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ENDS OF THE FIRST COMPLEMENTARY SERIES OF GENERALIZED PRINCIPAL SERIES

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Abstract. We determine all irreducible non-tempered composition factors of induced representations appearing at the ends of the first complementary series of generalized principal series representation of either symplectic, special odd-orthogonal, or unitary group over a non-archimedean local field.

1. Introduction

The generalized principal series present a well-studied and particularly important class of induced representations of classical groups over non-archimedean local fields. These are representations of the form $\pi \rtimes \sigma$, induced from the representation $\pi \otimes \sigma$ of the maximal parabolic subgroup having an irreducible essentially square-integrable representation π on the general linear group part and a discrete series σ on the classical group part. Reducibility points of such representations of symplectic and odd orthogonal groups have been determined in [16], while the unitary group case has been handled in [8], but more detailed description of the composition factors is still missing, except in some particular cases, as in [15].

By the classical result of [19], an irreducible essentially square-integrable representation of the general linear group is of the form $\delta([\nu^a \rho, \nu^b \rho])$, and is attached to the segment $[\nu^a \rho, \nu^b \rho]$, where ρ is an irreducible cuspidal representation of the general linear group. We are primarily interested in determining the irreducible non-tempered composition factors appearing at the ends of the first complementary series, i.e., we start from the unitary generalized principal series $\delta([\nu^{-a} \rho, \nu^a \rho]) \rtimes \sigma$, and describe the non-tempered irreducible subquotients of $\delta([\nu^{-a+s} \rho, \nu^{a+s} \rho]) \rtimes \sigma$, for minimal positive s such that this induced representation reduces.

It is well-known that all the irreducible representations appearing at the ends of the complementary series are unitarizable, so we construct a class

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of unitarizable irreducible non-tempered representations. Furthermore, the situation studied should be rather similar to the one studied in [15], providing short and approachable composition series.

One possible approach to the investigation of the generalized principal series is via the Mœglin-Tadić classification of discrete series, provided in [12,14], and in the odd GSpin case in [4]. This classification now holds unconditionally, due to [1], [13, Théorème 3.1.1] and [2, Theorem 7.8]. Having this classification at hand, we provide a uniform approach for symplectic, special odd-orthogonal and unitary groups. We start from the results of [8], or [16], and rely on algebraic methods, which are based mainly on the calculation of the Jacquet modules using the structural formula provided in [17].

Our approach might be regarded as a further development of the methods of [5], [7] and [15]. The obtained results show that induced representations appearing at the end of the first complementary series contain at most four mutually non-isomorphic irreducible non-tempered subquotients, which all appear in the composition with the multiplicity one. Also, it appears that most of the irreducible non-tempered subquotients have been constructed in [8] and [16]. The similar construction has also appeared to be useful for determining the reducibility points in more general situations ([6], [9], [10]).

We note that all our results and proofs can also be used for the odd GSpin groups over a non-archimedean local field of characteristic zero without any change, based on the discrete series classification provided in [4] and the structural formula given in the odd GSpin case in [3].

Let us now describe the contents of the paper in more detail. In the following section we introduce the notation and present some preliminaries. In the third section we provide several technical results, and discuss the special case $\mathrm{Jord}_\rho(\sigma) = \emptyset$. The case of non-empty $\mathrm{Jord}_\rho(\sigma)$ is studied in the final two sections. In the fourth section we determine all the non-tempered irreducible subquotients appearing at the end of the first complementary series $\delta([\nu^{-a+s}\rho, \nu^{a+s}\rho]) \rtimes \sigma$, when $-a + s \leq 0$, using a case-by-case consideration. In the final section we describe all the non-tempered irreducible subquotients when $-a + s > 0$.

2. Preliminaries

Through the paper, we denote by F a non-archimedean local field. Let us fix one of the following series $\{G_n\}$ of classical groups over F.

In the odd orthogonal group case, let Y_0 denote a fixed anisotropic orthogonal vector space over F of odd dimension, and let us consider the Witt tower based on Y_0. For n such that $2n+1 \geq \dim Y_0$, there is exactly one space V_n in the tower of dimension $2n+1$. We let G_n denote the special orthogonal group of this space.

If V_n denotes the symplectic space of dimension $2n$ in the corresponding Witt tower, we let G_n stand for the symplectic group of this space. We also consider the unitary groups $U(n, F'/F)$, for a separable quadratic extension $F'$of F. There is also an anisotropic unitary space Y_0 over F', and the Witt tower of unitary spaces V_n based on Y_0. Let G_n stand for the unitary group of the space V_n of dimension either $2n+1$ or $2n$.

In the symplectic and odd orthogonal case we set $F' = F$.

Let us fix the set of standard parabolic subgroups such that their Levi factors are of the form $M \cong GL(n_1, F') \times \cdots \times GL(n_k, F') \times G_{n'}$, where $GL(m, F')$ denotes the general linear group of rank m over F'. If δ_i is a representation of $GL(n_i, F')$ and τ a representation of $G_{n'}$, the normalized parabolically induced representation $\mathrm{Ind}_M^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$ will be denoted by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$. A similar notation is used to denote a parabolically induced representation of $GL(m, F')$.

Let $\mathrm{Irr}(G_n)$ stand for the set of all irreducible admissible representations of G_n. Let $R(G_n)$ stand for the Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \oplus_{n \geq 0} R(G_n)$. In a similar way we define $\mathrm{Irr}(GL(n, F'))$ and $R(GL) = \oplus_{n \geq 0} R(GL(n, F'))$.

For an irreducible smooth representation $\pi \in R(GL)$, let $\widetilde{\pi}$ stand for the contragredient representation of π. If $F = F'$, we say that π is F'/F-selfdual if $\pi \cong \widetilde{\pi}$. If $F \neq F'$, we denote by θ the non-trivial F-automorphism of F', let $\widehat{\pi}$ denote the representation $g \mapsto \widetilde{\pi}(\theta(g))$, and say that the representation π is F'/F-selfdual if $\pi \cong \widehat{\pi}$.

For $\sigma \in \mathrm{Irr}(G_n)$ and $1 \leq k \leq n$, we let $r_{(k)}(\sigma)$ stand for the normalized Jacquet module of σ with respect to the parabolic subgroup $P_{(k)}$ having the Levi subgroup equal to $GL(k, F') \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k, F')) \otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

We denote by ν a composition of the determinant mapping with the normalized absolute value on F'. Let $\rho \in R(GL)$ denote an irreducible supercuspidal representation. By a segment Δ we mean a set of the form $[\rho, \nu^m \rho] := \{\rho, \nu\rho, \ldots, \nu^m \rho\}$, for a non-negative integer m. The induced representation $\nu^m \rho \times \nu^{m-1} \times \cdots \times \rho$ has a unique irreducible subrepresentation ([19]), which we denote by $\delta(\Delta)$. Representation $\delta(\Delta)$ is essentially square-integrable, and by [19] every irreducible essentially square-integrable representation in $R(GL)$ can be obtained in this way.

In the following result we state the structural formula which is crucial for our calculations with the Jacquet modules ([17]).

LEMMA 2.1. *Let $\rho \in R(GL)$ be an irreducible cuspidal F'/F-selfdual representation and $k, l \in \mathbb{R}$ such that $k + l \in \mathbb{Z}_{\geq 0}$. Let $\sigma \in R(G)$ be an irreducible* *admissible representation. Write* $\mu^*(\sigma) = \sum_{\pi,\sigma'} \pi \otimes \sigma'$. *Then the following holds:*

$$\mu^*(\delta([\nu^{-k}\rho, \nu^l\rho]) \rtimes \sigma) = \sum_{i=-k-1}^{l} \sum_{j=i}^{l} \sum_{\tau,\sigma'} \delta([\nu^{-i}\rho, \nu^k\rho]) \times \delta([\nu^{j+1}\rho, \nu^l\rho]) \times \pi \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'.$$

We omit $\delta([\nu^x\rho, \nu^y\rho])$ *if* $x > y$.

Let us recall the subrepresentation version of the Langlands classification for the general linear groups.

Let $\delta \in R(GL)$ stand for an irreducible essentially square-integrable representation. Then there is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable, and $e(\delta([\nu^a\rho, \nu^b\rho])) = (a+b)/2$.

Suppose that $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \ldots \leq e(\delta_k)$. Then the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1, \delta_2, \ldots, \delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, it appears with multiplicity one in the composition series of $\delta_1 \times \delta_2 \times \cdots \times \delta_k$, and every irreducible representation $\pi \in R(GL)$ is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k)$, the representations $\delta_1, \delta_2, \ldots, \delta_k$ being unique up to a permutation.

We also use the subrepresentation version of the Langlands classification for classical groups, which enables us to realize a non-tempered irreducible representation π of G_n as the unique irreducible subrepresentation of an induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where τ is an irreducible tempered representation of some G_t, and $\delta_1, \delta_2, \ldots, \delta_k \in R(GL)$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \cdots \leq e(\delta_k) < 0$. Then we write $\pi = L(\delta_1, \delta_2, \ldots, \delta_k; \tau)$.

By the Mœglin-Tadić classification of discrete series [12, 14], a discrete series $\sigma \in G_n$ corresponds to an admissible triple which consists of the Jordan block, the partial cuspidal support, and the ϵ-function. Elements appearing in the Jordan block are of the form (a, ρ), where a is a positive integer of the appropriate parity and $\rho \in R(GL)$ is a F'/F-selfdual representation. The ϵ-function is defined on the subset of $\mathrm{Jord} \cup \mathrm{Jord} \times \mathrm{Jord}$ and attains values in $\{\pm 1\}$. More details on those invariants can be found in [14, 16] and in [4], where slightly different approach, which also covers the classical group case, has been used.

For a discrete series $\sigma_{ds} \in R(G)$, we denote by $\mathrm{Jord}(\sigma_{ds})$ the corresponding set of the Jordan blocks, and by $\epsilon_{\sigma_{ds}}$ the corresponding ϵ-function. For an irreducible F'/F-selfdual cuspidal representation ρ of $GL(n, F')$ let $\mathrm{Jord}_\rho(\sigma_{ds}) = \{x : (x, \rho) \in \mathrm{Jord}(\sigma_{ds})\}$. If $\mathrm{Jord}_\rho(\sigma_{ds}) \neq \emptyset$ and $x \in \mathrm{Jord}_\rho(\sigma_{ds})$, set $x\_ = \max\{y \in \mathrm{Jord}_\rho(\sigma_{ds}) : y < x\}$, if it exists. To define the ϵ-function on the ordered pairs of the form $((a_1,\rho_1),(a_2,\rho_2)) \in \mathrm{Jord}\times\mathrm{Jord}$, it is enough to define it on the ordered pairs of the form $((x\_,\rho),(x,\rho))$.

3. Some technical results

In this section we recall and obtain some useful technical results. The first one follows from [18, Theorem 8.2].

Lemma 3.1. *Let $\sigma_1 \in R(G)$ denote a discrete series representation, and let $(a,\rho) \in \mathrm{Jord}(\sigma_1)$ be such that $a\_$ is defined and $a\_ \le a-4$. Then for every x such that $\frac{a-x}{2}$ is an integer and $a\_+4 \le x \le a$, there exists a discrete series σ_2 such that σ_1 is a subrepresentation of $\delta([\nu^{\frac{x-1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho]) \rtimes \sigma_2$. Furthermore, if an irreducible constituent of the form $\delta([\nu^{\frac{x-1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho]) \otimes \pi$ appears in $\mu^*(\sigma_1)$, then $\pi \cong \sigma_2$ and $\delta([\nu^{\frac{x-1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho]) \otimes \sigma_2$ appears in $\mu^*(\sigma_1)$ with multiplicity one.*

Lemma 3.2. *Let $\sigma \in R(G)$ denote a discrete series representation and let $\rho \in R(GL)$ denote an irreducible cuspidal F'/F-selfdual representation. Let $a \in \frac{1}{2}\mathbb{Z}$, $a>0$, such that $2a-1 \in \mathrm{Jord}_\rho(\sigma)$. Suppose that for a non-negative integer k we have $2a+2i+1 \notin \mathrm{Jord}_\rho(\sigma)$ for $i=0,1,\dots,k$. Then the induced representation $\delta([\nu^a\rho,\nu^{a+k}\rho]) \rtimes \sigma$ contains a discrete series subrepresentation, which we denote by σ_{ds}, and in $R(G)$ we have*

$$\delta([\nu^a\rho,\nu^{a+k}\rho]) \rtimes \sigma = L(\delta([\nu^{-a-k}\rho,\nu^{-a}\rho]);\sigma) + \sigma_{ds}.$$

Proof. Discrete series σ_{ds} is constructed in [18, Theorem 8.2].

Let us now describe the non-tempered irreducible subquotients of $\delta([\nu^a\rho,\nu^{a+k}\rho]) \rtimes \sigma$. Every such irreducible subquotient is of the form

$$L(\delta([\nu^{x_1}\rho_1,\nu^{y_1}\rho_1]),\dots,\delta([\nu^{x_m}\rho_m,\nu^{y_m}\rho_m]);\tau),$$

where $x_i+y_i<0$ for $i=1,\dots,m$, and $x_i+y_i \le x_{i+1}+y_{i+1}$ for $i=1,\dots,m-1$. Since $L(\delta([\nu^{x_1}\rho_1,\nu^{y_1}\rho_1]),\dots,\delta([\nu^{x_m}\rho_m,\nu^{y_m}\rho_k]);\tau)$ is a subrepresentation of

$$\delta([\nu^{x_1}\rho_1,\nu^{y_1}\rho_1]) \times L(\delta([\nu^{x_2}\rho_2,\nu^{y_2}\rho_2]),\dots,\delta([\nu^{x_m}\rho_m,\nu^{y_m}\rho_m]);\tau),$$

it follows that $\mu^*(\delta([\nu^a\rho,\nu^{a+k}\rho]) \rtimes \sigma)$ contains

$$\delta([\nu^{x_1}\rho_1,\nu^{y_1}\rho_1]) \otimes L(\delta([\nu^{x_2}\rho_2,\nu^{y_2}\rho_2]),\dots,\delta([\nu^{x_m}\rho_m,\nu^{y_m}\rho_m]);\tau).$$

By Lemma 2.1, there are i,j, $a-1 \le i \le j \le a+k$ and an irreducible constituent $\pi_1 \otimes \pi_2$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^{x_1}\rho_1,\nu^{y_1}\rho_1]) \le \delta([\nu^{-i}\rho,\nu^{-a}\rho]) \times \delta([\nu^{j+1}\rho,\nu^{a+k}\rho]) \times \pi_1$$

and

$$L(\delta([\nu^{x_2}\rho_2,\nu^{y_2}\rho_2]),\dots,\delta([\nu^{x_m}\rho_m,\nu^{y_m}\rho_m]);\tau) \le \delta([\nu^{i+1}\rho,\nu^{j}\rho]) \rtimes \pi_2.$$

Since $x_1+y_1<0$, the square-integrability of σ implies $\rho_1 \cong \rho$, $i \ge a$, and $j=a+k$. Thus, π_1 is of the form $\delta([\nu^{x_1}\rho,\nu^{-i-1}\rho]) \times \delta([\nu^{-a+1}\rho,\nu^{y_1}\rho])$. Since σ is a discrete series, it follows that $x_1=-i$. If $y_1 \ne -a$, $\mu^*(\sigma)$

contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{y_1}\rho]) \otimes \pi_2$ where $-i + y_1 = x_1 + y_1 < 0$. Square-integrability of σ gives $y_1 \geq a$, and $i \leq a+k$ implies $y_1 < a+k$. Since $\mu^*(\sigma)$ contains an irreducible constituent of the form $\nu^{y_1}\rho \otimes \pi$, this contradicts the definition of $\mathrm{Jord}_\rho(\sigma)$. Thus, $y_1 = -a$ and

$$L(\delta([\nu^{x_2}\rho_2, \nu^{y_2}\rho_2]), \ldots, \delta([\nu^{x_m}\rho_m, \nu^{y_m}\rho_m]); \tau) \leq \delta([\nu^{i+1}\rho, \nu^{a+k}\rho]) \rtimes \sigma.$$

Suppose that $i \neq a+k$. Using the cuspidal support considerations we directly obtain $m \geq 2$. Also, $\mu^*(\delta([\nu^{i+1}\rho, \nu^{a+k}\rho]) \rtimes \sigma)$ contains $\delta([\nu^{x_2}\rho_2, \nu^{y_2}\rho_2]) \otimes L(\delta([\nu^{x_3}\rho_3, \nu^{y_3}\rho_3]), \ldots, \delta([\nu^{x_m}\rho_m, \nu^{y_m}\rho_m]); \tau)$. Repeating the same reasoning as before, we conclude $y_2 = -i-1$, which contradicts $x_1+y_1 \leq x_2+y_2$. Thus, $i = a+k$, $m = 1$ and $\tau \cong \sigma$. In other words, every non-tempered irreducible subquotient of $\delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$ is isomorphic to $L(\delta([\nu^{-a-k}\rho, \nu^{-a}\rho]); \sigma)$, and it is an integral part of the Langlands classification that it appears in the composition series of $\delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$ with multiplicity one.

Using the cuspidal support considerations, we deduce that every irreducible tempered subquotient of $\delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$ is a discrete series, whose set of the Jordan blocks equals $\mathrm{Jord}(\sigma_{ds})$. By Lemma 3.1, the Jacquet module of such a discrete series with respect to the appropriate parabolic subgroup contains an irreducible constituent of the form $\delta([\nu^a\rho, \nu^{a+k}\rho]) \otimes \pi$. It follows from the structural formula that the only irreducible constituent of $\mu^*(\delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma)$ of such a form is $\delta([\nu^a\rho, \nu^{a+k}\rho]) \otimes \sigma$ and it appears there with multiplicity one. Thus, $\delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$ contains a unique discrete series subquotient and the lemma is proved. □

Through the rest of the paper we fix an irreducible cuspidal unitarizable representation $\rho \in R(GL)$ and a discrete series $\sigma \in R(G)$. It is well-known, and also follows from [8, 16] that $\delta([\nu^x\rho, \nu^y\rho]) \rtimes \sigma$ is irreducible if ρ is not F'/F-selfdual, so we can also assume that ρ is F'/F-selfdual. We denote by $(\mathrm{Jord}(\sigma), \epsilon_\sigma, \sigma_{cusp})$ the admissible triple corresponding to σ. Also, let us denote by α a unique non-negative real number such that $\nu^\alpha\rho \rtimes \sigma_{cusp}$ reduces. It follows from [1] and [13, Théorème 3.1.1] that $\alpha \in \frac{1}{2}\mathbb{Z}$.

In this section we also comment the simple case when $\mathrm{Jord}_\rho(\sigma) = \emptyset$. Note that this implies $\alpha \in \{0, \frac{1}{2}\}$. Let $a \in \frac{1}{2}\mathbb{Z}$, $a \geq 0$. It is well-known that the induced representation $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma$ is irreducible if and only if $a - \alpha$ is not an integer. Furthermore, if $a - \alpha$ is an integer then $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma$ is a direct sum of two mutually non-isomorphic tempered representations.

Suppose that $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma$ is irreducible. Then $\delta([\nu^{-a+\frac{1}{2}}\rho, \nu^{a+\frac{1}{2}}\rho]) \rtimes \sigma$ reduces and presents the end of the first complementary series. In the same way as in the proof of Lemma 3.2, we can conclude that

$$L(\delta([\nu^{-a-\frac{1}{2}}\rho, \nu^{a-\frac{1}{2}}\rho]); \sigma)$$

is a unique irreducible non-tempered subquotient of $\delta([\nu^{-a+\frac{1}{2}}\rho, \nu^{a+\frac{1}{2}}\rho]) \rtimes \sigma$, and it appears with multiplicity one.

4. Ends of the first complementary series in the non-positive case

In this section we start with a description of the non-tempered irreducible representations appearing at the ends of the first complementary series. Thus, in what follows we assume that $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$.

Let us first recall the irreducibility criterion in this case, given in [8, Theorem 3.5.]:

Theorem 4.1. *Suppose that* $0 \leq a \leq b$ *and for* α *such that* $\nu^\alpha \rho \rtimes \sigma_{cusp}$ *reduces we have* $a - \alpha \in \mathbb{Z}$. *The induced representation* $\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma$ *is irreducible if and only if* $\{2a+1, 2b+1\} \subseteq \mathrm{Jord}_\rho(\sigma)$ *and for every* $x \in \mathrm{Jord}_\rho(\sigma) \cap \langle 2a+1, 2b+1]$ *such that* $x\_$ *is defined we have* $\epsilon((x\_, \rho), (x, \rho)) = -1$.

We continue with a useful lemma.

Lemma 4.2. *Let* $a \in \frac{1}{2}\mathbb{Z}$, $a > 0$, *such that* $2a - 1 \in \mathrm{Jord}_\rho(\sigma)$. *The induced representation* $\nu^a \rho \rtimes \sigma$ *is irreducible if and only if* $2a+1 \in \mathrm{Jord}_\rho(\sigma)$ *and* $\epsilon_\sigma((2a-1, \rho), (2a+1, \rho)) = -1$. *If* $\nu^a \rho \rtimes \sigma$ *reduces, in* $R(G)$ *we have*

$$\nu^a \rho \rtimes \sigma = L(\nu^{-a}\rho; \sigma) + \tau,$$

where τ *is an irreducible tempered representation, which is a discrete series representation if and only if* $2a+1 \notin \mathrm{Jord}_\rho(\sigma)$.

Proof. Irreducibility criterion is a special case of [8, Theorem 5.4].

In the same way as in the proof of Lemma 3.2 we deduce that every non-tempered irreducible subquotient of $\nu^a \rho \rtimes \sigma$ is isomorphic to $L(\nu^{-a}\rho; \sigma)$, and it is an integral part of the Langlands classification that it appears in the composition series of $\nu^a \rho \rtimes \sigma$ with multiplicity one.

If $2a+1 \notin \mathrm{Jord}_\rho(\sigma)$, the claim of the lemma follows from Lemma 3.2.

If $2a+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2a-1, \rho), (2a+1, \rho)) = 1$, an irreducible subquotient τ of $\nu^a \rho \rtimes \sigma$ has been constructed in [8, Lemma 3.3]. In that case, σ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^a\rho]) \rtimes \sigma_1$, for a discrete series σ_1. Furthermore, $\delta([\nu^{-a+1}\rho, \nu^a\rho]) \otimes \sigma_1$ is a unique irreducible constituent of $\mu^*(\sigma)$ of the form $\delta([\nu^{-a+1}\rho, \nu^a\rho]) \otimes \pi_1$, and appears there with multiplicity one.

Using the cuspidal support considerations, we conclude that every irreducible tempered subquotient of $\nu^a \rho \rtimes \sigma$ is a subrepresentation of an induced representation of the form $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \pi$. Since $\delta([\nu^{-a}\rho, \nu^a\rho]) \otimes \sigma_1$ is a unique irreducible constituent of $\mu^*(\nu^a \rho \rtimes \sigma)$ of the form $\delta([\nu^{-a}\rho, \nu^a\rho]) \otimes \pi'$, and appears there with multiplicity one, the lemma follows. ∎

Proposition 4.3. *Suppose that* $2a+1, 2a+5 \notin \mathrm{Jord}_\rho(\sigma)$ *and* $2a+3 \in \mathrm{Jord}_\rho(\sigma)$. *Let* τ_1 *denote an irreducible tempered representation such that* σ *is a subrepresentation of* $\nu^{a+1}\rho \rtimes \tau_1$, *and let* τ_2 *denote an irreducible tempered* *subrepresentation of* $\nu^{a+2}\rho \rtimes \sigma$. *All irreducible non-tempered subquotients of* $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma), L(\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]); \tau_1), L(\delta([\nu^{-a-1}\rho, \nu^{a}\rho]); \tau_2).$$

All three irreducible non-tempered subquotients appear with multiplicity one in the composition series.

PROOF. It is well-known that $L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma)$ appears in the composition series of $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ with multiplicity one. Irreducible subquotients $L(\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]); \tau_1)$ and $L(\delta([\nu^{-a-1}\rho, \nu^{a}\rho]); \tau_2)$ of the induced representation $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ have been obtained in the proof of [8, Lemma 3.2.].

Let us prove that in this way we have determined all irreducible non-tempered subquotients of $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$. We denote an irreducible subquotient by

$$L(\delta([\nu^{y_1}\rho_1', \nu^{z_1}\rho_1']), \ldots, \delta([\nu^{y_k}\rho_k', \nu^{z_k}\rho_k']); \tau'),$$

where $y_1 + z_1 \leq \cdots \leq y_k + z_k < 0$, ρ_i' cuspidal for all $i = 1, \ldots, k$, and τ' tempered.

Then $\mu^*(\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma)$ contains

$$\delta([\nu^{y_1}\rho_1', \nu^{z_1}\rho_1']) \otimes L(\delta([\nu^{y_2}\rho_2', \nu^{z_2}\rho_2']), \ldots, \delta([\nu^{y_k}\rho_k', \nu^{z_k}\rho_k']); \tau'),$$

and the structural formula implies that there are $-a-1 \leq i \leq j \leq a+2$ and an irreducible constituent $\pi \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^{y_1}\rho_1', \nu^{z_1}\rho_1']) \leq \delta([\nu^{-i}\rho, \nu^{a}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{a+2}\rho]) \times \pi$$

and

$$L(\delta([\nu^{y_2}\rho_2', \nu^{z_2}\rho_2']), \ldots, \delta([\nu^{y_k}\rho_k', \nu^{z_k}\rho_k']); \tau') \leq \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \sigma'.$$

The square-integrability of σ gives $\rho_1' \cong \rho$, $z_1 \geq a$, and $i \in \{a+1, a+2\}$. If $(i, z_1) = (a+2, a)$, we get $L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma)$.

If $i = a+2$ and $z_1 \neq a$, we get $z_1 = a+1$, and [11, Lemma 8.3] gives $k = 1$ and $\tau' \cong \tau_1$, so we obtain $L(\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]); \tau_1)$.

Suppose that $i = a+1$. Then $z_1 = a$ and

$$L(\delta([\nu^{y_2}\rho_2', \nu^{z_2}\rho_2']), \ldots, \delta([\nu^{y_k}\rho_k', \nu^{z_k}\rho_k']); \tau') \leq \nu^{a+2}\rho \rtimes \sigma.$$

Lemma 4.2 implies that either $(k, \tau') = (1, \tau_2)$ or $(k, y_2, z_2, \rho_2') = (2, -a-2, -a-2, \rho)$. Using $y_2 + z_2 \geq y_1 + z_1 = -1$ we get $(k, \tau') = (1, \tau_2)$ which gives $\delta([\nu^{-a-1}\rho, \nu^{a}\rho]) \otimes \tau_2$. Consequently, we have obtained all irreducible non-tempered subquotients of $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$.

Using the structural formula, together with Lemma 3.1, one can easily deduce that $\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]) \otimes \tau_1$ appears with multiplicity one in $\mu^*(\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma)$. Also, using structural formula and Lemma 3.2 we obtain that $\delta([\nu^{-a-1}\rho, \nu^{a}\rho]) \otimes \tau_2$ appears with multiplicity one in $\mu^*(\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma)$.

Thus, all irreducible non-tempered subquotients appear in the composition series with multiplicity one. ∎

Proofs of the following three propositions follow in a similar way as in the previously considered case, and are being left to the reader.

PROPOSITION 4.4. *Suppose that* $2a+1 \notin \mathrm{Jord}_\rho(\sigma)$, $2a+3, 2a+5 \in \mathrm{Jord}_\rho(\sigma)$.

(1) *Suppose that* $\epsilon_\sigma((2a+3,\rho),(2a+5,\rho)) = 1$. *Let* τ_1 *denote an irreducible tempered representation such that* σ *is a subrepresentation of* $\nu^{a+1}\rho \rtimes \tau_1$, *and let* τ_2 *denote a unique irreducible tempered subrepresentation of* $\nu^{a+2}\rho \rtimes \sigma$. *All irreducible non-tempered subquotients of* $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma), L(\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]); \tau_1), L(\delta([\nu^{-a-1}\rho, \nu^{a}\rho]); \tau_2).$$

(2) *Suppose that* $\epsilon_\sigma((2a+3,\rho),(2a+5,\rho)) = -1$. *Let* τ_1 *denote an irreducible tempered representation such that* σ *is a subrepresentation of* $\nu^{a+1}\rho \rtimes \tau_1$. *All irreducible non-tempered subquotients of the induced representation* $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma), L(\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]); \tau_1).$$

In both cases, all irreducible non-tempered subquotients appear in the composition series with multiplicity one.

PROPOSITION 4.5. *Suppose that* $2a+1, 2a+3 \in \mathrm{Jord}_\rho(\sigma)$ *and* $2a+5 \notin \mathrm{Jord}_\rho(\sigma)$.

(1) *Suppose that* $\epsilon_\sigma((2a+1,\rho),(2a+3,\rho)) = 1$. *Let* τ_1 *denote an irreducible tempered representation such that* σ *is a subrepresentation of* $\nu^{a+1}\rho \rtimes \tau_1$, *and let* τ_2 *denote an irreducible tempered subrepresentation of* $\nu^{a+2}\rho \rtimes \sigma$. *All irreducible non-tempered subquotients of* $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma), L(\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]); \tau_1), L(\delta([\nu^{-a-1}\rho, \nu^{a}\rho]); \tau_2).$$

(2) *Suppose that* $\epsilon_\sigma((2a+1,\rho),(2a+3,\rho)) = -1$. *Let* τ_2 *denote a unique irreducible tempered subrepresentation of* $\nu^{a+2}\rho \rtimes \sigma$. *All irreducible non-tempered subquotients of* $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma), L(\delta([\nu^{-a-1}\rho, \nu^{a}\rho]); \tau_2).$$

In both cases, all irreducible non-tempered subquotients appear in the composition series with multiplicity one.

PROPOSITION 4.6. *Suppose that* $2a+1, 2a+3, 2a+5 \in \mathrm{Jord}_\rho(\sigma)$, *and that* $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ *reduces.*

(1) *Suppose that* $\epsilon_\sigma((2a+1,\rho),(2a+3,\rho)) = \epsilon_\sigma((2a+3,\rho),(2a+5,\rho)) = 1$. *Let* τ_1 *denote an irreducible tempered representation such that* σ *is a subrepresentation of* $\nu^{a+1}\rho \rtimes \tau_1$, *and let* τ_2 *denote a unique irreducible tempered subrepresentation of* $\nu^{a+2}\rho \rtimes \sigma$. *All irreducible non-tempered subquotients of*

$$\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$$

are

$$L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma), L(\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]); \tau_1), L(\delta([\nu^{-a-1}\rho, \nu^{a}\rho]); \tau_2).$$

(2) *Suppose that* $\epsilon_\sigma((2a+1,\rho),(2a+3,\rho)) = 1$ *and* $\epsilon_\sigma((2a+3,\rho),(2a+5,\rho)) = -1$. *Let* τ_1 *stand for an irreducible tempered representation such that* σ *is a subrepresentation of* $\nu^{a+1}\rho \rtimes \tau_1$. *All irreducible non-tempered subquotients of* $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma), L(\delta([\nu^{-a-2}\rho, \nu^{a+1}\rho]); \tau_1).$$

(3) *Suppose that* $\epsilon_\sigma((2a+1,\rho),(2a+3,\rho)) = -1$ *and* $\epsilon_\sigma((2a+3,\rho),(2a+5,\rho)) = 1$. *Let* τ_2 *stand for a unique irreducible tempered subrepresentation of* $\nu^{a+2}\rho \rtimes \sigma$. *All irreducible non-tempered subquotients of* $\delta([\nu^{-a}\rho, \nu^{a+2}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-a-2}\rho, \nu^{a}\rho]); \sigma), L(\delta([\nu^{-a-1}\rho, \nu^{a}\rho]); \tau_2).$$

In each case, all irreducible non-tempered subquotients appear in the composition series with multiplicity one.

To handle the remaining case, we need one technical result.

LEMMA 4.7. *Let* $x, y \in \frac{1}{2}\mathbb{Z}$ *such that* $0 \leq x \leq y-2$. *Let* $\sigma_{ds} \in R(G)$ *stand for a discrete series such that* $2x+1 \notin \mathrm{Jord}_\rho(\sigma_{ds})$, $2y-1, 2y+1 \in \mathrm{Jord}_\rho(\sigma_{ds})$ *and* $\epsilon_{\sigma_{ds}}((2y-1,\rho),(2y+1,\rho)) = 1$. *If* τ *is an irreducible subrepresentation of* $\delta([\nu^{-x}\rho, \nu^{x}\rho]) \rtimes \sigma_{ds}$, *then* $\nu^y\rho \rtimes \tau$ *has a unique irreducible tempered subrepresentation, which contains an irreducible constituent of the form* $\nu^y\rho \times \nu^y\rho \otimes \pi$ *in the Jacquet module with respect to the appropriate parabolic subgroup.*

PROOF. In $R(G)$ we have $\delta([\nu^{-x}\rho, \nu^{x}\rho]) \rtimes \sigma_{ds} = \tau + \tau'$, where τ and τ' are mutually non-isomorphic irreducible tempered subrepresentations.

By Lemma 4.2, there is an irreducible tempered subrepresentation $\tau^{(1)}$ of $\nu^y\rho \rtimes \sigma_{ds}$. In $R(G)$ we have $\delta([\nu^{-x}\rho, \nu^{x}\rho]) \rtimes \tau^{(1)} = \tau_1 + \tau_{-1}$, where τ_1 and τ_{-1} are mutually non-isomorphic irreducible tempered subrepresentations. Thus, for $i \in \{1, -1\}$ we have the following embedding and isomorphism:

$$\tau_i \hookrightarrow \delta([\nu^{-x}\rho, \nu^{x}\rho]) \times \nu^y\rho \rtimes \sigma_{ds} \cong \nu^y\rho \times \delta([\nu^{-x}\rho, \nu^{x}\rho]) \rtimes \sigma_{ds}.$$

So, for $i \in \{1, -1\}$ there is $\pi_i \in \{\tau, \tau'\}$ such that τ_i is a subrepresentation of $\nu^y\rho \rtimes \pi_i$. Since $\delta([\nu^{-x}\rho, \nu^{x}\rho]) \otimes \sigma_{ds}$ is a unique irreducible constituent of the form $\delta([\nu^{-x}\rho, \nu^{x}\rho]) \otimes \pi$ appearing in $\mu^*(\pi_i)$, and appears there with multiplicity one, using the structural formula and Lemma 4.2 we obtain that

$\delta([\nu^{-x}\rho, \nu^x\rho]) \otimes \tau^{(1)}$ appears in $\mu^*(\nu^y\rho \rtimes \pi_i)$ with multiplicity one. Consequently, $\pi_1 \not\cong \pi_{-1}$ and there is an $i \in \{1, -1\}$ such that τ_i is a unique irreducible subrepresentation of $\nu^y\rho \rtimes \tau$.

Also, from $\epsilon_{\sigma_{ds}}((2y-1, \rho), (2y+1, \rho)) = 1$ we deduce that there is an irreducible tempered representation $\tau^{(2)}$ such that σ_{ds} is a subrepresentation of $\nu^y\rho \rtimes \tau^{(2)}$. This leads to an embedding

$$\tau_i \hookrightarrow \nu^y\rho \times \nu^y\rho \times \delta([\nu^{-x}\rho, \nu^x\rho]) \rtimes \tau^{(2)},$$

for $i \in \{1, -1\}$. Consequently, $\mu^*(\tau_i)$ contains an irreducible constituent of the form $\nu^y\rho \times \nu^y\rho \otimes \pi$, for $i \in \{1, -1\}$. □

Proposition 4.8. *Let $a, b \in \frac{1}{2}\mathbb{Z}$ such that $a \leq b - 3$, and such that $\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma$ reduces. Suppose that for $x = a+1, a+2, \ldots, b-1$ we have $2x+1 \in \mathrm{Jord}_\rho(\sigma)$, and for $y = a+2, \ldots, b-1$ we have $\epsilon_\sigma((2y-1, \rho), (2y+1, \rho)) = -1$.*

(1) *Suppose that either $2a+1 \notin \mathrm{Jord}_\rho(\sigma)$ or $2a+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2a+1, \rho), (2a+3, \rho)) = 1$, and either $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$ or $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$. Let τ_1 denote an irreducible tempered representation such that σ is a subrepresentation of $\nu^{a+1}\rho \rtimes \tau_1$, τ_3 denote a unique irreducible tempered subrepresentation of $\nu^b\rho \rtimes \sigma$, and τ_4 denote a unique irreducible tempered subrepresentation of $\nu^b\rho \rtimes \tau_1$. All irreducible non-tempered subquotients of $\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma$ are*

$$L(\delta([\nu^{-b}\rho, \nu^a\rho]); \sigma), L(\delta([\nu^{-b}\rho, \nu^{a+1}\rho]); \tau_1),$$
$$L(\delta([\nu^{-b+1}\rho, \nu^a\rho]); \tau_3), L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4).$$

(2) *Suppose that either $2a+1 \notin \mathrm{Jord}_\rho(\sigma)$ or $2a+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2a+1, \rho), (2a+3, \rho)) = 1$, and $2b+1 \in \mathrm{Jord}_\rho(\sigma)$, with $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = -1$. Let τ_1 denote an irreducible tempered representation such that σ is a subrepresentation of $\nu^{a+1}\rho \rtimes \tau_1$. All irreducible non-tempered subquotients of $\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma$ are*

$$L(\delta([\nu^{-b}\rho, \nu^a\rho]); \sigma), L(\delta([\nu^{-b}\rho, \nu^{a+1}\rho]); \tau_1).$$

(3) *Suppose that $2a+1 \in \mathrm{Jord}_\rho(\sigma)$, with $\epsilon_\sigma((2a+1, \rho), (2a+3, \rho)) = -1$, and either $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$ or $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$. Let τ_3 denote a unique irreducible tempered subrepresentation of $\nu^b\rho \rtimes \sigma$. All irreducible non-tempered subquotients of $\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma$ are*

$$L(\delta([\nu^{-b}\rho, \nu^a\rho]); \sigma), L(\delta([\nu^{-b+1}\rho, \nu^a\rho]); \tau_3).$$

In each case, all the irreducible non-tempered subquotients appear with multiplicity one in the composition series.

Proof. Let us comment only the representation $L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4)$ appearing as an irreducible subquotient in (1), other parts of the proposition follow directly from the proofs of [8, Lemmas 3.2., 3.4]. If $2a+1 \notin \mathrm{Jord}_\rho(\sigma)$, a tempered representation τ_4 is constructed in Lemma 4.2, and if $2a+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2a+1, \rho), (2a+3, \rho)) = 1$, then τ_4 is constructed in Lemma 4.7. Note that $\mu^*(L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4))$ contains an irreducible constituent of the form $\nu^{a+1}\rho \otimes \pi$.

We have the following embeddings and an isomorphism:

$$\begin{aligned} L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4) &\hookrightarrow \delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]) \rtimes \tau_4 \\ &\hookrightarrow \delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]) \times \nu^b\rho \rtimes \tau_1 \\ &\cong \nu^b\rho \times \delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]) \rtimes \tau_1. \end{aligned}$$

Thus, the Frobenius reciprocity implies that $\mu^*(L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4))$ contains an irreducible constituent of the form $\nu^b\rho \otimes \pi$.

From

$$\begin{aligned} L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4) &\leq \delta([\nu^{-a-1}\rho, \nu^{b-1}\rho]) \rtimes \tau_4 \\ &\leq \delta([\nu^{-a-1}\rho, \nu^{b-1}\rho]) \times \nu^b\rho \rtimes \tau_1, \end{aligned}$$

follows that there is an irreducible subquotient π_1 of $\delta([\nu^{-a-1}\rho, \nu^{b-1}\rho]) \times \nu^b\rho$ such that $L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4)$ is an irreducible subquotient of $\pi_1 \rtimes \tau_1$. If $\mu^*(\tau_1)$ does not contain an irreducible constituent of the form $\nu^b\rho \otimes \pi$, it follows at once that the Jacquet module of π_1 with respect to the appropriate parabolic subgroup contains an irreducible constituent of such a form, so $\pi_1 \cong \delta([\nu^{-a-1}\rho, \nu^b\rho])$. If $\mu^*(\tau_1)$ contains an irreducible constituent of the form $\nu^b\rho \otimes \pi$, it follows that $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$, and in the same way as before we obtain that $\mu^*(L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4))$ contains an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$. Definition of τ_1 implies that $\mu^*(\tau_1)$ does not contain an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$, so again the Jacquet module of π_1 with respect to the appropriate parabolic subgroup has to contain an irreducible constituent of the form $\nu^b\rho \otimes \pi$, so $\pi_1 \cong \delta([\nu^{-a-1}\rho, \nu^b\rho])$.

Thus, we have

$$L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4) \leq \delta([\nu^{-a-1}\rho, \nu^b\rho]) \rtimes \tau_1 \leq \delta([\nu^{-a}\rho, \nu^b\rho]) \times \nu^{a+1}\rho \rtimes \tau_1,$$

so there is an irreducible subquotient π_2 of $\nu^{a+1}\rho \rtimes \tau_1$ such that $L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4)$ is contained in $\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \pi_2$.

Since $\mu^*(L(\delta([\nu^{-b+1}\rho, \nu^{a+1}\rho]); \tau_4))$ contains an irreducible constituent of the form $\nu^{a+1}\rho \otimes \pi$, using the structural formula we obtain that $\mu^*(\pi_2)$ contains an irreducible constituent of such a form, and using Lemma 4.2 we deduce $\pi_2 \cong \sigma$. □

5. Ends of the first complementary series in the positive case

Let us first recall the irreducibility criterion in the case $\frac{1}{2}$, given in [8, Theorem 4.6.]:

Theorem 5.1. *Suppose that* $\frac{1}{2} \leq b$, $b - \frac{1}{2}$ *is a non-negative integer, and for* α *such that* $\nu^{\alpha}\rho \rtimes \sigma_{cusp}$ *reduces we have* $\alpha - \frac{1}{2} \in \mathbb{Z}$. *The induced representation* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma$ *is irreducible if and only if* $2b+1 \in \mathrm{Jord}_{\rho}(\sigma)$, $\epsilon(\min(\mathrm{Jord}_{\rho}(\sigma)), \rho) = -1$, *and for every* $x \in \mathrm{Jord}_{\rho}(\sigma) \cap [\min(\mathrm{Jord}_{\rho}(\sigma)), 2b+1]$ *such that* $x\_$ *is defined we have* $\epsilon((x\_, \rho), (x, \rho)) = -1$.

Further, let us recall the irreducibility criterion in the remaining case, given in [8, Theorem 5.4.]:

Theorem 5.2. *Suppose that* $1 \leq a \leq b$ *and for* α *such that* $\nu^{\alpha}\rho \rtimes \sigma_{cusp}$ *reduces we have* $a - \alpha \in \mathbb{Z}$. *The induced representation* $\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma$ *is irreducible if and only if one of the following holds:*

(1) $[2a-1, 2b+1] \cap \mathrm{Jord}_{\rho}(\sigma) = \emptyset$.
(2) $2b+1 \in \mathrm{Jord}_{\rho}(\sigma)$ *and for every* $x \in [2a+1, 2b+1] \cap \mathrm{Jord}_{\rho}(\sigma)$ *such that* $x\_$ *is defined and* $x\_ \geq 2a-1$ *we have* $\epsilon((x\_, \rho), (x, \rho)) = -1$.

The following result can be proved in the same way as Lemma 4.2, using [8, Lemma 4.2] and the proof of [8, Lemma 4.4].

Proposition 5.3. *Suppose that* $\mathrm{Jord}_{\rho}(\sigma)$ *consists of even integers. Then* $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ *is irreducible if and only if* $2 \in \mathrm{Jord}_{\rho}(\sigma)$ *and* $\epsilon_{\sigma}(2, \rho) = -1$. *If* $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ *reduces, in* $R(G)$ *we have*

$$\nu^{\frac{1}{2}}\rho \rtimes \sigma = L(\nu^{-\frac{1}{2}}\rho; \sigma) + \tau,$$

where τ *is an irreducible tempered representation, which is a discrete series representation if and only if* $2 \notin \mathrm{Jord}_{\rho}(\sigma)$.

Now we state a result which can be obtained in the same way as the ones from previous section, using the proof of [8, Lemma 4.4].

Proposition 5.4. *Suppose that* $2 \in \mathrm{Jord}_{\rho}(\sigma)$ *and* $4 \notin \mathrm{Jord}_{\rho}(\sigma)$. *Let* $\sigma^{(1)}$ *denote a unique discrete series subrepresentation of* $\nu^{\frac{3}{2}}\rho \rtimes \sigma$.

(1) *Suppose that* $\epsilon_{\sigma}(2, \rho) = 1$. *Let* $\sigma^{(2)}$ *stand for a discrete series such that* σ *is a subrepresentation of* $\nu^{\frac{1}{2}}\rho \rtimes \sigma^{(2)}$. *All irreducible non-tempered subquotients of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma), L(\nu^{-\frac{1}{2}}\rho; \sigma^{(1)}), L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{1}{2}}\rho]); \sigma^{(2)}).$$

(2) *Suppose that* $\epsilon_{\sigma}(2, \rho) = -1$. *All irreducible non-tempered subquotients of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma), L(\nu^{-\frac{1}{2}}\rho; \sigma^{(1)}).$$

In both cases, all the irreducible non-tempered subquotients appear in the composition series with multiplicity one.

Proof of the following proposition is left to the reader.

PROPOSITION 5.5. *Suppose that* $2, 4 \in \mathrm{Jord}_\rho(\sigma)$*, and that* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma$ *reduces.*

(1) *Suppose that* $\epsilon_\sigma(2, \rho) = 1$ *and* $\epsilon_\sigma((2, \rho), (4, \rho)) = 1$*. Let* $\sigma^{(2)}$ *denote a discrete series such that* σ *is a subrepresentation of* $\nu^{\frac{1}{2}}\rho \rtimes \sigma^{(2)}$*, and let* τ_5 *stand for a unique irreducible tempered subrepresentation of* $\nu^{\frac{3}{2}}\rho \rtimes \sigma$*. All irreducible non-tempered subquotients of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma$ *are*
$$L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma), L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{1}{2}}\rho]); \sigma^{(2)}), L(\nu^{-\frac{1}{2}}\rho; \tau_5).$$

(2) *Suppose that* $\epsilon_\sigma(2, \rho) = 1$ *and* $\epsilon_\sigma((2, \rho), (4, \rho)) = -1$*. Let* $\sigma^{(2)}$ *denote a discrete series such that* σ *is a subrepresentation of* $\nu^{\frac{1}{2}}\rho \rtimes \sigma^{(2)}$*. All irreducible non-tempered subquotients of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma$ *are*
$$L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma), L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{1}{2}}\rho]); \sigma^{(2)}).$$

(3) *Suppose that* $\epsilon_\sigma(2, \rho) = -1$ *and* $\epsilon_\sigma((2, \rho), (4, \rho)) = 1$*. Let* τ_5 *stand for a unique irreducible tempered subrepresentation of* $\nu^{\frac{3}{2}}\rho \rtimes \sigma$*. All irreducible non-tempered subquotients of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma$ *are*
$$L(\delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma), L(\nu^{-\frac{1}{2}}\rho; \tau_5).$$

In each case, all the irreducible non-tempered subquotients appear in the composition series with multiplicity one.

PROPOSITION 5.6. *Let* $b \in \frac{1}{2}\mathbb{Z}$*,* $b \geq \frac{5}{2}$*, be such that* $b - \frac{1}{2}$ *is an integer. Suppose that* $\{2, 4, \ldots, 2b-1\} \subseteq \mathrm{Jord}_\rho(\sigma)$ *and that for* $x = 4, 6, \ldots, 2b-1$ *we have* $\epsilon_\sigma((x-2, \rho), (x, \rho)) = -1$*. Suppose that* $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma$ *reduces.*

(1) *Suppose that* $\epsilon_\sigma(2, \rho) = 1$*,* $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ *and* $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = -1$*. Let* $\sigma^{(2)}$ *denote a discrete series such that* σ *is a subrepresentation of* $\nu^{\frac{1}{2}}\rho \rtimes \sigma^{(2)}$*. All irreducible non-tempered subquotients of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma$ *are*
$$L(\delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma), L(\delta([\nu^{-b}\rho, \nu^{\frac{1}{2}}\rho]); \sigma^{(2)}).$$

(2) *Suppose that* $\epsilon_\sigma(2, \rho) = 1$*, and either* $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$ *or* $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ *and* $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$*. Let* $\sigma^{(2)}$ *denote a discrete series such that* σ *is a subrepresentation of* $\nu^{\frac{1}{2}}\rho \rtimes \sigma^{(2)}$*, let* τ_3 *denote a unique irreducible tempered subrepresentation of* $\nu^b\rho \rtimes \sigma$*, and let* τ_6 *denote a unique irreducible tempered subrepresentation of* $\nu^b\rho \rtimes \sigma^{(2)}$*. All irreducible non-tempered subquotients of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma$ *are*
$$L(\delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma), L(\delta([\nu^{-b}\rho, \nu^{\frac{1}{2}}\rho]); \sigma^{(2)}),$$
$$L(\delta([\nu^{-b+1}\rho, \nu^{-\frac{1}{2}}\rho]); \tau_3), L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6).$$

(3) *Suppose that* $\epsilon_\sigma(2,\rho) = -1$, *and either* $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$ *or* $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ *and* $\epsilon_\sigma((2b-1,\rho),(2b+1,\rho)) = 1$. *Let* τ_3 *denote a unique irreducible tempered subrepresentation of* $\nu^b\rho \rtimes \sigma$. *All irreducible non-tempered subquotients of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]); \sigma), L(\delta([\nu^{-b+1}\rho, \nu^{-\frac{1}{2}}\rho]); \tau_3).$$

In each case, all the irreducible non-tempered subquotients appear in the composition series with multiplicity one.

Proof. Let us only discuss the subquotient $L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6)$ in the case when $\epsilon_\sigma(2,\rho) = 1$, and either $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$ or $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2b-1,\rho),(2b+1,\rho)) = 1$. Other parts of the proof can be deduced in the same way as before, using the proofs of [8, Lemmas 4.4, 4.5.].

Note that $\mu^*(L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6))$ contains irreducible constituents of the form $\nu^{\frac{1}{2}}\rho \otimes \pi_1$ and of the form $\nu^b\rho \otimes \pi_2$, since τ_6 is a subrepresentation of $\nu^b\rho \rtimes \sigma^{(2)}$. Also, $\mu^*(L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6))$ contains an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$ if and only if $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2b-1,\rho),(2b+1,\rho)) = 1$. From

$$L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6) \leq \delta([\nu^{-\frac{1}{2}}\rho, \nu^{b-1}\rho]) \rtimes \tau_6 \leq \delta([\nu^{-\frac{1}{2}}\rho, \nu^{b-1}\rho]) \times \nu^b\rho \rtimes \sigma^{(2)},$$

in the same way as in the proof of Proposition 4.8 we get

$$L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6) \leq \delta([\nu^{-\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^{(2)}.$$

Thus,

$$L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6) \leq \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \nu^{\frac{1}{2}}\rho \rtimes \sigma^{(2)},$$

so there is an irreducible subquotient π of $\nu^{\frac{1}{2}}\rho \rtimes \sigma^{(2)}$ such that $L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6)$ is contained in $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \pi$. Since $\mu^*(L(\delta([\nu^{-b+1}\rho, \nu^{\frac{1}{2}}\rho]); \tau_6))$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \pi_1$, using Proposition 5.3 we obtain $\pi \cong \sigma$. □

Remaining case is covered by the following proposition, which can be proved following the same lines as in the previously considered cases, using proofs of [8, Lemmas 5.2., 5.3].

Proposition 5.7. *Let* $a, b \in \frac{1}{2}\mathbb{Z}$ *such that* $b - a \in \mathbb{Z}$ *and* $1 \leq a \leq b-1$. *Suppose that for* $x = a-1, a, \ldots, b-1$ *we have* $2x+1 \in \mathrm{Jord}_\rho(\sigma)$, *and that for* $y = a, \ldots, b-1$ *we have* $\epsilon_\sigma((2y-1,\rho),(2y+1,\rho)) = -1$. *Suppose that* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ *reduces, and let* τ_3 *stand for a unique irreducible tempered subrepresentation of* $\nu^b\rho \rtimes \sigma$. *All irreducible non-tempered subquotients of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$ *are*

$$L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \sigma), L(\delta([\nu^{-b+1}\rho, \nu^{-a}\rho]); \tau_3),$$

both appearing in the composition series with multiplicity one.

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Krajevi prvih komplementarnih serija generaliziranih osnovnih serija

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Sažetak. U radu ćemo odrediti sve ireducibilne netemperirane kompozicione faktore induciranih reprezentacija koje se pojavljuju u krajevima prvih komplementarnih serija generaliziranih osnovnih serija simplektičke, specijalne neparno ortogonalne ili unitarne grupe nad nearhimedskim lokalnim poljem.

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