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GENERALIZED HORN FUNCTION $H^{\lambda}_{4,p,q,\nu}$ AND RELATED BOUNDING INEQUALITIES WITH APPLICATIONS TO STATISTICS

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Abstract. Motivated by recent unified version of the Euler's Beta integral form with a MacDonald function in the integrand, we generalize the Horn double hypergeometric function $H_4[x, y]$. We then establish integral representations of the Euler and Laplace type including some other representations involving Bessel $J_\nu(z)$ and modified Bessel functions $I_\nu(z)$ for the generalized Horn double hypergeometric function $H^{\lambda}_{4,p,q,\nu}$. Several functional upper bounds for the $H^{\lambda}_{4,p,q,\nu}$ including the extended Gaussian hypergeometric $F^{\lambda}_{p,q,\nu}$, the extended Kummer's confluent hypergeometric $\Phi^{\lambda}_{p,q,\nu}$ are obtained by using functional bounds for extended Euler's Beta function $\mathrm{B}^{\lambda}_{p,q,\nu}(x, y)$. Various other bounding inequalities are obtained *via* Luke's, von Lommel's, Minakshisundaram and Szász and Olenko bounds. As an application, we define a Horn hypergeometric probability distribution to obtain certain statistical interference.

1. Introduction and preliminaries

The second–order modified homogeneous Bessel differential equation

$$z^2\,\omega''(z) + z\,\omega'(z) - (z^2+\nu^2)\,\omega(z) = 0$$

has linearly independent solutions $I_\nu(z)$ and $K_\nu(z)$ are called modified Bessel functions of the first and second kinds of the order ν. The MacDonald function or Hankel function(so-called modified Bessel function of the second kind) of the order ν is defined as [24, p. 251, Eq. 10.27.4]

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\pi\nu)}, \quad \nu\notin\mathbb{Z}; \qquad I_\nu(z) = \sum_{n\geq 0}\frac{\left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(\nu+1+n)\,n!},$$

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where I_ν is the modified Bessel function of the first kind, see [24, p. 249, Eq. 10.25.2]. Here, the symbol Γ being the familiar Euler's Gamma integral

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-t}\, t^{s-1}\, \mathrm{d}t, \qquad \Re(s) > 0\,.$$

We point out that $I_\nu(x)$ is real when $\nu \in \mathbb{R}$ and $\arg(x) = 0$.

Bearing in mind that for a fixed ν [24, p. 255, Eq. 10.40.2]

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, \mathrm{e}^{-z}\Big(1 + \frac{4\nu^2-1}{8z} + \mathcal{O}\big(z^{-2}\big)\Big), \qquad z \to \infty, \tag{1.1}$$

because of the parity with respect to the real order $\nu + \frac{1}{2}$ the asymptotic expansion is valid for all fixed real ν and in that case we have

$$K_{\nu+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, \mathrm{e}^{-z}\Big(1 + \frac{\nu(\nu+1)}{2z} + \mathcal{O}\big(z^{-2}\big)\Big) = \sqrt{\frac{\pi}{2z}}\, \mathrm{e}^{-z}\big(1 + \mathcal{O}(z^{-1})\big),$$

when $z \to \infty$. Throughout the paper, the usual conventions: $\mathbb{Z}^-$, $\mathbb{R}_+$, $\mathbb{C}$ denote the sets of negative integers, positive real and complex numbers, respectively, then $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$.

The Beta function (or Euler function of the first kind) is defined as

$$\mathrm{B}(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1}\, \mathrm{d}t, \qquad \min\{\Re(u), \Re(v)\} > 0. \tag{1.2}$$

In a recent paper [25] Parmar and Pogány give a unified approach to the generalized Beta function involving MacDonald function K_ν in kernel reads

$$\mathrm{B}^\lambda_{p,q,\nu}(u, v) = \sqrt{\frac{2}{\pi}} \int_0^1 t^{u-1}(1-t)^{v-1}\, \sqrt{h_\theta(t)}\, K_{\nu+\frac{1}{2}}(h_\theta(t))\, \mathrm{d}t\,, \tag{1.3}$$

where

$$h_\theta(t) = \frac{p}{t^\lambda} + \frac{q}{(1-t)^\lambda}\,, \qquad \theta = (p, q, \lambda)\,.$$

Here $\lambda > 0$, $\min\{\Re(p), \Re(q)\} > 0$ and $\min\{u, v\} > \frac{\lambda}{2} > 0$ and $\nu \in \mathbb{R}$. They used generalized Beta function (1.3) to extend the Gaussian and the Kummer confluent hypergeometric functions to establish their functional bounding inequalities, Turán inequalities and the raw moments and moment inequalities by defining a new probability Beta distribution.

Specifying the values of parameters p, q, λ and ν, the generalized Beta function $\mathrm{B}^\lambda_{p,q,\nu}(u, v)$ given in (1.3) covers various well-known forms of extended Beta functions. In fact (1.3) is a so–called Beta function transform and maps a suitable input function ψ into a multiparameter function [13]

$$\psi \mapsto \int_0^1 t^{u-1}(1-t)^{v-1}\, \psi(t)\, \mathrm{d}t\,.$$

In defining integral (1.3) we have $\psi(t) = \sqrt{h_\theta(t)}\, K_{\nu+\frac12}\big(h_\theta(t)\big)$, being $\sqrt{h_\theta(t)}$ the necessarily implemented correcting factor function (up to the multiplicative constant $\sqrt{2/\pi}$), see (1.1). In turn, the constraint $\min\{u,v\} > \frac{\lambda}{2} > 0$ follows immediately by re–writing $\sqrt{h_\theta(t)}$ in (1.3) into a convenient form.

Setting the values of the parameters p, q, ν, λ in (1.3) we get various known and frequently studied members of the Beta functions' family. So, when $\lambda = 1$ and $q = p$ we arrive at the so–called (p,ν)-extended Beta function introduced by Parmar *et al.* [27, p. 93, Eq. (13)]:

$$\mathrm{B}_{p,\nu}(u,v) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{u-\frac32}\,(1-t)^{v-\frac32}\, K_{\nu+\frac12}\left(\frac{p}{t(1-t)}\right) \mathrm{d}t,$$

where $\Re(p) \geq 0$; $\min\{\Re(u), \Re(v)\} > 0$ and $\sqrt{p}$ takes its principal value. This kind Beta function is recently considered by Milovanović *et al.* in [21] for establishing Gautschi–Pinelis type upper bounds for the MacDonald function and the (p,ν)–extended Beta function.

For $\lambda = 1,\ \nu = 0$ and using the fact $K_{\frac12}(z) = \sqrt{\pi/(2z)}\,\mathrm{e}^{-z}$, we arrive at the so–called (p,q)–extended Beta function, *viz.*

$$\mathrm{B}_{p,q}(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\,\mathrm{d}t\,,$$

the case when $\min\{\Re(u), \Re(v)\} > 0$ and $\min\{\Re(p), \Re(q)\} \geq 0$ was studied by Choi *et al.* in [3]. In turn, if we put $q = p$ and $\nu = 0$ in (1.3), it reduces to the generalized extended Beta function

$$\mathrm{B}^{\lambda}_{p,p,\frac12}(u,v) = \int_0^1 t^{u-1}\,(1-t)^{v-1}\,\mathrm{e}^{-p(t^{-\lambda}+(1-t)^{-\lambda})}\,\mathrm{d}t\,, \qquad \lambda > 0, \Re(p) > 0,$$

which should be distinguished from the generalization of the Beta function studied by Lee *et al.* [15, p. 189, Eq. (1.13)]:

$$\mathrm{B}(u,v;p;m) = \int_0^1 t^{u-1}\,(1-t)^{v-1}\,\mathrm{e}^{-p\ t^{-m}(1-t)^{-m}}\,\mathrm{d}t\,, \qquad m > 0, \Re(p) > 0.$$

Another special case occurs for $\lambda = 1$, $q = p$ and $\nu = 0$, when we get the p–extended Beta function [2, p. 20, Eq. (1.7)]

$$\mathrm{B}_p(u,v) = \int_0^1 t^{u-1}\,(1-t)^{v-1}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t, \quad \Re(p) \geq 0\,; \min\{\Re(u), \Re(v)\} > 0.$$

Finally, if we use the fact (1.1) and set $\nu = 0$ and $p, q \searrow 0$, then $\mathrm{B}^{\lambda}_{p,q,\nu}(u,v) \to \mathrm{B}(u,v)$ gives Euler's integral (1.2).

In this article the authors investigate the Horn double hypergeometric function $H^{\lambda}_{4,p,q,\nu}[x,\, y]$ by considering the definition of extended Beta function $\mathrm{B}^{\lambda}_{p,q,\nu}(u,v)$ in (1.3). Further various integral representations including Euler's and Laplace–Mellin type, as well as certain integral representations involving

Bessel and modified Bessel functions are established. Also, we derive several functional upper bounds for defined extended functions. Finally, related probability distribution is introduced and present some statistical properties including Turán–type inequalities for the newly defined extension of the Horn double hypergeometric function $H^{\lambda}_{4,p,q,\nu}[x,\, y]$.

2. Generalized Horn function

We begin by the definition of the generalized hypergeometric function with r numerator and s denominator parameters, as the series, reads

$$ {}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;z) = \sum_{k\geq 0} \frac{\prod\limits_{j=1}^{r}(a_j)_k}{\prod\limits_{j=1}^{s}(b_j)_k}\,\frac{z^k}{k!}\,, \tag{2.1} $$

where $(\delta)_n = \delta(\delta+1)\cdots(\delta+n-1) = \Gamma(\delta+n)/\Gamma(\delta)$, $\big(\delta \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ n\in\mathbb{N}_0\big)$ and $(\delta)_0 = 1$ denotes the raising/shifted factorial or Pochhammer symbol, $a_j \in \mathbb{C}$, $j \in \overline{1,r} := \{1,\, 2,\, \ldots,\, r\}$ and $b_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $j\in\overline{1,s}$. Particular cases for $r=2$, $s=1$ and $r=1$, $s=1$ are the Gaussian hypergeometric function and Kummer's confluent hypergeometric function

$$ {}_2F_1\big(a_1,a_2;\, b_1;\, z\big) = \sum_{k\geq 0}\frac{(a_1)_k(a_2)_k}{(b_1)_k}\,\frac{z^k}{k!}, $$

$$ \Phi\big(a_1;b_1;\, z\big) = {}_1F_1\big(a_1;b_1;\, z\big) = \sum_{k\geq 0}\frac{(a_1)_k}{(b_1)_k}\,\frac{z^k}{k!}, $$

respectively.

In terms of the extended Beta function $\mathrm{B}^{\lambda}_{p,q,\nu}(u,v)$ in (1.3), Parmar and Pogány [25, Eq. (4)] introduced unified extensions of the Gauss's hypergeometric function

$$ F^{\lambda}_{p,q,\nu}(a,\, u;\, v;\, z) = \sum_{k\geq 0}(a)_k\,\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(u+k,\, v-u)}{\mathrm{B}(u,\, v-u)}\,\frac{z^k}{k!}\,, \tag{2.2} $$

provided $\lambda > 0, p,\, q\geq 0$; $a\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $\Re(v) > \Re(u) > 0, |z| < 1$, and the confluent hypergeometric function [25, Eq. (5)]

$$ \Phi^{\lambda}_{p,q,\nu}(u;\, v;\, z) = \sum_{k\geq 0}\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(u+k,\, v-u)}{\mathrm{B}(u,\, v-u)}\,\frac{z^k}{k!}\,. \tag{2.3} $$

where $\lambda > 0$, $p,\, q\geq 0$; $\Re(v) > \Re(u) > 0$, respectively.

Connections to the definition of Beta function, and extensions of a number of known higher transcendental functions can now be generalized. Here we are interested to generalize the definition of Horn double hypergeometric function

H_4 pioneered by Horn in [7], also see [29, p. 24 and p. 59]. Namely, for all $a,\, u \in \mathbb{C}$ and $v, v' \in \mathbb{C} \setminus \mathbb{Z}_0^-$,

$$H_4[\, a, u; v, v'; x, y] = \sum_{k,m\geq 0} \frac{(a)_{2k+m}(u)_m}{(v)_k(v')_m}\, \frac{x^k}{k!}\, \frac{y^m}{m!}; \quad 2\sqrt{|x|}+|y|<1\,. \tag{2.4}$$

Now by making use of the transformation

$$\frac{(u)_m}{(v')_m} = \frac{\mathrm{B}(u+m, v'-u)}{\mathrm{B}(u, v'-u)}, \qquad \Re(v') > \Re(u) > 0,\ m \in \mathbb{N}_0,$$

in which the numerator of the Beta function is replaced by the extended Beta function $\mathrm{B}^{\lambda}_{p,q,\nu}(u,v)$ we introduce the following generalized Horn double hypergeometric function $H_4[\cdot,\cdot]$. Consider $a,\, u \in \mathbb{C}$ and $v, v' \in \mathbb{C} \setminus \mathbb{Z}_0^-$ in

$$H^{\lambda}_{4,p,q,\nu}[a, u; v, v'; x, y] = \sum_{k,m\geq 0} \frac{(a)_{2k+m}}{(v)_k}\, \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(u+m, v'-u)}{\mathrm{B}(u, v'-u)}\, \frac{x^k}{k!}\, \frac{y^m}{m!}\,, \tag{2.5}$$

for $\lambda > 0$, $p, q > 0$; $2\sqrt{|x|} + |y| < 1$ and $\Re(v') > \Re(u) > 0$. Clearly, the case $\lambda = 1$, $p = 0 = q$ and $\nu = 0$ in (2.5) gives the classical Horn double hypergeometric function (2.4).

2.1. *Integral Representations.*

THEOREM 2.1. *For all $\lambda > 0$, $\Re(p) > 0, \Re(q) > 0$; $\Re(v') > \Re(u) > 0$ when $p = 0 = q$, we have the integral representation*

$$\begin{aligned} H^{\lambda}_{4,p,q,\nu}[a, u; v, v'; x, y] &= \frac{\sqrt{2/\pi}}{\mathrm{B}(u, v'-u)} \int_0^1 t^{u-1}(1-t)^{v'-u-1}(1-yt)^{-a} \\ &\quad \cdot {}_2F_1\left(\frac{a}{2}, \frac{a}{2}+\frac{1}{2}; v; \frac{4x}{(1-yt)^2}\right) \sqrt{h_\theta(t)}\, K_{\nu+\frac{1}{2}}(h_\theta(t))\, \mathrm{d}t\,. \end{aligned} \tag{2.6}$$

PROOF. By making use of the identity

$$(a)_{2k+m} = (a)_{2k}\ (a+2k)_m$$

and the extended Gauss's hypergeometric function (2.2), the extended Horn double hypergeometric function (2.5) can be expressed as a single series:

$$H^{\lambda}_{4,p,q,\nu}[a, u; v, v'; x, y] = \sum_{k\geq 0} \frac{(a)_{2k}}{(v)_k}\, F^{\lambda}_{p,q,\nu}\big(a+2k, u; v'; y\big)\, \frac{x^k}{k!}. \tag{2.7}$$

Applying the integral representation of the extended Gauss's hypergeometric function [25, Eq. (6)]

$$F^{\lambda}_{p,q,\nu}(a, u; v; z) = \frac{\sqrt{2/\pi}}{\mathrm{B}(u, v-u)} \int_0^1 \frac{t^{u-1}(1-t)^{v-u-1}}{(1-zt)^a} \sqrt{h_\theta(t)}\, K_{\nu+\frac{1}{2}}\big(h_\theta(t)\big)\, \mathrm{d}t\,,$$

where the parameters $\lambda > 0, \Re(p) > 0, \Re(q) > 0$; also $|\arg(1-z)| < \pi$, $\Re(v) > \Re(u) > 0$ when $p = 0 = q$, to (2.7), one finds

$$H^{\lambda}_{4,p,q,\nu}[a, u; v; v'; x, y] = \frac{\sqrt{2/\pi}}{\mathrm{B}(u, v'-u)} \sum_{k\geq 0} \frac{(a)_{2k}}{(v)_k} \frac{x^k}{k!} \int_0^1 \frac{t^{u-1}(1-t)^{v'-u-1}}{(1-yt)^{a+2k}} \cdot \sqrt{h_\theta(t)}\, K_{\nu+\frac12}\big(h_\theta(t)\big)\,\mathrm{d}t\,.$$

Changing the order of summation and integration, which is guaranteed under the theorem's conditions, using the identity

$$(\alpha)_{2k} = 4^k \left(\frac{\alpha}{2}\right)_k \left(\frac{\alpha+1}{2}\right)_k, \qquad \alpha \in \mathbb{C},\, k \in \mathbb{N}_0$$

and the Gauss's hypergeometric function we get the stated integral representation (2.6). □

Theorem 2.2. *For all $\lambda > 0$, $\Re(p) > 0$, $\Re(q) > 0$; whilst $\Re(a) > 0$ when $p = 0 = q$, there holds the Laplace type integral expression for $H^{\lambda}_{4,p,q,\nu}$:*
(2.8)

$$H^{\lambda}_{4,p,q,\nu}[a, u; v, v'; x, y] = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}\mathrm{e}^{-t}\, {}_0F_1(-; v; xt^2)\Phi^{\lambda}_{p,q,\nu}(u; v'; yt)\mathrm{d}t.$$

Proof. Using the shorthand $H^{\lambda}_{4,p,q,\nu} := H^{\lambda}_{4,p,q,\nu}[a, u; v, v'; x, y]$, the integral representation

$$(\tau)_n = \frac{1}{\Gamma(\tau)} \int_0^\infty t^{\tau+n-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \qquad \Re(\tau) > 0,\, n \in \mathbb{N}_0$$

for the Pochhammer symbol $(a)_{2k+m}$ in (2.5) and interchanging the order of summations and integral, we get

$$\begin{aligned} H^{\lambda}_{4,p,q,\nu} &= \frac{1}{\Gamma(a)} \sum_{k,m\geq 0} \int_0^\infty t^{a-1}\mathrm{e}^{-t} \frac{1}{(v)_k} \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(u+m, v'-u)}{\mathrm{B}(u, v'-u)} \frac{(xt^2)^k}{k!} \frac{(yt)^m}{m!}\mathrm{d}t \\ &= \int_0^\infty \frac{t^{a-1}}{\Gamma(a)}\,\mathrm{e}^{-t} \sum_{k\geq 0} \frac{1}{(v)_k} \frac{(xt^2)^k}{k!} \cdot \sum_{m\geq 0} \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(u+m, v'-u)}{\mathrm{B}(u, v'-u)} \frac{(yt)^m}{m!}\,\mathrm{d}t. \end{aligned}$$

Now, applying (2.1) and the related definition (2.3) in both sums constituting the integrand, we prove the assertion (2.8). □

Remark 2.3. The Bessel function $J_\mu(z)$ and the modified Bessel function $I_\mu(z)$, both of the first kind and of the order μ are expressible in terms of the confluent hypergeometric function ${}_0F_1(\cdot)$ as follows [24, p. 228, Entries 10.16.9, 10.39.9]

(2.9) $$J_\mu(z) = \frac{(\frac{z}{2})^\mu}{\Gamma(\mu+1)}\, {}_0F_1\left(-; \mu+1; -\frac{z^2}{4}\right),$$

$$I_\mu(z) = \frac{(\frac{z}{2})^\mu}{\Gamma(\mu+1)}\, {}_0F_1\left(-;\mu+1;\frac{z^2}{4}\right), \tag{2.10}$$

where $-\mu \notin \mathbb{N}$ in both cases.

Now, applying the relationships (2.9) and (2.10) to (2.8), we deduce integral representations for the extended Horn double hypergeometric function in (2.5) asserted by Corollary 2.4 below.

Corollary 2.4. *The following Laplace–Mellin type transforms hold true:*

(2.11)

$$H^{\lambda}_{4,p,q,\nu}[a,u;v,v';-x,y] = \frac{\Gamma(v)x^{\frac{1-v}{2}}}{\Gamma(a)} \int_0^\infty t^{a-v}\mathrm{e}^{-t} J_{v-1}(2\sqrt{x}t)\Phi^{\lambda}_{p,q,\nu}(u;v';yt)\mathrm{d}t$$

(2.12)

$$H^{\lambda}_{4,p,q,\nu}[a,u;v,v';x,y] = \frac{\Gamma(v)x^{\frac{1-v}{2}}}{\Gamma(a)} \int_0^\infty t^{a-v}\mathrm{e}^{-t} I_{v-1}(2\sqrt{x}t)\Phi^{\lambda}_{p,q,\nu}(u;v';yt)\mathrm{d}t.$$

All parameters and variables are restricted so that the representations are convergent: $\Re(a)>0, \Re(v)>0, \Re(a-v)>-1$, $v\in\mathbb{C}\backslash\mathbb{N}_0$ *and* $x\in\mathbb{C}\backslash(-\infty,0]$.

3. Functional bounds for $H^{\lambda}_{4,p,q,\nu}$

This section explores bounding inequalities for the extended Horn double hypergeometric function $H^{\lambda}_{4,p,q,\nu}$. The first auxiliary lemma is a simple sharp estimate using (1.3).

Lemma 3.1. [25, Eq. (8)] *Let* $p,q>0$, $\lambda\in(0,1)\cup(1,\infty)$, $\nu\in\mathbb{R}$. *Then for all* $2\min\{u,v\}>\lambda>0$ *we have*

$$\begin{aligned}\mathrm{B}^{\lambda}_{p,q,\nu}(u,v) &\le \frac{\sqrt{2pq}\,K_{\nu+\frac12}\left(\left(p^{\frac{1}{\lambda+1}}+q^{\frac{1}{\lambda+1}}\right)^{\lambda+1}\right)}{\sqrt{\pi}\left(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}}\right)^{\frac{\lambda-1}{2}}}\,\mathrm{B}\big(u-\tfrac{\lambda}{2},v-\tfrac{\lambda}{2}\big)\\ &=: \Omega^{\lambda}_{\nu}(p,q)\mathrm{B}\big(u-\tfrac{\lambda}{2},v-\tfrac{\lambda}{2}\big)\,.\end{aligned} \tag{3.1}$$

Theorem 3.2. *For all* $p,q>0$ *and* $\lambda\in(0,1)\cup(1,\infty)$, $\nu\in\mathbb{R}$, *or when* $p=0=q$, $\Re(v)>\Re(u)>0$ *we have*

$$\big|F^{\lambda}_{p,q,\nu}(a,u;v;\,z)\big| \le \Xi^{\lambda}_{p,q,\nu}(u,v)\ {}_2F_1\big(a,u-\tfrac{\lambda}{2};\,v-\lambda;\,|z|\big)\,, \tag{3.2}$$

$$\big|\Phi^{\lambda}_{p,q,\nu}(u;v;z)\big| \le \Xi^{\lambda}_{p,q,\nu}(u,v)\ \Phi\big(u-\tfrac{\lambda}{2};\,v-\lambda;\,|z|\big)\,, \tag{3.3}$$

$$\big|H^{\lambda}_{4,p,q,\nu}[a,u;v,v';x,y]\big| \le \Xi^{\lambda}_{p,q,\nu}(u,v')\ H_4[a,u-\tfrac{\lambda}{2};v,v'-\lambda;|x|,|y|]\,,$$

provided $2\min\{u,v-u,v'-u\}>\lambda$. *Here, the multiplication constant reads*

$$\Xi^{\lambda}_{p,q,\nu}(u,\tau) = \frac{\sqrt{2pq}\,K_{\nu+\frac12}\left(\left(p^{\frac{1}{\lambda+1}}+q^{\frac{1}{\lambda+1}}\right)^{\lambda+1}\right)}{\sqrt{\pi}\left(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}}\right)^{\frac{\lambda-1}{2}}}\,\frac{\mathrm{B}\big(u-\frac{\lambda}{2},\tau-u-\frac{\lambda}{2}\big)}{\mathrm{B}(u,\tau-u)}\,,$$

where $\tau = u$ *appears in* (3.2) *and* (3.3), *in turn,* $\tau = v'$ *holds for the Horn function bound.*

PROOF. We prove only (3.2). Applying (3.1) to the extended Gaussian hypergeometric function (2.2) we get

$$\begin{aligned}\left|F^{\lambda}_{p,q,\nu}(a,u;v;z)\right| &\le \Omega^{\lambda}_{\nu}(p,q)\sum_{k\ge 0}(a)_k\frac{\mathrm{B}(u-\frac{\lambda}{2}+k, v-u-\frac{\lambda}{2})}{\mathrm{B}(u,v-u)}\,\frac{|z|^k}{k!}\\ &= \frac{\Omega^{\lambda}_{\nu}(p,q)}{\mathrm{B}(u,v-u)}\,\mathrm{B}\big(u-\tfrac{\lambda}{2}, v-u-\tfrac{\lambda}{2}\big)\,{}_2F_1(a,u-\tfrac{\lambda}{2};v-\lambda;|z|)\\ &= \Xi^{\lambda}_{p,q,\nu}(u,v,v')\ {}_2F_1(a,u-\tfrac{\lambda}{2};v-\lambda;|z|)\,,\end{aligned}$$

where we have used the series representation of Gaussian hypergeometric function ${}_2F_1$ in above last step. This proves (3.2). The other inequalities can be verified using similar arguments as in the proof of (3.2). ☐

3.1. *Functional bounds obtained via integral representations.* In this subsection, we investigate the bounds of the extended Horn double hypergeometric function $H^{\lambda}_{4,p,q,\nu}$. To accomplish this, we review and recall certain inequalities pertaining to the generalized hypergeometric function, Bessel function and modified Bessel function as follows:

- For $b_j \ge a_j > 0$, $j = \overline{1,r}$ and $x \in \mathbb{R}_+$, the following Luke's two-sided inequalities for ${}_rF_r$ hold true [19, Theorem 16, Eq. (5.6)]

  $$\mathrm{e}^{\theta x} < {}_rF_r\big(a_1,\dots,a_r;b_1,\dots,b_r;\,x\big) < 1-\theta\big(1-\mathrm{e}^{x}\big),$$

  where

  $$\theta = \frac{\max\limits_{1\le j\le r} a_j}{\min\limits_{1\le j\le r} b_j}.$$

  For $b \ge a > 0$, the bilateral inequalities for Kummer's confluent hypergeometric function $\Phi(a;b;x) = {}_1F_1(a;b;x)$ is given as

$$\mathrm{e}^{\frac{a}{b}x} < \Phi(a;b;\,x) < 1-\frac{a}{b}\big(1-\mathrm{e}^{x}\big). \tag{3.4}$$

- Bounding inequalities for J_ν and I_ν:
  (i) von Lommel's bounds [30, pp. 31 and 406], [16], [17, pp. 548–549]

$$|J_\nu(t)| \le 1, \qquad |J_{\nu+1}(t)| \le \frac{1}{\sqrt{2}}, \qquad \nu\in\mathbb{R}_+,\ t\in\mathbb{R}; \tag{3.5}$$

  (ii) Minakshisundaram and Szász bound [8, Eq. (1.8)], [22, pp. 36-37]; cf. [30, p. 16]

$$|J_\nu(t)| \le \frac{1}{\Gamma(\nu+1)}\left(\frac{|t\,|}{2}\right)^{\nu}, \qquad \nu\ge 0,\ t\in\mathbb{R}; \tag{3.6}$$

(iii) For $\nu \geq 0$ and $t \in \mathbb{R}$ there are Landau bounds [14]

$$|J_\nu(t)| \leq b_L\, \nu^{-1/3}, \qquad b_L := \sqrt[3]{2} \sup_{t\geqslant 0} \mathrm{Ai}(t), \tag{3.7}$$

$$|J_\nu(t)| \leq c_L\, |t|^{-1/3}, \qquad c_L := \sup_{t\geqslant 0} t^{1/3} J_0(t)\,, \tag{3.8}$$

where $\mathrm{Ai}(\cdot)$ stands for the Airy function

$$\mathrm{Ai}(t) = \frac{\pi}{2}\sqrt{\frac{t}{3}}\left(J_{-1/3}\left\{2(t/3)^{3/2}\right\} + J_{-1/3}\left\{2(t/3)^{3/2}\right\}\right). \tag{3.9}$$

(iv) Olenko's bound [23, Theorem 2.1]

$$\sup_{t\geq 0} \sqrt{t}\, |J_\nu(t)| \leq b_L \sqrt{\nu^{1/3} + \frac{\tau_1}{\nu^{1/3}} + \frac{3\tau_1^2}{10\nu}} =: d_O, \qquad \nu > 0\,, \tag{3.10}$$

where τ_1 is the smallest positive zero of the Airy–function Ai in (3.9) and b_L is the Landau's constant in (3.7). This bound is asymptotically precise and the constant b_L is the best possible.

(v) Luke [19, Eq.(6.25)] gave the following inequality for the modified Bessel function I_μ:

$$I_\mu(t) < \frac{\left(\frac{t}{2}\right)^\mu}{\Gamma(\mu+1)} \cosh t, \qquad t > 0,\ \mu > -\tfrac{1}{2}\,. \tag{3.11}$$

Now, we state our second set of bounded inequalities for $H^{\lambda}_{4,p,q,\nu}$.

Theorem 3.3. *For all* $p, q > 0, a+1 > v > 0, v' \geq u > 0,\ x \geq 0, y < 1$*; or* $p = 0 = q, a+1 > v > 0,\ 2\min\{u, v'-u\} > \lambda > 0,\ x \geq 0, y \in [0,1)$, $2\sqrt{x} + y < 1$,

$$\begin{aligned}\left|H^{\lambda}_{4,p,q,\nu}[a,u;v,v';-x,y]\right| &\leq \frac{1}{a}\ \Xi^{\lambda}_{p,q,\nu}(u,v')\ \mathrm{B}(v, a-v+1)\ x^{\frac{1-v}{2}} \\ &\quad\cdot\left[1 - \frac{u-\frac{\lambda}{2}}{v'-\lambda}\big(1-(1-y)^{-a+v-1}\big)\right].\end{aligned} \tag{3.12}$$

If $p, q > 0$*;* $a+1 > v > 1, v' \geq u > 0$*;* $x \geq 0,\ y < 1$*; or* $p = 0 = q$, $a+1 > v > 0$*;* $2\min\{u, v'-u\} > \lambda > 0,\ y \in [0,1),\ 2\sqrt{x}+y < 1$*, there holds*

$$\begin{aligned}\left|H^{\lambda}_{4,p,q,\nu}[a,u;v,v';-x,y]\right| &\leq \Xi^{\lambda}_{p,q,\nu}(u,v')\ \frac{b'_L\ \mathrm{B}(v,a-v+1)\ x^{\frac{1-v}{2}}}{a\ \sqrt[3]{v-1}} \\ &\quad\cdot\left[1 - \frac{u-\frac{\lambda}{2}}{v'-\lambda}\left(1-(1-y)^{-a+v-1}\right)\right],\end{aligned} \tag{3.13}$$

where $b'_L := \sqrt[3]{2}\sup_{t\geq 0}\mathrm{Ai}(2\sqrt{x}t)$*. Finally, for all* $p, q \geq 0, a > 0, v > \frac{1}{2}$, $\min\{u, v'-u\} > \frac{\lambda}{2} > 0,\ 0 < x < \frac{1}{4},\ 2\sqrt{x}+y < 1$*, it is*

$$\left|H^{\lambda}_{4,p,q,\nu}[a,u;v,v';x,y]\right| \leq \frac{\Xi^{\lambda}_{p,q,\nu}(u,v')}{v'-\lambda}\ \left[\frac{v'-u-\frac{\lambda}{2}}{(1-2\sqrt{x})^a} + \frac{u-\frac{\lambda}{2}}{(1-2\sqrt{x}-y)^a}\right].$$

Proof. Applying the estimate (3.3) in Theorem 3.2 to the integral representations (2.11) and (2.12), respectively, we obtain

$$R_1 = \left\{ \begin{array}{c} \big|H^{\lambda}_{4,p,q,\nu}[a,u;v,v';-x,y]\big| \\ \big|H^{\lambda}_{4,p,q,\nu}[a,u;v,v';x,y]\big| \end{array} \right\} \le \Xi^{\lambda}_{p,q,\nu}(u,v')\,\frac{\Gamma(v)\,x^{\frac{1-v}{2}}}{\Gamma(a)}$$

$$\cdot \int_0^\infty \mathrm{e}^{-t}\,t^{a-v} \left\{ \begin{array}{c} |J_{v-1}(2\sqrt{x}t)| \\ |I_{v-1}(2\sqrt{x}t)| \end{array} \right\} \Phi(u-\tfrac{\lambda}{2}; v'-\lambda; yt)\,\mathrm{d}t. \tag{3.14}$$

Employing Luke's upper bound (3.4) in (3.14) gives the following estimate:

$$R_1 \le \Xi^{\lambda}_{p,q,\nu}(u,v')\,\frac{\Gamma(v)\,x^{\frac{1-v}{2}}}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-t}\,t^{a-v} \left\{ \begin{array}{c} |J_{v-1}(2\sqrt{x}t)| \\ |I_{v-1}(2\sqrt{x}t)| \end{array} \right\}$$

$$\cdot \Big[1-\frac{u-\frac{\lambda}{2}}{v'-\lambda}\big(1-\mathrm{e}^{yt}\big)\Big]\,\mathrm{d}t =: R_2\Big(\begin{array}{c} J \\ I \end{array} \Big). \tag{3.15}$$

Using the first one in (3.5) evaluating the upper display in (3.15), $R_2(J)$ say, we find

$$R_2(J) \le \Xi^{\lambda}_{p,q,\nu}(u,v')\,\frac{\Gamma(v)\,x^{\frac{v-1}{2}}}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-t}\,t^{a-v}\,\Big[1-\frac{u-\frac{\lambda}{2}}{v'-\lambda}\big(1-\mathrm{e}^{yt}\big)\Big]\,\mathrm{d}t,$$

which, upon employing the known integral

$$\int_0^\infty \mathrm{e}^{-\mu\,t}\,t^{\tau-1}\,\mathrm{d}t = \frac{\Gamma(\tau)}{\mu^\tau}, \qquad \Re(\tau)>0,\ \mu>0,$$

to evaluate the right sided integral and combining the result into (3.15) yields the desired inequality (3.12). Next, by utilizing the first Landau's result (3.7) we can derive the inequality (3.13) in a similar manner.

Applying the inequality: $\cosh t \le \mathrm{e}^t$ for $t \ge 0$ to (3.11) offers the following inequality:

$$I_\mu(t) < \frac{\big(\frac{t}{2}\big)^\mu\,\mathrm{e}^t}{\Gamma(\mu+1)}, \qquad t>0,\ 2\mu>-1,$$

which gives

$$|I_{v-1}(2\sqrt{x}t)| < \frac{x^{\frac{v-1}{2}}\,t^{v-1}}{\Gamma(v)}\,\mathrm{e}^{2\sqrt{x}t}, \qquad x>0,\ t>0,\ 2v>1. \tag{3.16}$$

Employing (3.16) to the $R_2(I)$ using similar process as in the proof of (3.12) we infer the third bound. The incidental details are omitted. □

Now, we formulate our third set of bounding inequalities upon $H^{\lambda}_{4,p,q,\nu}$ function.

Theorem 3.4. *For $a,v>0$, $\min\{u,v'-u\} > \frac{\lambda}{2} > 0$ and $x>0$, $y\in[0,1)$, we have*

$$\big|H^{\lambda}_{4,p,q,\nu}[a,u;v,v';-x,y]\big| \le \Xi^{\lambda}_{p,q,\nu}(u,v')\,\Big[1-\frac{u-\frac{\lambda}{2}}{v'-\lambda}\big(1-(1-y)^{-a}\big)\Big].$$

Moreover, $a, v > 0$, $\min\{u, v' - u\} > \frac{\lambda}{2} > 0$, *whilst* $x > 0$, $y \in [0, 1)$ *implies*

$$\Big|H^{\lambda}_{4,p,q,\nu}[a, u; v, v'; -x, y]\Big| \le \Xi^{\lambda}_{p,q,\nu}(u, v')\, \frac{\Gamma(v)}{\Gamma(a)}$$
$$\cdot \begin{cases} \dfrac{c_L\, x^{-\frac{v}{2}+\frac{1}{3}}}{\sqrt[3]{2}}\, \Gamma(a - v + \frac{2}{3}) \Big[1 - \frac{u-\frac{\lambda}{2}}{v'-\lambda}\big(1 - (1 - y)^{-a+v-\frac{2}{3}}\big)\Big], \\ \frac{d_O\, x^{-\frac{v}{2}+\frac{1}{4}}}{\sqrt{2}}\, \Gamma(a - v + \frac{1}{2}) \Big[1 - \frac{u-\frac{\lambda}{2}}{v'-\lambda}\big(1 - (1 - y)^{-a+v-\frac{1}{2}}\big)\Big], \end{cases}$$

provided $3(a - v) + 2 > 0$ *for the first bound; the second one holds under the restriction* $2(a-v)+1 > 0$*. Finally, we note that in view of* (2.5) *when* $p = 0$ *we assume* $2\sqrt{x} + y < 1$ *in both bounds.*

Proof. Firstly, we point out that the estimates of Bessel function in (3.6), (3.8) and (3.10) are of the magnitude $|J_{v-1}(t)| \le \mathfrak{C}\, t^{\kappa}$ where $\mathfrak{C} \in \{[2^{v-1}\Gamma(v)]^{-1}, c_L, d_O\}$ and $\kappa \in \{v - 1, -\frac{1}{3}, -\frac{1}{2}\}$, respectively. Now, the application of these estimate (3.6) to the integral representation (2.11) results

$$\begin{aligned} R_1' = \big|H^{\lambda}_{4,p,q,\nu}[a, u; v, v'; -x, y]\big| &\le \Xi^{\lambda}_{p,q,\nu}(u, v')\, \frac{\Gamma(v)}{\Gamma(a)\,|x|^{\frac{v-1}{2}}} \\ &\quad\cdot \int_0^\infty e^{-t}\, t^{a-v}\, \big|J_{v-1}(2\sqrt{x}t)\big|\, \Phi(u - \tfrac{\lambda}{2}; v' - \lambda; yt)\, dt \\ &\le \Xi^{\lambda}_{p,q,\nu}(u, v')\, \frac{\mathfrak{C}\,\Gamma(v)}{\Gamma(a)|x|^{\frac{v-\kappa-1}{2}}} \int_0^\infty e^{-t}\, t^{a+\kappa-v} \left[1 - \frac{u - \frac{\lambda}{2}}{v' - \lambda}\big(1 - e^{yt}\big)\right] dt \\ &= \Xi^{\lambda}_{p,q,\nu}(u, v')\, \frac{\mathfrak{C}\,\Gamma(v)}{\Gamma(a)|x|^{\frac{v-\kappa-1}{2}}} \Bigg\{ \left(1 - \frac{u - \frac{\lambda}{2}}{v' - \lambda}\right) \int_0^\infty e^{-t} t^{a+\kappa-v}\, dt \\ &\quad + \frac{u - \frac{\lambda}{2}}{v' - \lambda} \int_0^\infty e^{-(1-y)t} t^{a+\kappa-v}\, dt \Bigg\} \\ &= \Xi^{\lambda}_{p,q,\nu}(u, v')\, \frac{\mathfrak{C}\,\Gamma(v)\Gamma(a + \kappa - v + 1)}{\Gamma(a)|x|^{\frac{v-\kappa-1}{2}}(v' - \lambda)} \left\{ v' - u - \frac{\lambda}{2} + \frac{u - \frac{\lambda}{2}}{(1 - y)^{a+\kappa-v}} \right\}. \end{aligned}$$

Then, choosing either $\mathfrak{C} = [2^{v-1}\Gamma(v)]^{-1}$, c_L or d_O and $\kappa = v-1$, $\kappa = -\frac{1}{3}, -\frac{1}{2}$, mutually, we realize the bounds affiliated to the Minakshisundaram and Szász, the second Landau's and Olenko's estimates, respectively. ∎

4. Applications to Statistical Distribution

Special functions are important in studying probability distribution and statistical inference (see for instance [4, Chapter 7], and [20, Chapters 6, 8], [5, 9–12, 26]). Recently, researchers have been studying McKay Bessel-type distributions, which are related to special functions, such as Horn confluent functions (see [5, 9, 10, 18]). The extended Horn double hypergeometric function (2.8) is expected to have many applications, similar to the generalized Beta and Gamma functions. One potential application is in statistics, and it can also be applied in inequality theory to derive novel bilateral bounds for the generalized Horn function $H^{\lambda}_{4,p,q,\nu}$ using probabilistic methods.

Consider the random variable ξ defined on a standard probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, where Ω is a sample space, $\mathfrak{F}$ is the event space in Ω, and $\mathbf{P}$ is a probability function, characterized by the following probability density function throughout (*abbr.* density):

$$f_\xi(t) = \begin{cases} C_{p,q}(\rho,\tau)\; t^{\tau-1}\mathrm{e}^{-\rho t}\; {}_0F_1(-;v;xt^2)\; \Phi^{\lambda}_{p,q,\nu}(u;v';yt), & t > 0 \\ 0 & \text{elsewhere}, \end{cases}$$

where it is assumed that $\Re(\rho) > 0$, $\Re(\tau) > 0$, the positive arguments (x, y), and the parameters $p, q\,\lambda$, μ and u, v, v' are suitably constrained so that $f_\xi(t)$ remains nonnegative. The normalization constant reads

$$C_{p,q}(\rho,\tau) := \frac{\rho^{\tau}}{\Gamma(\tau)\, H^{\lambda}_{4,p,q,\nu}\Big[\tau, u; v, v'; \dfrac{x}{\rho^2}, \dfrac{y}{\rho}\Big]}.$$

We define the generalized Horn gamma distribution of the random variable (*abbr.*r.v.) ξ as $\mathrm{GHG}(\theta)$, where $\theta = (p, q, \lambda; \tau, u, v, v', \rho; x, y)$ is the parameter vector. Alternatively, we denote this as $\xi \sim f_\xi(t)$. Hereafter, we will derive some statistical functions for the r.v. $\xi \sim \mathrm{GHG}(\theta)$.

Raw Moments and Turán inequalities. The sth fractional-order moments m_s, $s > 0$ equal

$$m_s = \mathsf{E}\xi^s = \int_0^\infty u^s f_\xi(u)\,\mathrm{d}u = \frac{(\tau)_s}{\rho^\tau}\,\frac{H^{\lambda}_{4,p,q,\nu}\Big[\tau+s, u; v, v'; \dfrac{x}{\rho^2}, \dfrac{y}{\rho}\Big]}{H^{\lambda}_{4,p,q,\nu}\Big[\tau, u; v, v'; \dfrac{x}{\rho^2}, \dfrac{y}{\rho}\Big]}. \tag{4.1}$$

As the first application of (4.1), we derive a Turán-type inequality for the extended Horn double hypergeometric function $H^{\lambda}_{4,p,q,\nu}[\cdot]$ by virtue of the moment inequality, which holds for the nonnegative r.v. $\xi \sim f_\xi(t)$. Lukacs reported on the moment inequality [18, p. 28, Eq. (1.4.6)]

$$m^2_{s+r} \le m_s\, m_{s+2r}, \qquad (\min\{s, r\} > 0). \tag{4.2}$$

By inserting the expression (4.1) in (4.2), we obtain for all $2s > -\tau$, $s + 2r > -\tau$ the bounding inequality

$$\{H^{\lambda}_{4,p,q,\nu}[s+r]\}^2 \le \frac{\Gamma(\tau+s)\,\Gamma(\tau+s+2r)}{\Gamma^2(\tau+s+r)}\, H^{\lambda}_{4,p,q,\nu}[s]\cdot H^{\lambda}_{4,p,q,\nu}[s+2r],$$

where we take the shorthand

$$H^{\lambda}_{4,p,q,\nu}[a] = H^{\lambda}_{4,p,q,\nu}\Big[\tau+a, u; v, v'; \frac{x}{\rho^2}, \frac{y}{\rho}\Big].$$

Also, another statement by Lukacs [18, p. 393, a)] asserts that for $0 < r \le s$, the moment inequality $m_{s+r}^2 \le m_{2s}\, m_{2r}$ holds, which can be inferred using the Cauchy–Bunyakovsky–Schwarz inequality. This inequality implies a variant of the Turán–type inequality, *viz.*

$$\{H_{4,p,q,\nu}^{\lambda}[s+r]\}^2 \le \frac{\Gamma(\tau+2s)\,\Gamma(\tau+2r)}{\Gamma^2(\tau+s+r)}\, H_{4,p,q,\nu}^{\lambda}[2s]\cdot H_{4,p,q,\nu}^{\lambda}[2r]\,,$$

provided $2\min\{s,r\} > -\tau$.

Characteristic Function. The Fourier transform of the density $f_\xi(t)$ is the characteristic function (ch.f.) $\varphi_\xi(t)$ of the r.v. ξ. Hence,

$$\begin{aligned}\varphi_\xi(w) &= \mathsf{E}\mathrm{e}^{\,\mathrm{i}w\xi} = \int_0^\infty \mathrm{e}^{\,\mathrm{i}wt} f_\xi(t)\,\mathrm{d}t \\ &= C_{p,q}(\rho,\tau)\int_0^\infty \mathrm{e}^{-(\rho-\mathrm{i}w)t}t^{\tau-1}{}_0F_1(-;v;xt^2)\;\Phi_{p,q,\nu}^{\lambda}(u;v';yt)\,\mathrm{d}t\,.\end{aligned}$$

Therefore the ch.f. becomes

$$\varphi_\xi(w) = \frac{\rho^\tau\, H_{4,p,q,\nu}^{\lambda}\left[\tau,u;v,v';\dfrac{x}{(\rho-\mathrm{i}w)^2},\dfrac{y}{\rho-\mathrm{i}w}\right]}{(\rho-\mathrm{i}w)^\tau\, H_{4,p,q,\nu}^{\lambda}\left[\tau,u;v,v';\dfrac{x}{\rho^2},\dfrac{y}{\rho}\right]}\,. \tag{4.3}$$

The surprising summation result in the following theorem establishes a connection between the density and the ch.f. through the corresponding integer–order moments.

Theorem 4.1. *The parameter vector $\theta = (p,q,\lambda;\tau,u,v,v',\rho;x,y) > 0$ ensures*

$$\begin{aligned}&\sum_{n\ge 0}(\tau)_n\, H_{4,p,q,\nu}^{\lambda}\left[\tau+n,u;v,v';\frac{x}{\rho^2},\frac{y}{\rho}\right]\,\frac{(\mathrm{i}w)^n}{n!} \\ &\qquad = \frac{\rho^{2\tau}}{(\rho-\mathrm{i}w)^\tau}\, H_{4,p,q,\nu}^{\lambda}\left[\tau,u;v,v';\frac{x}{(\rho-\mathrm{i}w)^2},\frac{y}{\rho-\mathrm{i}w}\right]\,.\end{aligned}$$

Proof. Since the Maclaurin series of the ch.f. reads [18, p. 41]

$$\varphi_\xi(w) = \sum_{n\ge 0} m_n\,\frac{(\mathrm{i}w)^n}{n!}\,,$$

by inserting (4.1) and (4.3) into this expansion, the routine steps lead to the assertion. ∎

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O poopćenoj Hornovoj hipergeometrijskoj funkciji $H_{4,p,q,\nu}^{\lambda}$ i pridruženim nejednakostima s primjenama u statistici

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Sažetak. U ovom radu izvedene su nove integralne reprezentacije za poopćeni Hornov hipergeometrijski red $H_{4,p,q,\nu}^{\lambda}$ dviju varijabli, kao i za poopćene – Gaussovsku hipergeometrijsku funkciju $F_{p,q,\nu}^{\lambda}$ i Kummerovu konfluentnu hipergeometrijsku funkciju $\Phi_{p,q,\nu}^{\lambda}$. Glavni alat ovih proširenja je poopćena Beta funkcija i njena funkcionalna gornja granica. Dobiveni rezultati kombinirani s granicama koje su postavili za Besselove funkcije Luke, von Lommel, Minakshisundaram i Szász, odnosno Olenko su se koristile za postavljanje funkcionalnih granica. Definira se slučajna varijabla tipa Horna, kao i njena vjerojatnostna razdioba. Pomoću momenata pozitivnog reda te varijable dodatno se dokazuju dvije Turánove nejednakosti.

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