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A NOTE ON THE CHEBYSHEV INEQUALITY AND RELATED INEQUALITIES FOR FIBONACCI NUMBERS

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Abstract. Some new results for Fibonacci sequence concerning the Chebyshev type inequalities are proved.

1. Introduction

The Chebyshev inequality is the important inequality in mathematical analysis which state that

$$\sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i x_i y_i \geq \sum_{j=1}^{n} p_j x_j \sum_{i=1}^{n} p_i y_i \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n)$ are n-tuples monotonic in the same direction, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is a positive n-tuple.

If $\mathbf{x}$ and $\mathbf{y}$ are monotonic in opposite direction then the reverse of the inequality (1.1) holds.

In either case equality holds if and only if either $x_1 = x_2 = \cdots = x_n$ or $y_1 = y_2 = \cdots = y_n$.

The Chebyshev inequality can be generalized for m nonnegative n-tuples $\mathbf{x}_j = (x_{j1}, x_{j2}, \ldots, x_{jn})$, $j = 1, 2, \ldots, m : m > 2)$ which are monotonic in the same direction. Then it holds

$$\Big(\sum_{i=1}^{n} p_i\Big)^{m-1} \sum_{i=1}^{n} p_i \Big(\prod_{j=1}^{m} x_{ji}\Big) \geq \prod_{j=1}^{m} \Big(\sum_{i=1}^{n} p_i x_{ji}\Big). \tag{1.2}$$

If all n-tuples $\mathbf{x}$ are positive, then the equality in (1.2) holds if and only if at least $m-1$ n-tuples among $\mathbf{x}_1, \ldots, \mathbf{x}_m$ have identical components.

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Let us denote difference of the Chebyshev inequality

$$T_n(\mathbf{x},\mathbf{y};\mathbf{p}) = \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i x_i y_i - \sum_{j=1}^{n} p_j x_j \sum_{i=1}^{n} p_i y_i. \tag{1.3}$$

We will also consider inequalities related to the Chebyshev inequality. The Grüss inequality provides bound for the difference in the Chebyshev inequality and the Karamata inequality is an analogous result for the ratio (see [7], pp. 296, 298 and [6], pp. 206, 212). There are a number of further refinements and generalizations of Grüss inequality.

Let us recall the definition of the Fibonacci sequence F_n: F_n is the n^{th} Fibonacci number defined by $F_0 = 0$, $F_1 = 1$ and for all $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

Furthermore, for Fibonacci numbers, let us state some known identities (see [4], p. 11 and p. 61):

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}, \tag{1.4}$$

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1, \tag{1.5}$$

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}, \tag{1.6}$$

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1. \tag{1.7}$$

$$\sum_{i=1}^{n} i\, F_i = F_{n+2} - F_{n+3} + 2, \tag{1.8}$$

$$\sum_{i=1}^{n} F_{4i-2} = F_{2n}^2, \tag{1.9}$$

$$\sum_{i=1}^{n} \binom{n}{i} F_{2i} = F_{2n}. \tag{1.10}$$

In this note we are inspired by the Popescu and Diaz Barrero result for Fibonacci numbers F_n published in [8]:

Theorem A1. *Let n be a positive integer and l be an integer. Then it holds*

$$(F_n F_{n+1})^2 \leq \sum_{i}^{n} F_i^l \sum_{i}^{n} F_i^{4-l}. \tag{1.11}$$

The authors used the Jensen inequality for convex functions and the proof reveals that (1.11) is valid for all $n \in \mathbb{N}$ and all $l \in \mathbb{R}$. Recently, Alzer and Luca in [2] obtained the following extension of this result by using the Cauchy-Schwarz inequality.

Theorem A2. *Let $r, s \in \mathbb{R}$ with $r + s \geq 4$. Then for $n \geq 1$, it holds*

$$(F_n F_{n+1})^2 \leq \sum_{i}^{n} F_i^r \sum_{i}^{n} F_i^s. \tag{1.12}$$

The sign of equality is valid in (1.12) if and only if $n = 1, 2$ or $n \geq 3$, $r = s = 2$.

Alzer and Kwong in [1] determined by using computer software all real parameters r and s such that inequality (1.12) holds for $n \geq 1$.

We use the reverse Chebyshev inequality to get the result in Theorem A1 and result in Theorem A2 for special case $r + s = 4$.

Theorem A3. *Let $n \in \mathbb{N}$ and $c \in \mathbb{R}$. Then it holds*

$$(F_n F_{n+1})^2 \leq \sum_{j=1}^{n} F_j^{2+c} \sum_{i=1}^{n} F_i^{2-c}. \tag{1.13}$$

Equality holds if and only if either $n = 1, 2$ or $n \geq 3$, $c = 0$.

Proof. Let us use the reverse Chebyshev inequality (1.1) for n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n)$ which are monotonic in the opposite direction and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is a positive n-tuple with the following substitutions: $p_i = F_i^2$, $x_i = F_i^c$ and $y_i = F_i^{-c}$ for $i = 1, 2, \ldots, n$ and $c \in \mathbb{R}$:

$$\sum_{j=1}^{n} F_j^2 \sum_{i=1}^{n} F_i^2 F_i^c F_i^{-c} \leq \sum_{j=1}^{n} F_j^2 F_j^c \sum_{i=1}^{n} F_i^2 F_i^{-c}$$

$$\sum_{j=1}^{n} F_j^2 \sum_{i=1}^{n} F_i^2 \leq \sum_{j=1}^{n} F_j^{2+c} \sum_{i=1}^{n} F_i^{2-c}.$$

By using the identities (1.4) we get the inequality (1.13).

The condition for the equality in the Chebyshev inequality give us the condition for the equality in (1.13). □

Remark 1.1. For $l = c + 2$ in (1.13) we get the inequality in Theorem A1 and for $r = 2 + c$ and $s = 2 - c$ we get the inequality in Theorem A2 if $r + c = 4$.

2. Chebyshev inequality for Fibonacci numbers

Theorem 2.1. *Let $n \in \mathbb{N}$, and $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ be a positive n-tuple with $P_n = \sum_{i=1}^{n} p_i$. Let f and g be real valued functions. If f and g are* *monotonic in the same direction, then it holds*

$$P_n \sum_{i=1}^{n} p_i f(F_i)\, g(F_i) \geq \sum_{j=1}^{n} p_j\, f(F_j) \sum_{i=1}^{n} p_i\, g(F_i). \tag{2.1}$$

If f and g are monotonic in opposite direction then the reverse of the inequality (2.1) holds.

Proof. We use the Chebyshev inequality for n-tuples $\mathbf{x}=(x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ and for positive n-tuple $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ with the following substitutions: $x_i = f(F_i)$ and $y_i = g(F_i),\ i = 1, 2, \ldots, n$ for functions f and g which are monotonic in the same direction. ∎

Corollary 2.2. *Let $n \in \mathbb{N}$. If r and $s \in \mathbb{R}$ such that $r\,s > 0$, then it holds*

$$F_n F_{n+1} \sum_{i=1}^{n} F_i^{2+r+s} \geq \sum_{j=1}^{n} F_j^{2+r} \sum_{i=1}^{n} F_i^{2+s}. \tag{2.2}$$

If r and $s \in \mathbb{R}$ such that $r\,s < 0$, then the reverse of inequality (2.2) holds.

Proof. We apply (2.1) for functions $f(x) = x^r$ and $g(x) = x^s$ such that $rs > 0$ with substitutions $p_i = F_i^2$ and $x_i = F_i^r$, $y_i = F_i^s$. The identities (1.4) give us the inequality (2.2). ∎

Remark 2.3. For $r = c$, $s = -c$ for $c \in \mathbb{R}$ we get a result in Theorem A3.

Theorem 2.4. *Let $n \in \mathbb{N}$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is a positive n-tuple and $P_n = \sum\limits_{i=1}^{n} p_i$. Let $f_1, f_2, f_3, \ldots, f_m$, $m > 2$, be nonnegative real valued functions. If $f_1, f_2, f_3, \ldots, f_m$ are monotonic in the same direction, then it holds*

$$\big(P_n\big)^{m-1} \sum_{i=1}^{n} p_i \Big(\prod_{j=1}^{m} f_j(F_i)\Big) \geq \prod_{j=1}^{m} \Big(\sum_{i=1}^{n} p_i\, f_j(F_i)\Big). \tag{2.3}$$

Proof. Let us use the Chebyshev inequality (1.2) for m nonnegative n-tuples $\mathbf{x}_j = (x_{j1}, x_{j2}, \ldots, x_{jn})$, $j = 1, 2, \ldots, m : m > 2)$ which are monotonic in the same direction, with the following substitutions: $x_{ji} = f_j(F_i)$, $j = 1, \ldots, m$ $i = 1, 2, \ldots, n$ for functions $f_j\, j = 1, \ldots, m$ which are positive and monotonic in the same direction. ∎

COROLLARY 2.5. *Let* $n \in \mathbb{N}$ *and* $r_1, r_2, r_3, \ldots, r_m \in \mathbb{R}$, $m > 2$. *If* $\prod_{j=1}^{m} r_j > 0$, *then it holds*

$$(F_n F_{n+1})^{m-1} \sum_{i=1}^{n} F_i^{2+\sum_{j=1}^{m} r_j} \geq \prod_{j=1}^{m} \Big(\sum_{i=1}^{n} F_i^{2+r_j} \Big). \tag{2.4}$$

PROOF. Let us use the Chebyshev inequality (1.2) for m nonnegative n-tuples $\mathbf{x}_j = (x_{j1}, x_{j2}, \ldots, x_{jn})$, $j = 1, 2, \ldots, m$, $m > 2$, which are monotonic in the same direction, with the following substitutions: $x_{ji} = f_j(F_i)$, $j = 1, \ldots, m$ $i = 1, 2, \ldots, n$ for functions $f_j(x) = x^{r_j}$, $j = 1, \ldots, m$. If $\prod_{j=1}^{m} r_j > 0$, then functions f_j are positive and monotonic in the same direction. We are setting in (1.2): $p_i = F_i^2$ and $x_{ji} = F_i^{r_j}$ for $j = 1, \ldots, m; i = 1, \ldots, n$.

By using the identities (1.4) we get the inequality (2.4). ∎

REMARK 2.6. As special cases of Theorem 2.1 and Theorem 2.4 we can establish new inequalities if we select for weights $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ the following substitutions and corresponding $P_n = \sum_{i}^{n} p_i$ according identities (1.4) – (1.10), respectively:

$$p_i = F_i^2,\ p_i = F_i,\ p_i = F_{2i-1}, p_i = F_{2i},\ p_i = i\,F_i,\ p_i = F_{4i-2},\ p_i = \binom{n}{i} F_{2i}.$$

3. CHEBYSHEV INEQUALITY FOR LUCAS NUMBERS

Let us recall the definition of the Lucas numbers L_n: L_n is the n^{th} Lucas number defined by $L_0 = 2$, $L_1 = 1$ and for all $n \geq 1$,

$$L_n = L_{n-1} + L_{n-2}$$

or, alternatively,

$$L_n = F_{n+1} + F_{n-1}.$$

Furthermore, let us state some known identities for Lucas numbers (see [4], p. 98):

$$\sum_{i=1}^{n} L_i^2 = L_n L_{n+1} - 2, \tag{3.1}$$

$$\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2, \tag{3.2}$$

$$\sum_{i=1}^{n} L_{2i} = L_{2n+1} - 1, \tag{3.3}$$

$$\sum_{i=1}^{n} i\, L_i = nL_{n+2} - L_{n+3} + 4. \tag{3.4}$$

By using reverse Chebyshev inequality we can obtain inequality related to Theorem A3 for Lucas numbers.

THEOREM A4. *Let $n \in \mathbb{N}$ and $c \in \mathbb{R}$. Then it holds*

$$\big(L_n L_{n+1} - 2\big)^2 \leq \sum_{j=1}^{n} L_j^{2+c} \sum_{i=1}^{n} L_i^{2-c}. \tag{3.5}$$

REMARK 3.1. As special case of Theorem A4 we can establish new inequalities for Lucas numbers if we select for weights $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ the following substitutions and corresponding $P_n = \sum_i^n p_i$ according identities (3.1) – (3.4), respectively:

$$p_i = L_i^2,\ p_i = L_{2i-1},\ p_i = L_{2i},\ p_i = i\, L_i.$$

We can state similar result as Theorem 2.1 and Theorem 2.4 for Lucas numbers.

For mixed identities of Fibonacci and Lucas number (see [4], p. 110):

$$\sum_{i=1}^{n} \binom{n}{i} F_{n-i} F_i = \frac{1}{5}(2^n L_n - 2), \tag{3.6}$$

$$\sum_{i=1}^{n} \binom{n}{i} L_{n-i} F_i = 2^n F_n. \tag{3.7}$$

we present the following corollaries.

COROLLARY 3.2. *Let $n \in \mathbb{N}$. If r and $s \in \mathbb{R}$ such that $r\,s > 0$ then for Fibonacci numbers and Lucas numbers it holds*

(3.8)

$$\frac{1}{5}(2^n L_n - 2) \sum_{i=1}^{n} \binom{n}{i} F_{n-i} F_i^{1+r+s} \geq \sum_{j=1}^{n} \binom{n}{i} F_{n-i} F_j^{1+r} \sum_{i=1}^{n} \binom{n}{i} F_{n-i} F_i^{1+s},$$

$$(3.9)\quad (2^n F_n)\sum_{i=1}^{n}\binom{n}{i}L_{n-i}F_i^{1+r+s} \geq \sum_{j=1}^{n}\binom{n}{i}L_{n-i}F_j^{1+r}\sum_{i=1}^{n}\binom{n}{i}L_{n-i}F_i^{1+s}.$$

If r and $s \in \mathbb{R}$ such that $r\,s < 0$, then for Fibonacci and Lucas numbers the reverse of inequality (3.8) and (3.9) holds.

Proof. We apply (2.1) for functions $f(x) = x^r$ and $g(x) = x^s$ such that $rs > 0$ with substitutions $p_i = \binom{n}{i}F_{n-i}F_i$ or $p_i = \binom{n}{i}L_{n-i}F_i$ and $x_i = F_i^r$, $y_i = F_i^s$. The identities (3.6) and (3.7) give us the inequality (3.8) and (3.9), respectively. □

4. Grüss inequality and Karamata inequality for Fibonacci numbers and Lucas numbers

The following theorem point out the Grüss inequality for Fibonacci numbers.

Theorem 4.1. *Let $n \in \mathbb{N}$, and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ be a positive n-tuple with $P_n = \sum_{i=1}^{n} p_i$. Let f and g be real valued functions such that it holds*

$$(4.1)\quad 0 < m_1 < M_1,\ 0 < m_2 < M_2, m_1 \leq f(F_i) \leq M_1,\ m_2 \leq g(F_i) \leq M_2.$$

Then it holds

$$(4.2)\quad \left|\frac{\sum_{i=1}^{n} p_i f(F_i)g(F_i)}{P_n} - \frac{\sum_{j=1}^{n} p_j f(F_j)}{P_n}\,\frac{\sum_{i=1}^{n} p_i g(F_i)}{P_n}\right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

Proof. As a complement of the Chebyshev inequality holds (discrete) weighted Grüss' inequality (see [7], p.296) holds, we use the Grüss inequality for n-tuples $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n)$ and for positive n-tuple $\mathbf{p} = (p_1, p_2, \dots, p_n)$ with the following substitutions: $x_i = f(F_i)$ and $y_i = g(F_i)$, $i = 1, 2, \dots, n$ for functions f and g such that the condition (4.1) is satisfied. □

Corollary 4.2. *Let $n \in \mathbb{N}$, $n > 2$.*
If r and $s \in \mathbb{R}$ such that $r\,s > 0$, then it holds

(4.3)
$$\left|\frac{1}{F_nF_{n+1}}\sum_{i=1}^{n}F_i^{2+r+s} - \frac{1}{(F_nF_{n+1})^2}\sum_{j=1}^{n}F_j^{2+r}\sum_{i=1}^{n}F_i^{2+s}\right| \leq \frac{1}{4}(F_n^r - 1)(F_n^s - 1).$$

If r and $s \in \mathbb{R}$ such that $r\,s < 0$, then it holds

$$\left|\frac{1}{F_nF_{n+1}}\sum_{i=1}^{n}F_i^{2+r+s} - \frac{1}{(F_nF_{n+1})^2}\sum_{j=1}^{n}F_j^{2+r}\sum_{i=1}^{n}F_i^{2+s}\right| \leq \frac{1}{4}(F_n^r - 1)(1 - F_n^s).$$

Proof. We apply (4.2) for functions $f(x) = x^r$ and $g(x) = x^s$ such that $r\,s > 0$ which satisfied (4.1) with substitutions $p_i = F_i^2$. The identities (1.4) give us the inequality (4.3). We proceed analogously for the case $r\,s < 0$. □

Corollary 4.3. *Let $n \in \mathbb{N}$, $n > 2$.*

If r and $s \in \mathbb{R}$ such that $r\,s > 0$, then for Fibonacci and Lucas numbers it holds

$$\Big|\frac{1}{F_nF_{n+1}}\sum_{i=1}^{n} F_i^{2+r}L_i^s - \frac{1}{(F_nF_{n+1})^2}\sum_{j=1}^{n} F_j^{2+r}\sum_{i=1}^{n} F_i^2L_i^s\Big| \le \frac{1}{4}(F_n^r - 1)(L_n^s - 1)$$

If r and $s \in \mathbb{R}$ such that $r\,s < 0$, then for Fibonacci and Lucas numbers it holds

$$\Big|\frac{1}{F_nF_{n+1}}\sum_{i=1}^{n} F_i^{2+r}L_i^s - \frac{1}{(F_nF_{n+1})^2}\sum_{j=1}^{n} F_j^{2+r}\sum_{i=1}^{n} F_i^2L_i^s\Big| \le \frac{1}{4}(F_n^r - 1)(1 - L_n^s)$$

Recall now the Karamata inequality (see [7], p. 298 and [6], p. 212):

$$K^{-2} \le \frac{\Big(\sum\limits_{i=1}^{n} p_ix_i\Big)\Big(\sum\limits_{i=1}^{n} p_iy_i\Big)}{P_n\sum\limits_{i=1}^{n} p_ix_i\,y_i} \le K^2, \tag{4.4}$$

$$K = \frac{\sqrt{m_1\,m_2} + \sqrt{M_1\,M_2}}{\sqrt{m_1\,M_2} + \sqrt{M_1\,m_2}} \ge 1$$

where $\mathbf{x} = (x_1, x_{2,}..., x_n), \mathbf{y} = (y_1, y_{2,}..., y_n)$ are n-tuples such that the condition

$$0 < m_1 < M_1,\ 0 < m_2 < M_2, m_1 \le x_k \le M_1,\ m_2 \le y_k \le M_2; \tag{4.5}$$

holds and $\mathbf{p} = (p_1, p_{2,}..., p_n)$ is a positive n-tuple with $P_n = \sum\limits_{j=1}^{n} p_j$.

We point out the Karamata inequality for Fibonacci numbers.

Theorem 4.4. *Let $n \in \mathbb{N}$, and $\mathbf{p} = (p_1, p_{2,}..., p_n)$ be a positive n-tuple with $P_n = \sum\limits_{i=1}^{n} p_i$. Let f and g be real valued functions such that it holds*

$$0 < m_1 < M_1,\ 0 < m_2 < M_2, m_1 \le f(F_i) \le M_1,\ m_2 \le g(F_i) \le M_2. \tag{4.6}$$

Then it holds

$$K^{-2} \le \frac{\sum\limits_{i=1}^{n} p_if(F_i)\cdot\sum\limits_{i=1}^{n} p_ig(F_i)}{P_n\sum\limits_{i=1}^{n} p_if(F_i)\,g(F_i)} \le K^2, \tag{4.7}$$

$$K = \frac{\sqrt{m_1\,m_2} + \sqrt{M_1\,M_2}}{\sqrt{m_1\,M_2} + \sqrt{M_1\,m_2}} \ge 1.$$

PROOF. We use the Karamata inequality (see [7], p. 298 and [6], p. 212) for n-tuples $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n)$ and for positive n-tuple $\mathbf{p} = (p_1, p_2, \dots, p_n)$ with the following substitutions: $x_i = f(F_i)$ and $y_i = g(F_i),\ i = 1, 2, \dots, n$ for functions f and g such that the condition (4.6) is satisfied. □

COROLLARY 4.5. *Let $n \in \mathbb{N}$.*
If r and $s \in \mathbb{R}$ such that $r\,s > 0$, then it holds

$$K^{-2} \le \frac{\sum\limits_{i=1}^{n} F_i^{2+r} \cdot \sum\limits_{i=1}^{n} F_i^{2+s}}{F_n F_{n+1} \sum\limits_{i=1}^{n} F_i^{2+r+s}} \le K^2, \tag{4.8}$$

$$K = \frac{1 + \sqrt{F_n^{r+s}}}{\sqrt{F_n^s} + \sqrt{F_n^r}} \ge 1$$

If r and $s \in \mathbb{R}$ such that $r\,s < 0$, then it holds

$$K^{-2} \le \frac{\sum\limits_{i=1}^{n} F_i^{2+r} \cdot \sum\limits_{i=1}^{n} F_i^{2+s}}{F_n F_{n+1} \sum\limits_{i=1}^{n} F_i^{2+r+s}} \le K^2, \tag{4.9}$$

$$K = \frac{\sqrt{F_n^s} + \sqrt{F_n^r}}{1 + \sqrt{F_n^{r+s}}} \ge 1$$

PROOF. We apply (4.4) for functions $f(x) = x^r$ and $g(x) = x^s$ such that $r\,s > 0$ with substitutions $p_i = F_i^2$ and $x_i = F_i^r,\ y_i = F_i^s$. The identities (1.4) give us the inequality (4.8). We proceed analogously for the case $r\,s < 0$. □

For Fibonacci and Lucas numbers it holds the following Karamata inequality.

COROLLARY 4.6. *Let $n \in \mathbb{N}$.*
If r and $s \in \mathbb{R}$ such that $r\,s > 0$, then it holds

$$K^{-2} \le \frac{\sum\limits_{i=1}^{n} F_i^{2+r} \cdot \sum\limits_{i=1}^{n} F_i^2 L_i^s}{F_n F_{n+1} \sum\limits_{i=1}^{n} F_i^{2+r} L_i^s} \le K^2, \tag{4.10}$$

$$K = \frac{1 + \sqrt{F_n^r L_n^s}}{\sqrt{L_n^s} + \sqrt{F_n^r}} \ge 1.$$

If r and $s \in \mathbb{R}$ such that $r\,s < 0$, then it holds

$$K^{-2} \le \frac{\sum_{i=1}^{n} F_i^{2+r} \cdot \sum_{i=1}^{n} F_i^2 L_i^s}{F_n F_{n+1} \sum_{i=1}^{n} F_i^{2+r} L_i^s} \le K^2, \tag{4.11}$$

$$K = \frac{\sqrt{L_n^s} + \sqrt{F_n^r}}{1 + \sqrt{F_n^r L_n^s}} \ge 1.$$

Proof. We apply (4.4) for functions $f(x) = x^r$ and $g(x) = x^s$ such that $r\,s >$ with substitutions $p_i = F_i^2$ and $x_i = F_i^r$, $y_i = L_i^s$. The identities (1.4) give us the inequality (4.10). □

5. Extension of Grüss inequality for Fibonacci numbers and Lucas number

For Fibonacci numbers the following interpolation result holds.

Theorem 5.1. *Let $n \in \mathbb{N}$, and $T_n(\mathbf{a}, \mathbf{b}; \mathbf{p})$ defined by (1.3). If functions f and g are monotonic in the same direction, then for n-tuples $\mathbf{a} = (f(F_1), f(F_2), \cdots, f(F_n))$ and $\mathbf{b} = (g(F_1), g(F_2), \cdots, g(F_n))$ it holds*

$$T_n(\mathbf{a}, \mathbf{b}; p) \ge T_{n-1}(\mathbf{a}, \mathbf{b}; p) \ge \cdots \ge T_2(\mathbf{a}, \mathbf{a}; p) \ge 0. \tag{5.1}$$

Proof. We use the refinement of the Chebyshev inequality (see [6], p. 275) for T_n with positive $p_{ij} = p_i p_j$. □

Theorem 5.2. *Let $n \in \mathbb{N}$, and $T_n(\mathbf{a}, \mathbf{b}; \mathbf{p})$ defined by (1.3).*

(i) If functions f and g are monotonic in the same direction, then for n-tuples $\mathbf{a} = (f(F_1), f(F_2), \cdots, f(F_n))$ and $\mathbf{b} = (g(F_1), g(F_2), \cdots, g(F_n))$ such that $f(F_{k+1}) - f(F_k) \ge m$ and $g(F_{k+1}) - g(F_k) \ge r$, $k = 1, \cdots, n-1$ it holds

$$T_n(\mathbf{a}, \mathbf{b}; \mathbf{p}) \ge mrT_n(\mathbf{e}, \mathbf{e}; \mathbf{p}) \ge 0, \tag{5.2}$$

where $\mathbf{e} = (0, 1, 2, \cdots, n-1)$.

(ii) If functions f and g are monotonic in the opposite direction then

$$T_n(\mathbf{a}, \mathbf{b}; \mathbf{p}) \le mrT_n(\overline{\mathbf{e}}, \overline{\mathbf{e}}; \mathbf{p}) \le 0, \tag{5.3}$$

where $\overline{\mathbf{e}} = (n-1, n, \cdots, 1, 0)$.

Proof. We use the refinement of the Chebyshev inequality (see [7], p. 207) for T_n with positive p_i for Fibonacci numbers. □

Theorem 5.3. *Let $n \in \mathbb{N}$, and $\mathbf{p} = (p_1, p_{2,\ldots,} p_n)$ be a positive n-tuple with $P_n = \sum_{i=1}^{n} p_i$. Let f and g be real valued functions such that it holds*

$$0 < m_1 < M_1,\ 0 < m_2 < M_2, m_1 \le f(F_i) \le M_1,\ m_2 \le g(F_i) \le M_2. \tag{5.4}$$

Then it holds

$$\left|\frac{\sum_{i=1}^{n} p_i f(F_i) g(F_i)}{P_n} - \frac{\sum_{j=1}^{n} p_j f(F_j)}{P_n} \frac{\sum_{i=1}^{n} p_i g(F_i)}{P_n}\right| \leq \tag{5.5}$$
$$\frac{1}{4}(M_1 - m_1)(M_2 - m_2) \max_{J \subset I_n} P(J)\,(1 - P(J)),$$

where $I_n = \{1, 2, \cdots, n\}$ *and* $P(J) = \frac{1}{P_n} \sum_{k \in J} p_k$ *for* $J \subset I_n$.

Proof. We use the extension of the Grüss inequality (see Corollary 2.6 in [5] and [3]) for $D(\mathbf{x}, \mathbf{y}; \mathbf{p}) = \frac{1}{P_n^2} T_n(\mathbf{x}, \mathbf{y}; \mathbf{p})$ with positive p_i. □

Theorem 5.4. *Let* $n \in \mathbb{N}$, *and* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ *be a positive* n*-tuple with* $P_n = \sum_{i=1}^{n} p_i$. *Let* f *and* g *be real valued functions such that* f *is monotonically decreasing (or increasing) and it holds*

$$0 < m_1 < M_1,\ 0 < m_2 < M_2, m_1 \leq f(F_i) \leq M_1,\ m_2 \leq g(F_i) \leq M_2. \tag{5.6}$$

Then it holds

$$\left|\frac{\sum_{i=1}^{n} p_i f(F_i) g(F_i)}{P_n} - \frac{\sum_{j=1}^{n} p_j f(F_j)}{P_n} \frac{\sum_{i=1}^{n} p_i g(F_i)}{P_n}\right| \leq \tag{5.7}$$
$$\frac{1}{P_n^2}(M_1 - m_1)(M_2 - m_2) \max_{1 \leq k \leq n-1} P_k\,(P_n - P_k).$$

Proof. We use the extension of the Grüss inequality (see Corollary 2.7 in [5] and [3]) for $D(\mathbf{x}, \mathbf{y}; \mathbf{p}) = \frac{1}{P_n^2} T_n(\mathbf{x}, \mathbf{y}; \mathbf{p})$ with positive p_i. □

Corollary 5.5. *Let* $n \in \mathbb{N}$, $n > 2$.
If r *and* $s \in \mathbb{R}$ *such that* $r\,s > 0$, *then it holds*

$$\left|\frac{1}{F_n F_{n+1}} \sum_{i=1}^{n} F_i^{2+r+s} - \frac{1}{(F_n F_{n+1})^2} \sum_{j=1}^{n} F_j^{2+r} \sum_{i=1}^{n} F_i^{2+s}\right| \leq$$
$$\frac{1}{F_n^2 F_{n+1}^2}(F_n^r - 1)(F_n^s - 1) \max_{1 \leq k \leq n-1} F_k F_{k+1}\,(F_n F_{n+1} - F_k F_{k+1}).$$

If r *and* $s \in \mathbb{R}$ *such that* $r\,s < 0$, *then it holds*

$$\left|\frac{1}{F_n F_{n+1}} \sum_{i=1}^{n} F_i^{2+r+s} - \frac{1}{(F_n F_{n+1})^2} \sum_{j=1}^{n} F_j^{2+r} \sum_{i=1}^{n} F_i^{2+s}\right| \leq$$
$$\frac{1}{F_n^2 F_{n+1}^2}(F_n^r - 1)(1 - F_n^s) \max_{1 \leq k \leq n-1} F_k F_{k+1}\,(F_n F_{n+1} - F_k F_{k+1}).$$

REMARK 5.6. As special cases of Theorem 4.1, Theorem 4.4, Theorem 5.1, Theorem 5.2, Theorem 5.3 and Theorem 5.4 we can establish new inequalities if we select for weights $\mathbf{p} = (p_1, p_{2,} ..., p_n)$ the following substitutions and corresponding $P_n = \sum_i^n p_i$ according identities (1.4) – (1.10), respectively:

$$p_i = F_i^2,\, p_i = F_i,\, p_i = F_{2i-1}, p_i = F_{2i},\, p_i = i\, F_i,\, p_i = F_{4i-2},\, p_i = \binom{n}{i} F_{2i}.$$

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Bilješka o Čebiševljevoj nejednakosti i povezanim nejednakostima za Fibonaccijeve brojeve

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Sažetak. U radu su dokazani novi rezultati za Fibonaccijeve brojeve koji se odnose na Čebiševljevu nejednakost i s njom povezane nejednakosti.

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