable metric space, let K be a nonempty compact set in (X, d) and let $r > 0$. Then there exists $l \in \mathbb{N}$ such that $K \subseteq_r J_l$.*

Proposition 4.19. *Let (X, d, α) be a computable metric space and let A, B be disjoint nonempty compact sets in (X, d). Then there exists $r > 0$ such that r is an (A, B)-separator.*

Lemma 4.20. *Let (X, d, α) be a computable metric space, let $A \subseteq X$, $j \in \mathbb{N}$ and $r > 0$ such that $A \subseteq_r J_j$. Then*

$$\operatorname{fdiam}(j) < \operatorname{diam}(A) + 4r.$$

Proposition 4.21. *Let (X, d, α) be a computable metric space, let $a \in \mathbb{N}$ and let K be a compact set in (X, d) such that $K \subseteq J_a$. Then there exists $r > 0$ such that r is an (K, a)-augmentator.*

Lemma 4.22. *Let (X, d, α) be a computable metric space. Let $\mathcal{K}$ be a nonempty finite collection of nonempty compact sets in (X, d). Let A be a finite subset of $\mathbb{N}$ and $\epsilon > 0$. Then for every $K \in \mathcal{K}$ there exists $i_K \in \mathbb{N}$ such that for all $K, L \in \mathcal{K}, a \in A$ the following holds:*

1. $K \subseteq J_{i_K}$;
2. $K \cap L = \emptyset \implies J_{i_K} \diamond J_{i_L}$;
3. $K \subseteq J_a \implies J_{i_K} \subseteq_F J_a$;
4. $\operatorname{fdiam}(i_K) < \operatorname{diam} K + \epsilon$.

Proof. We define $\Delta = \{(K, K') \mid K, K' \in \mathcal{K}, K \cap K' = \emptyset\}$. By Proposition 4.19 for every $(K, K') \in \Delta$ there exists $\mu_{K,K'} > 0$ which is a $(K, K')-$ separator. If $\Delta = \emptyset$ we define $\mu_S = 1$, otherwise $\mu_S = \min\{\mu_{K,K'} \mid (K, K') \in \Delta\}$ (existence of the minimum follows from the finiteness of $\mathcal{K}$).

Next, we define $\Gamma = \{(K, a) \in \mathcal{K} \times A \mid K \subseteq J_a\}$. By Proposition 4.21 for each $(K, a) \in \Gamma$ there exists $\mu_{K,a}$ which is an (K, a)-augmentator. If $\Gamma = \emptyset$ we define $\mu_A = 1$, otherwise $\mu_A = \min\{\mu_{K,a} \mid (K, a) \in \Gamma\}$. Finiteness of $\mathcal{K}$ and A implies the finiteness of Γ, which ensures the existence of the minimum.

Now we set $\mu = \min\{\mu_S, \mu_A, \epsilon/8\}$.

1. By Lemma 4.18 for each $K \in \mathcal{K}$ there exists $i_K \in \mathbb{N}$ such that $K \subseteq_\mu J_{i_K}$.
2. Let $K, L \in \mathcal{K}$ be disjoint. Then $(K, L) \in \Delta$, so μ is a (K, L)-separator. Since $K \subseteq_\mu J_{i_K}$ and $L \subseteq_\mu J_{i_L}$, we have that $J_{i_K} \diamond J_{i_L}$ follows.
3. Let $K \in \mathcal{K}, a \in A$ be such that $K \subseteq J_a$. Then $(K, a) \in \Gamma$ and μ is an (K, a)-augmentator. Now $K \subseteq_\mu J_{i_K}$ implies $J_{i_K} \subseteq_F J_a$.
4. We know from Lemma 4.20 and from $K \subseteq_\mu J_{i_K}$ that $\operatorname{fdiam}(i_K) \leq \operatorname{diam} K + 4\mu$. Then $\operatorname{fdiam}(i_K) \leq \operatorname{diam} K + 4 \cdot \epsilon/8 < \operatorname{diam} K + \epsilon$.

∎

Lemma 4.23. *Let (X, d, α) be a computable metric space. Suppose S is a circularly chainable continuum with $a \in S$. Let $n \in \mathbb{N}$ such that $S \subseteq \cup\mathcal{H}_n$. Let $\epsilon > 0$. Then there exists $l \in \mathbb{N}$ such that:*

1. $S \subseteq \mathcal{H}_l$;
2. $\mathcal{H}_l$ *is a formal circular chain;*
3. $\operatorname{fmesh}(l) < \epsilon$;
4. $\mathcal{H}_l \leq \mathcal{H}_n$;
5. $a \in J_{(l)_0}$.

Proof. The set $\operatorname{links}(\mathcal{H}_n) = \{J_{(n)_0}, \ldots, J_{(n)_{\overline{n}}}\}$ is an open cover of S in (X, d). Therefore, $\{J_{(n)_0} \cap S, \ldots, J_{(n)_{\overline{n}}} \cap S\}$ is an open cover of S in $(S, d|_{S \times S})$. There exists $\lambda > 0$ which is the Lebesgue number of this cover, i.e. for every A subset of S such that $\operatorname{diam} A < \lambda$ there exists $i \in \{0, \ldots, \overline{n}\}$ such that $A \subseteq J_{(n)_i}$.

Let $\mathcal{K} = (K_0, \ldots K_m)$ be a compact $\min\{\epsilon/2, \lambda\}$-circular chain in S which covers S. Such circular chain exists because of Proposition 3.4. If $a \in K_0$, then we proceed. If $a \in K_i$ for some $i > 0$, now we rotate $\mathcal{K}$ and notice that now a is contained in the zeroth link of $\mathcal{K}_{+i}$ Without loss of generality we proceed with denoting $\mathcal{K}_{+i}$ by $\mathcal{K}$.

Since $\operatorname{diam} K_i < \lambda$, for each $i \in \{0, \dots, m\}$ there exists $w_i \in \{0, \dots, \overline{n}\}$ such that $K_i \subseteq J_{(n)_{w_i}}$. Now we can apply Lemma 4.22 with $\mathcal{K} = \{K_0, \dots, K_m\}$, $A = [n]$ and the parameter $\epsilon/2$. For each $K_j \in \mathcal{K}$ we denote the corresponding i_{K_j} by k_j. There exists $l \in \mathbb{N}$ such that $((l)_0, \dots, (l)_{\overline{l}}) = (k_0, \dots, k_m)$. Now, $\mathcal{H}_l = (J_{k_0}, \dots J_{k_m})$.

1. For each $i \in \{0, \dots, m\}$ we have $K_i \subseteq J_{k_i}$. Since $\mathcal{K}$ covers S, so does $\mathcal{H}_l$.
2. Let $u, v \in \{0, \dots, m\}$ such that $1 < |u - v| < m$. Since $\mathcal{K}$ is a circular chain, K_u and K_v are disjoint. Then by Lemma 4.22 $J_{k_u} \diamond J_{k_v}$, therefore $\mathcal{H}_l$ is a formal circular chain.
3. By Lemma 4.22, for every $i \in \{0, \dots, m\}$, $\operatorname{fdiam}(k_i) < \operatorname{diam} K_i + \epsilon/2 < \epsilon$. Therefore $\operatorname{fmesh}(l) < \epsilon$.
4. Let us fix an arbitrary $i \in \{0, \dots, m\}$. Since $K_i \subseteq J_{(n)_{w_i}}$, then again by Lemma 4.22 $J_{k_i} \subseteq J_{(n)_{w_i}}$, i.e. $\mathcal{H}_l \leq \mathcal{H}_n$.
5. By rotation of $\mathcal{K}$, $a \in K_0 \subseteq J_{(l)_0}$.

$\square$

THEOREM 4.24. *Let (X, d, α) be a computable metric space. Let K be a circularly chainable continuum in (X, d) and let $\epsilon_0 > 0$ be such that no compact ϵ_0-chain covers K. Let $a, b \in K$ such that $a \neq b$. Then there exists $l \in \mathbb{N}$ such that $\mathcal{H}_l$ is a formal circular chain which covers K,* $\operatorname{fmesh}(l) < \min\{1, \epsilon_0\}$ *and $a \in J_{(l)_0}, b \in J_{(l)_k}$, where $k \in \{2, \dots, \overline{l} - 1\}$.*

PROOF. Let $r = d(a, b)$. We fix $j \in \mathbb{N}$ such that $K \subseteq J_j$. Then there exists $n \in \mathbb{N}$ such that $((n)_0, \dots, (n)_{\overline{n}}) = (j)$. Thus $K \subseteq \bigcup \mathcal{H}_n$ follows.

We apply Lemma 4.23 with $\epsilon = \min\{1, \epsilon_0, r/4\}$: there exists $l \in \mathbb{N}$ such that $\mathcal{H}_l$ is a formal circular chain which covers K with $a \in J_{(l)_0}$ and $\operatorname{fmesh}(l) < \epsilon$.

Since $\mathcal{H}_l$ covers K, we have $b \in J_{(l)_k}$ for some $k \in \{0, \dots, \overline{l}\}$. But we claim that $\mathcal{H}_l$ is a circular chain as well. Namely, if there exist neighbouring links which are disjoint, then there exists an ϵ_0-chain which covers K, but K is not ϵ_0-chainable. From this it follows $|i - j| \leq 1$ or $|i - j| = \overline{l} \implies J_{(l)_i} \cup J_{(l)_j} \neq \emptyset$. For this reason $J_{(l)_0}$ intersects the sets $J_{(l)_0}, J_{(l)_1}$ and $J_{(l)_{\overline{l}}}$. Let us now assume $k \in \{0, 1, \overline{l}\}$. Now $J_{(l)_0}$ and $J_{(l)_k}$ intersect and therefore $d(a, b) \leq \operatorname{diam}(J_{(l)_0} \cup J_{(l)_k}) \leq \operatorname{diam} J_{(l)_0} + \operatorname{diam} J_{(l)_k} < 2\epsilon \leq \frac{r}{2}$, which is a contradiction with $d(a, b) = r$. Therefore $k \in \{2, \dots, \overline{l} - 1\}$. $\square$

Let (X, d, α) be a computable metric space. We fix K, a semicomputable circularly chainable, but not chainable, continuum in (X, d). Let $\epsilon_0 > 0$ be such that no compact ϵ_0-chain covers K. Without loss of generality we can assume that $\epsilon_0 \in \mathbb{Q}$.

We define the sets Γ and Ω:

$$\Gamma = \{(l,k,l',k') \in \mathbb{N}^4 \mid \mathcal{H}_l, \mathcal{H}_{l'} \text{ formal circular chains which cover } K, \\ \mathcal{H}_{l'}^{k'} \leq \mathcal{H}_l^k, \text{fmesh}(l') < \frac{1}{2}\text{fmesh}(l), k \leq \bar{l}, k' \leq \overline{l'}\};$$

$$\Omega = \{(l,k) \in \mathbb{N}^2 \mid \mathcal{H}_l \text{ formal circular chain, fmesh}(l) < \epsilon_0, \\ K \subseteq \bigcup \mathcal{H}_l, 2 \leq k \leq \bar{l} - 1\}.$$

Proposition 4.25. *The sets Γ and Ω are c.e.*

Proof. From Proposition 4.10 and Proposition 2.3 we know that $\{(l,k) \in \mathbb{N}^2 \mid K \subseteq \bigcup \mathcal{H}_l\}$ is c.e. Similarly, using Propositions 4.9 and 2.3 we know that $\{(l,k) \in \mathbb{N}^2 \mid \mathcal{H}_l \text{ a formal circular chain}\}$ is c.e. The set $\{(l,k) \in \mathbb{N}^2 \mid \text{fmesh}(l) < \epsilon_0\}$ is c.e. due to Proposition 2.4. The fact that $\{(l,k) \in \mathbb{N}^2 \mid 2 \leq k \leq \bar{l} - 1\}$ is computable and therefore c.e. is obvious. The set Ω is an intersection of c.e. sets and therefore c.e.

We similarly conclude for Γ. The only challenging part is to prove that $\{(l,k,l',k') \in \mathbb{N}^4 \mid \mathcal{H}_{l'}^{k'} \leq \mathcal{H}_l^k\}$ is c.e. We first prove that $\Gamma_1 = \{(l,l') \in \mathbb{N}^2 \mid \mathcal{H}_{l'} \leq \mathcal{H}_l\}$ is c.e. The set $\{(l,i,l',j) \in \mathbb{N}^4 \mid J_i \subseteq_F J_j \text{ and } j \in [l]\}$ is c.e due to Proposition 4.9 and Proposition 2.5. But then, using Proposition 2.3, so is $\Gamma_2 = \{(l,i,l') \in \mathbb{N}^3 \mid \exists j \in \mathbb{N} \text{ such that } J_i \subseteq_F J_j \text{ and } j \in [l]\}$. We then conclude:

$$\begin{aligned} (l,l') \in \Gamma_1 &\iff \forall i \in [l'] \; \exists j \in [l] \text{ such that } J_i \subseteq_F J_j \\ &\iff \forall i \in [l'] \; (l,i,l') \in \Gamma_2 \\ &\iff \{(l,i,l') \in \mathbb{N}^3 \mid i \in [l']\} \subseteq \Gamma_2. \end{aligned}$$

The function $\Phi : \mathbb{N}^2 \to \mathcal{P}(\mathbb{N}^3)$, $\Phi(l,l') = \{(l,i,l') \in \mathbb{N}^3 \mid i \in [l']\}$ is an e.f.v. function. Then $(l,l') \in \Gamma_1 \iff \Phi(l,l') \subseteq \Gamma_2$. It follows from Proposition 2.5 that Γ_1 is c.e. We proceed similarly with $\{(l,k,l',k') \in \mathbb{N}^4 \mid (J_{(l')_0}, \ldots, J_{(l')_{k'}})$ formally refines $(J_{(l)_0}, \ldots, J_{(l)_k})\}$, and conclude that Γ is c.e. ∎

Proposition 4.26. *For every $(l,k) \in \Omega$ there exists $(l',k') \in \Omega$ such that $(l,k,l',k') \in \Gamma$.*

Proof. Let us fix an arbitrary $(l,k) \in \Omega$. Now, $\mathcal{H}_l = (J_{(l)_0}, \ldots, J_{(l)_{\bar{l}}})$ is a formal circular chain which covers K, and $2 \leq k \leq \bar{l} - 1, \text{fmesh}(l) < \epsilon_0$. Let $m = \bar{l}$ and $\mathcal{C} = (C_0, \ldots, C_m) = (J_{(l)_0}, \ldots, J_{(l)_{\bar{l}}})$. Now $\mathcal{C}$ is a circular chain which covers K and $\text{mesh}\, \mathcal{C} < \epsilon_0$. We choose some compact circular chain $\mathcal{E} = (E_0, \ldots, E_n)$ in K which covers K such that $\mathcal{E} \leq \mathcal{C}$ and $\text{mesh}\, \mathcal{E} < \frac{1}{4}\text{fmesh}(l)$ (this is possible due to Proposition 3.16).

By Theorem 3.19 there exists a subchain $\mathcal{E}'$ of $\mathcal{E}$ which strictly refines $(C_0, \ldots, C_k)$ or $-(C_0, \ldots, C_k)$. We have $\mathcal{E}' = \mathcal{E}_{u,v}$. Our goal is to align 1) the orientation and 2) the starting link of $\mathcal{E}$ with $(C_0, \ldots, C_k)$. More precisely, we seek a circular chain $\mathcal{D} = (D_0, \ldots, D_n)$ such that links$(\mathcal{D})$ = links$(\mathcal{E})$ and its subchain $\mathcal{D}' = \mathcal{D}_{0,k'}$ such that links$(\mathcal{D}')$ = links$(\mathcal{E}')$ and $\mathcal{D}'$ refines $\mathcal{C}_{0,k}$ with $D_0 \subseteq C_0, D_{k'} \subseteq C_k$, for some $k' \leq n-1$. That way we keep the strict refinement of the subchains, but we gain more control over their orientations.

1) If $E_u \subseteq C_0$ and $E_v \subseteq C_k$, we proceed with $\mathcal{D} = \mathcal{E}$. If $E_v \subseteq C_0$ and $E_u \subseteq C_k$, then we note that $-\mathcal{E}' \leq -\mathcal{E}$, so without loss of generality we proceed with $\mathcal{D} = -\mathcal{E}$.

2) Without loss of generality we can put $\mathcal{D}' = (D_0, \ldots, D_{k'})$ for some $k' \leq n-1$. Else, if $\mathcal{D}' = \mathcal{D}_{u,v}$ for some $u \neq 0$, we consider the circular chain $\mathcal{D}_{+u}$ instead of $\mathcal{D}$ and note that $\mathcal{D}' = (\mathcal{D}_{+u})_{0,k'}$ for some $k' \leq n-1$.

Now we apply Lemma 4.22 with $K = \{D_0, \ldots, D_n\}$, $A = \{(l)_0, \ldots, (l)_{\bar{l}}\}$ and $\epsilon = \frac{1}{4}\operatorname{fmesh}(l)$. For each $i \in \{0, \ldots, n\}$ there exists $j_i \in \mathbb{N}$ such that $D_i \subseteq J_{j_i}$. We can now find $l' \in \mathbb{N}$ such that $((l')_0, \ldots, (l')_{\overline{l'}}) = (j_0, \ldots, j_n)$. Then $\mathcal{H}_{l'} = (J_{(l')_0}, \ldots, J_{(l')_{\overline{l'}}}) = (J_{j_0}, \ldots, J_{j_n})$. Since $\mathcal{D}$ covers K, so does $\mathcal{H}_{l'}$. If we choose $a, b \in \{0, \ldots n\}$ such that $1 < |a-b| < n$, then $D_a \cap D_b = \emptyset$, and by Lemma 4.22 $J_{j_a} \diamond J_{j_b}$, i.e. $\mathcal{H}_{l'}$ is a formal circular chain. For each $i \in \{0, \ldots, n\}$ there exists $w_i \in \{0, \ldots, \bar{l}\}$ such that $D_i \subseteq J_{(l)_{w_i}}$ and therefore, by Lemma 4.22 $J_{(l')_i} \subseteq_F J_{(l)_{w_i}}$, so we have $\mathcal{H}_{l'} \leq \mathcal{H}_l$. Next, we note that $\operatorname{fdiam}(j_i) < \operatorname{diam} D_i + \epsilon < \frac{1}{4}\operatorname{fmesh}(l) + \frac{1}{4}\operatorname{fmesh}(l) = \frac{1}{2}\operatorname{fmesh}(l)$, hence $\operatorname{fmesh}(l') < \frac{1}{2}\operatorname{fmesh}(l)$. This entails $\operatorname{fmesh}(l') < \epsilon_0$.

We know that $(D_0, \ldots, D_k')$ refines $(C_0, \ldots, C_k)$ with $D_0 \subseteq C_0, D_{k'} \subseteq C_k$ and we want to show that $(J_{(l')_0}, \ldots, J_{(l')_{k'}})$ formally refines $(J_{(l)_0}, \ldots, J_{(l)_k})$ with $J_{(l')_0} \subseteq_F J_{(l)_0}, J_{(l')_{k'}} \subseteq_F J_{(l)_k}$. By Lemma 4.22 $D_0 \subseteq J_{(l)_0}$ and $D_{k'} \subseteq J_{(l)_k}$ imply $J_{(l')_0} \subseteq_F J_{(l)_0}$ and $J_{(l')_{k'}} \subseteq_F J_{(l)_k}$. Let $i \in \{0, \ldots, k'\}$. By refinement, there exists $w_i \in \{0, \ldots, k\}$ such that $D_i \subseteq J_{(l)_{w_i}}$. That implies $J_{(l')_i} \subseteq_F J_{(l)_{w_i}}$. Therefore $\mathcal{H}^{k'}_{l'} \leq \mathcal{H}^k_l$ and we have proven $(l, k, l', k') \in \Gamma$.

To prove $(l', k') \in \Omega$ it remains to show that $2 \leq k' \leq \overline{l'} - 1$. We note that $J_{(l)_0} \cap J_{(l)_k} = \emptyset$ because of $2 \leq k \leq \bar{l} - 1$. But now, as mentioned, $J_{(l')_0} \subseteq_F J_{(l)_0}$ and $J_{(l')_{k'}} \subseteq_F J_{(l)_k}$ holds. We conclude $J_{(l')_0} \cap J_{(l')_{k'}} = \emptyset$ and therefore $2 \leq k' \leq \overline{l'} - 1$ and $(l', k') \in \Omega$. ∎

Here we state a well-known result from computable analysis:

Proposition 4.27 ([9, Proposition 5.2.]). *Let $k \in \mathbb{N} \setminus \{0\}, T \subseteq \mathbb{N}^k$ and $a \in T$. Suppose $\varphi : T \to \mathbb{N}^k$ is a partial recursive function such that $\varphi(T) \subseteq T$. Let $f : \mathbb{N} \to \mathbb{N}^k$ be the function defined by*

$$f(0) = a, \quad f(y+1) = \varphi(f(y)).$$

Then f is computable.

Here we state a useful auxiliary result. Its proof can be found in [6].

Lemma 4.28 ([6, Lemma 41.]). *Let* (X, d) *be a metric space which has compact closed balls. Let* $\mathcal{C}^k = (C_0^k, \dots, C_{m_k}^k), k \in \mathbb{N}$ *be a sequence of open chains such that* $\mathrm{Cl}(C_0^{k+1}), \dots, \mathrm{Cl}(C_{m_{k+1}}^{k+1})$ *strictly refines* $(C_0^k, \dots, C_{m_k}^k)$*, and* $\mathrm{mesh}(\mathcal{C}^k) < 2^{-k}, \forall k \in \mathbb{N}$. *Let*

$$S = \bigcap_{k \in \mathbb{N}} (\mathrm{Cl}(C_0^{k+1}) \cup \dots \cup \mathrm{Cl}(C_{m_{k+1}}^{k+1})).$$

Then S *is a continuum chainable from* a *to* b*, where* $a \in \bigcap_{k \in \mathbb{N}} C_0^k, b \in \bigcap_{k \in \mathbb{N}} C_{m_k}^k$.

Before the main theorem, we state another useful result:

Lemma 4.29 ([9, Lemma 6.6.]). *Let* (X, d, α) *be a computable metric space and let* S *be a nonempty compact set in* (X, d)*. Suppose there exists a computable function* $f : \mathbb{N} \to \mathbb{N}$ *such that, for each* $k \in \mathbb{N}$, $\mathrm{fmesh}(f(k)) < 2^{-k}, S \subseteq \bigcup \mathcal{H}_{f(k)}$ *and each of the sets in the finite sequence* $\mathcal{H}_{f(k)}$ *intersects* S*. Then* S *is a computable set.*

Theorem 4.30. *Let* (X, d, α) *be a computable metric space. Suppose* $K \subseteq X$ *is a circularly chainable, but not chainable, semicomputable continuum. Let* $a, b \in K$ *such that* $a \neq b$*. Then for every* $\epsilon > 0$ *there exist computable points* $a', b' \in K$ *such that* $d(a, a') < \epsilon, d(b, b') < \epsilon$ *and a computable subcontinuum* L *of* K *chainable from* a' *to* b'*.*

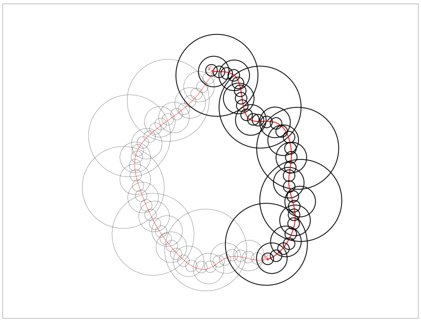


Figure 10. Recursive refinements

Proof. Let $\epsilon > 0$.

By Proposition 4.25 we have that Γ and Ω are c.e. By Proposition 4.26 we have that for every $(l, k) \in \Omega$ there exists $(l', k') \in \Omega$ such that $(l, k, l'k') \in \Gamma$. It follows from Proposition 2.3 that there exists a partial computable function $\varphi : \Omega \to \mathbb{R}^2$ such that $\varphi(\Omega) \subseteq \Omega$ and $(l, k, \varphi(l, k)) \in \Gamma$ for each $(l, k) \in \Omega$.

We should note that by Theorem 4.24 there exists $(l_0, k_0) \in \Omega$ such that $\text{fmesh}(l_0) < \min\{1, \epsilon\}, a \in J_{(l_0)_0}, b \in J_{(l_0)_{k_0}}$. Therefore we can apply Proposition 4.27 and introduce a computable function $f : \mathbb{N} \to \mathbb{N}^2$ such that

$$f(0) = (l_0, k_0), \quad f(y+1) = \varphi(f(y)).$$

Next, we introduce the componentwise notation of the function f: $f(y) = (l_y, k_y)$. We note that $f(y) \in \Omega$ holds for every $y \in \mathbb{N}$. Therefore $(f(y), \varphi(f(y))) \in \Gamma$ for every $y \in \mathbb{N}$. By definition of f this implies $(f(y), f(y+1)) \in \Gamma$, and then $(l_y, k_y, l_{y+1}, k_{y+1}) \in \Gamma$ for every $y \in \mathbb{N}$. By definition of Γ we then have $\mathcal{H}_{l_{y+1}}^{k_{y+1}} \leq \mathcal{H}_{l_y}^{k_y}$, for every $y \in \mathbb{N}$. We also claim that $\text{fmesh}(l_y) < 2^{-y}, \forall y \in \mathbb{N}$. Namely, that follows from $\text{fmesh}(l_0) < 1$ and $\text{fmesh}(l_{y+1}) < \frac{1}{2}\,\text{fmesh}(l_y)$.

Next, we fix $y \in \mathbb{N}$. We define:

$$\mathcal{C}^y = (C_0^y, \ldots, C_{m_y}^y) = (J_{(l_y)_0} \cap K, \ldots, J_{(l_y)_{k_y}} \cap K).$$

We try to see if conditions of Lemma 4.28 are met.

We claim that $\mathcal{C}^y$ is a chain in K. Namely, from Remark 4.11 we know that $(J_{(l_y)_0} \cap K, \ldots, J_{(l_y)_{\overline{l_y}}} \cap K)$ is a circular chain in K. Hence $\mathcal{C}^y$ is its subchain.

Next, we claim that

$$\text{Cl}(C_0^{y+1}), \ldots \text{Cl}(C_{m_{y+1}}^{y+1}) \text{ refines } \mathcal{C}^y,$$
$$\text{with } \text{Cl}(C_0^{y+1}) \subseteq C_0^y, \text{Cl}(C_{m_{y+1}}^{y+1}) \subseteq C_{m_y}^y.$$

To prove this, first we apply Proposition 4.7 to $\mathcal{H}_{l_{y+1}}^{k_{y+1}} \leq \mathcal{H}_{l_y}^{k_y}$ to establish that

$$(\text{Cl}(J_{(l_{y+1})_0}), \ldots, \text{Cl}(J_{(l_{y+1})_{k_{y+1}}})) \text{ refines } (J_{(l_y)_0}, \ldots, J_{(l_y)_{k_y}}).$$

Because of this now

$$(\text{Cl}(J_{(l_{y+1})_0}) \cap K, \ldots, \text{Cl}(J_{(l_{y+1})_{k_{y+1}}}) \cap K) \text{ refines } (J_{(l_y)_0} \cap K, \ldots, J_{(l_y)_{k_y}} \cap K).$$

Now we use the simple topological fact $\text{Cl}(A \cap K) \subseteq \text{Cl}(A) \cap K$ to conclude

$$(\text{Cl}(J_{(l_{y+1})_0} \cap K), \ldots, \text{Cl}(J_{(l_{y+1})_{k_{y+1}}} \cap K)) \text{ refines } (J_{(l_y)_0} \cap K, \ldots, J_{(l_y)_{k_y}} \cap K),$$

from which $\text{Cl}(C_0^{y+1}), \ldots \text{Cl}(C_{m_{y+1}}^{y+1})$ refines $\mathcal{C}^y$ is evident.

Similarly, we conclude that $\text{Cl}(J_{(l_{y+1})_0} \cap K) \subseteq J_{(l_y)_0} \cap K$ and $\text{Cl}(J_{(l_{y+1})_{k_{y+1}}} \cap K) \subseteq J_{(l_y)_{k_y}} \cap K$. This, by definition of $\mathcal{C}^y$, entails $\text{Cl}(C_0^{y+1}) \subseteq C_0^y$ and $\text{Cl}(C_{m_{y+1}}^{y+1}) \subseteq C_{m_y}^y$.

By Lemma 4.28 we have obtained a continuum

$$L = \bigcap_{k \in \mathbb{N}} (\text{Cl}(C_0^{k+1}) \cup \cdots \cup \text{Cl}(C_{m_{k+1}}^{k+1})),$$

chainable from $a' \in \bigcap_{k\in\mathbb{N}} C_0^k$ to $b' \in \bigcap_{k\in\mathbb{N}} C_{m_k}^k$. By construction, $L \subseteq K$, so L is a subcontinuum of K. Next we notice that since $a, a' \in J_{(l_0)_0}$ and $\text{fmesh}(l_0) < \epsilon$, $d(a, a') < \epsilon$ follows. Similar holds for $d(b, b') < \epsilon$.

To prove that a' and b' are computable points first we fix $y \in \mathbb{N}$. Then $a' \in J_{(l_y)_0}$. The set $J_{(l_y)_0}$ is the union of rational balls I_k, where $k \in [(l_y)_0]$. We can explicitly write one such k, namely $k = ((l_y)_0)_0$. The center of this I_k, $\lambda_{((l_y)_0)_0}$ is also contained in $J_{(l_y)_0}$. Since $\text{fmesh}(l_y) < 2^{-y}$, this entails

$$d(a', \lambda_{((l_y)_0)_0}) < 2^{-y},$$

or, by definition of function λ,

$$d(a', \alpha_{\tau_1(((l_y)_0)_0)}) < 2^{-y}.$$

Hence a' is a computable point, and similar argument can be brought forth for b'.

Now it is sufficient to prove that L is computable. We know that $L \subseteq J_{(l_y)_0} \cup \cdots \cup J_{(l_y)_{k_y}}, \forall y \in \mathbb{N}$. For each $y \in \mathbb{N}$ there exists $w \in \mathbb{N}$ such that $((w)_0, \ldots, (w)_{\overline{w}}) = ((l_y)_0, \ldots, (l_y)_{k_y})$. The set

$$\{(y, w) \in \mathbb{N}^2 \mid ((w)_0, \ldots, (w)_{\overline{w}}) = ((l_y)_0, \ldots, (l_y)_{k_y})\}$$

is computable. Therefore there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $((f(y))_0, \ldots, (f(y))_{\overline{f(y)}}) = ((l_y)_0, \ldots, (l_y)_{k_y})$, for all $y \in \mathbb{N}$. Then $L \subseteq \bigcup \mathcal{H}_{f(y)}, \forall y \in \mathbb{N}$. For any given $y \in \mathbb{N}$ we know that $J_{(f(y))_0}$ and $J_{(f(y))_{\overline{f(y)}}}$ intersect L, since $a', b' \in L$. But, since L is connected, every link of $\mathcal{H}_{f(y)}$ intersects L, otherwise there would exist a separation of L. Then by Lemma 4.29 L is computable. ∎

Remark 4.31. First we note the most important part of the result: Theorem 4.30 ensures the existence of computable subcontinua of a semicomputable circularly chainable, but not chainable, continuum K.

One might wonder whether we have succeeded to approximate some semicomputable continuum with these computable subcontinua. The answer is that this result guarantees no such thing, because there is no obvious candidate for the semicomputable continuum being approximated. One such candidate might be K itself, but here we restate the fact that the topological properties of K ensure that it is computable as well. Any possible approximations are then not useful. We should also note that while there is a subcontinuum L of K chainable from a' to b', there need not be a subcontinuum of K chainable from a to b (see Example 3.6).

Instead, what we can approximate is any pair of given points a and b on K with computable points on a' and b' on K. On top of that, there exists a computable subcontinuum L of K chainable from a' and b'.

5. Computable intersection points

Our final result is motivated by the computable version of the **Intermediate value theorem**. We state it again here:

Theorem 5.1 ([14]). *A computable function $f : [0,1] \to \mathbb{R}$ such that $f(0) < 0$, $f(1) > 0$ has a computable zero.*

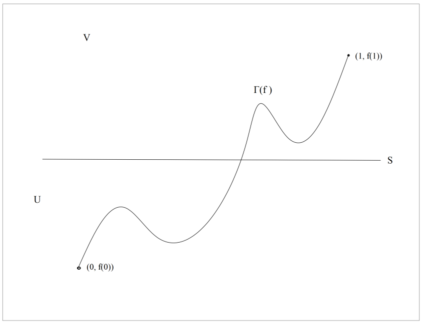


Figure 11. Computable intermediate value theorem

Remark 5.2. We now make the following observations and try to obtain a setting for possible generalizations of the theorem.

- Let S be the x-axis, and $\Gamma(f)$ the graph of f. Then $\Gamma(f) \cap S$ contains a computable point.
- Let U be the lower and V the upper half-plane. The graph $\Gamma(f)$ intersects both U and V.
- $\Gamma(f)$ is a computable subset of $\mathbb{R}^2$. It is also a continuum considering the Euclidean metric.
- This leads us to the following attempt: Suppose K is a continuum in $\mathbb{R}^2$ which intersects both U and V. Then K certainly intersects S. Does

$$K \text{ computable} \implies K \text{ intersects } S \text{ in a computable point} \tag{5.1}$$

  hold?

  In general, no! Let $f : [0,1] \to \mathbb{R}$ be a nonnegative computable function which has zeroes, but none of them is computable [15]. Let $K = \Gamma(f) \cup \Gamma(-f)$. Then K is a computable continuum, therefore it intersects S, but none of the points in $K \cap S$ are computable.
- Some conditions under which (5.1) holds have been examined in [9]. One such condition is that K is an arc.

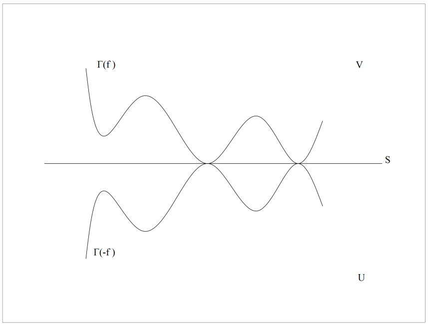


FIGURE 12. No computable intersection points

We restate this important result for clarity:

THEOREM 5.3 ([9, Theorem 5.3]). *Let (X, d, α) be a computable metric space and let U and V be disjoint c.e. open sets in X. Let $S = X \setminus (U \cup V)$. Suppose K is a continuum in X chainable from a to b, where $a \in U$ and $b \in V$. Suppose K is a computable set and $K \cap S$ is totally disconnected. Then $K \cap S$ contains a computable point.*

REMARK 5.4. A topological space X is said to be *totally disconnected* if every connected component of X is a one-point set. Note that any nonempty subspace of X is then totally disconnected as well.

We conclude with the main result of this section:

THEOREM 5.5. *Let (X, d, α) be a computable metric space and let U and V be disjoint c.e. open sets in X. Let $S = X \setminus (U \cup V)$. Suppose K is a circularly chainable, but not chainable, continuum in X which intersects both U and V. Suppose K is a semicomputable set and $K \cap S$ is totally disconnected. Then $K \cap S$ contains a computable point.*

PROOF. We choose arbitrary points $a \in K \cap U, b \in K \cap V$. Since U and V are open, there exist $r_1, r_2 > 0$ such that $B(a, r_1) \subseteq U$ and $B(b, r_2) \subseteq V$. Let $\epsilon = \min\{r_1, r_2\}$. By Theorem 4.30 it follows that there exist computable points $a', b' \in K$ such that $d(a, a') < \epsilon$, $d(b, b') < \epsilon$ and a computable subcontinuum L of K chainable from a' to b'. Note that $a' \in L \cap U, b' \in L \cap V$, and $L \cap S$ is totally disconnected. Therefore, following Theorem 1.3, $L \cap S$ contains a computable point, thus $K \cap S$ contains it as well. ∎

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Izračunljivi potkontinuumi cirkularno lančastih kontinuuma

David Tarandek

Sažetak. Ovaj rad istražuje cirkularno lančaste kontinuume, koji nisu lančasti, u izračunljivim metričkim prostorima. Za proizvoljan takav kontinuum K vrijedi da iz njegove poluizračunljivosti slijedi izračunljivost. U literaturi se često proučavaju uvjeti pod kojima poluizračunljivost skupa implicira njegovu izračunljivost, s naglaskom na topološka svojstva. Kada ti uvjeti nisu zadovoljeni, prirodno je istraživati aproksimativne pristupe. U ovom članku fokusiramo se na specifične izračunljive potkontinuume od K. Glavni rezultat jest da svake dvije točke na poluizračunljivom, cirkularno lančastom, no ne i lančastom, kontinuumu K možemo aproksimirati izračunljivim točkama tako da postoji izračunljivi potkontinuum L od K koji ih povezuje. Posljedično, ako su U i V disjunktni i izračunljivo prebrojivi otvoreni skupovi u izračunljivom metričkom prostoru, pokazujemo da, ako K siječe i U i V, tada njegov presjek s komplementom njihove unije nužno sadrži izračunljivu točku kada je taj presjek totalno nepovezan.

David Tarandek
Faculty of Architecture
University of Zagreb
10 000 Zagreb, Croatia
E-mail: david.tarandek@gmail.com

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