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The Cognitive Development of Students and Understanding the Equals Sign: A Review

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Abstract

For decades, the cognitive development of the equals sign has attracted attention from scholars, yet comprehensive reviews are lacking. This study aims to review research on four areas: students' understanding of the equals sign, factors influencing their understanding, interventions to promote understanding, and cross-cultural comparisons. It also aims to identify unresolved issues, gaps, and developmental trends. Based on a review of 85 papers from 1932 to 2024 from six databases, this study contributes by (a) summarizing students' understanding of the equals sign across age groups, various understanding levels, crucial factors influencing students' understanding, theoretical frameworks and measurement items and results widely seen in studies about interventions, and students' understanding from different countries; (b) recognizing the lack of research on high school and college students' understanding, the developmental paths of different understanding levels, specific influencing factors, and portable interventions to improve both procedural and conceptual knowledge of the equals sign; and (c) suggesting future research to explore the cognitive mechanisms of understanding the equals sign, integrate procedural and conceptual teaching, distinguish between reducing or preventing misunderstandings, develop a unified measurement tool, and examine factors behind cross-country differences in understanding. This study contributes to a more systematic theory and research framework in mathematics education.

Key words: *algebra; equation; mathematical equivalence; understanding*

Introduction

The equals sign ("=") is a symbolic representation of equality or mathematical equivalence (Devlin et al., 2023; Sumpter & Löwenhielm, 2024). Equality or mathematical equivalence indicates that the two sides of an equation have the same value, represent the same mathematical object, or are interchangeable (Jones et al., 2012). Thus, understanding the equal sign involves both sameness and interchangeability (Jones et al., 2013). In short, the equals sign holds a profound and multifaceted meaning.

The equals sign is crucial in mathematics and other disciplines, such as the sciences (Matthews & Fuchs, 2020). It serves as a fundamental component of algebraic thinking and reasoning (Falkner & Levi, 1999; Molina et al., 2009). Specifically, students' transition from arithmetic to algebra is closely tied to their understanding of the equals sign (Matthews et al., 2012). Moreover, an appropriate understanding of the equals sign in earlier grades predicts future mathematical competence in later grades, including achievement on standardized mathematics assessments (Davenport et al., 2023; Knuth et al., 2006; McNeil et al., 2019), solving equations (Knuth et al., 2006), and tackling word problems related to mathematical equivalence (Matthews & Fuchs, 2020). Thus, developing a more flexible understanding of this symbol and avoiding misconceptions are vital for students.

Unfortunately, children often struggle to grasp the concept of the equal sign (Behr, 1980; Lee & Pang, 2023). To make algebra accessible to all students, it is essential to understand how students perceive the equals sign and how this understanding can be improved (Simsek et al., 2022). Over the decades, numerous studies have accumulated on students' understanding of the equal sign. Despite this, very few attempts have been made to synthesise the extant research. We searched EBSCO using "math* equivalence," "equal* sign," "math* equality," and "equal symbol" as abstract search terms and only found two related reviews: one traditional review from 1981 (Kieran, 1981) and one meta-analysis focusing on feedback interventions (Fyfe & Brown, 2018).

To fill this gap, this study aims to comprehensively review the literature regarding students' cognitive development of the equals sign. Specifically, this review aims to (a) summarize students' understanding of the equals sign across different age groups and the various levels of understanding they exhibit; (b) provide an overview of the key factors influencing students' understanding of the equals sign and the effectiveness of interventions designed to improve their understanding; (c) outline the differences in students' understanding of the equals sign across various cultural contexts and the factors resulting in these differences; and (d) identify unresolved issues or gaps in these fields and to provide potential directions or recommendations for future research.

This review begins with a description of the search strategy. It then summarises how students of different age groups understand the equal sign and the varying levels of their understanding. Given that tailored interventions based on influencing factors are more likely to succeed, the review analyses research on factors affecting understanding before discussing interventions designed to improve it. Next, it reviews cross-cultural comparisons, highlighting differences in how students from various countries understand the equal sign and the factors contributing to these differences. At the end of each subsection, summaries are provided, unresolved issues or gaps are discussed, and future research directions are suggested. The review concludes with a summary of the findings and their implications.

Methodology

Unlike systematic reviews with strict criteria and more substantial evidence, we conducted a traditional review, emphasising a broader and interpretative synthesis. This approach enables detailed analysis and contextual discussion of included studies, providing deeper insights into themes and trends in research on students' cognitive development of the equals sign (Rozas & Klein, 2010).

Searches were conducted in September 2023 and updated in October 2024 using Web of Science and EBSCO (including the databases APA PsycArticles, Psychology and Behavioral Science Collection, Teacher Reference Center, ERIC, and Academic Search Premier). The search terms used were "equal* sign", "equal symbol", "math equivalence", "mathematical equivalence", "math equality", and "mathematical equality". Initially, we identified a total of 771 papers on this topic through abstract searches. A flowchart in Figure 1 shows the literature searching and screening process.

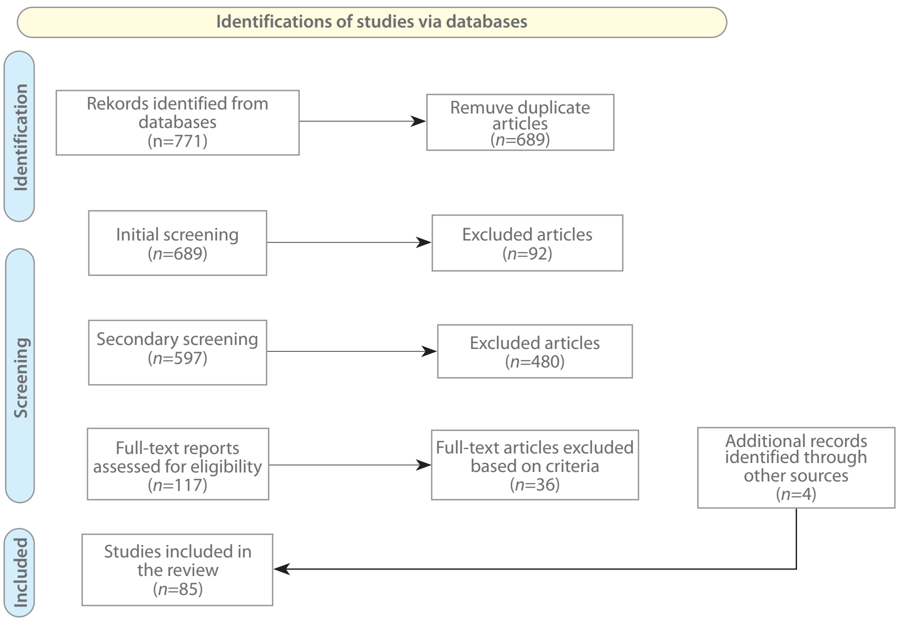

Figure 1. Flow chart of the article selection process

The inclusion criteria for articles were: (a) Academic journal articles; (b) Available abstracts; (c) English language; (d) Available full-text; and (e) the study had to examine students' understanding of the equal sign. The exclusion criteria for the literature were: studies that did not investigate at least one of the three aspects — influencing factors, interventions, or international comparisons of students' understanding of the equals sign — were excluded. To understand the origins and foundation of current studies on students' understanding of the equals sign, we searched for and retrieved the earliest available article on this topic, published in 1932 (Renwick, 1932). Therefore, the time frame of the included articles ranges from 1932 to 2024. In the first round of screening, 597 articles were included based on the criteria (a), (b), and (c). The second round of screening was conducted under the criteria (d) and the exclusion criteria. This excluded 480 articles and included 117 papers. After screening based on criterion (e) and deduplication, 81 articles were selected for inclusion. We also searched for and included four additional reports, resulting in a total of 85 papers ultimately included. Among them, 40 articles were coded as "students' understanding of the equals sign" (e.g., Freiman & Lee, 2004; Jones & Pratt, 2006; Knuth et al., 2008; Society for Research on Educational et al., 2010), 33 articles were coded as "pedagogical interventions designed to improve students' understanding" (e.g., Cook et al., 2008; Jacobs et al., 2007; Mann, 2004; Powell, 2015), 15 articles were coded as "influencing factors of this understanding" (e.g., Asquith et al., 2007; McNeil & Alibali, 2005; Powell, 2012), and 4 articles (Eichhorn et al., 2018; Jones et al., 2012; Li et al., 2008; Madej, 2022) and 2 reports (Capraro et al., 2007; Capraro et al., 2010) were coded as "cross-cultural comparisons".

Research on the understanding of the equals sign

Among the 40 articles on students' understanding of the equals sign, a significant portion focused on elementary (e.g., Baroody & Ginsburg, 1983; Behr, 1980; Matthews et al., 2012) and middle school students (e.g., Kieran & Martínez-Hernández, 2022; Rittle-Johnson et al., 2011; Sumpter & Löwenhielm, 2024), while only a small number explored the understanding of kindergarten (Blanton et al., 2018; Devlin et al., 2023; Falkner et al., 1999; Stephens et al., 2021), high school (Emre-Akdogan, 2023; Harrell, 2016; Simsek et al., 2019), and college students (Chesney et al., 2013; Fyfe et al., 2020; McNeil & Alibali, 2005a). The educational stage mentioned in this article refers to the respective national education systems, where the specific grades may vary by country. Next, we will provide an overview of students' understanding of each educational stage, using selected studies as examples. The reason for choosing these studies is that they can reflect the evolution of research or have a high citation rate. We then outline the different levels of students' understanding of the equals sign.

The Understanding of the equals sign across different age groups

The earliest research on the cognitive development of the equals sign that can be retrieved is Renwick's study from 1932, which reported pupils' misconceptions about the equals sign (Renwick, 1932). Since then, significant scholarly attention has focused on this topic. Researchers primarily employ instruments such as "True/false number sentences" (e.g., determining whether $3 = 1 + 2$ is correct or incorrect), "definitions of the equals sign" (e.g., asking students to explain the equals sign), "open number sentences" (e.g., filling in the blanks for $2 + \_\_ = 3 + 5$), and "equation encoding" (e.g., reconstructing equations like 7 + 1 = __ + 6, that is, reproducing several similar problems from students' memories) to test students' understanding of the equals sign (Hornburg et al., 2021; Matthews et al., 2012).

Renwick (1932) categorised students' understanding of the equals sign into operational and relational views. The operational view refers to perceiving the equals sign merely as a one-directional symbol for denoting the result of an operation and/or as a "do something" signal. In contrast, the relational view interprets it as a bidirectional symbol, indicating that the two sides of the equal sign have the same value or quantity. Renwick (1932) found that most participants aged 8-12 and some aged 13-14 held an operational view, indicating an insufficient understanding of the equals sign from elementary to middle school. This was assessed through True/false number sentences, where viewing 12 + 2 = 14 ÷ 2 = 7 as correct but considering 17 × 2 = 30 + 4 incorrect reflects an operational view.

Consistent with Renwick (1932), Behr (1980) found that children aged 6-12 predominantly held the operational view. They used open-number sentences, including traditional format (operations on the left side, e.g., 5 + 3 = __) and non-traditional format (operations on the right side, both sides, or no operations, e.g., 8 = __ + 6, 4 + __ = 5 + 9, or 7 = 7), as measurement tools. Traditional format sentences emphasise performing operations on one side of the equals sign to obtain the solution, whereas nontraditional format sentences highlight equality between both sides. With increasing age and education level, exposure to nontraditional formats also increases. However, Behr noted that even third-grade students encountering statements like a + b = c + d tend to accept only the a + b = c format. This reinforces the notion that students' understanding of the equals sign does not sufficiently mature with age and experience. This finding has been repeatedly demonstrated in other studies (e.g., Matthews & Fuchs, 2020; McNeil, Hornburg, Devlin, et al., 2019; Matthews et al., 2010).

As for middle school students, they have already begun studying algebra. Knuth et al. (2005) revealed that the proportions of 6-8 graders (n = 373) demonstrating the relational view were approximately 29%, 37%, and 46%, respectively, across the different grades. Although less than half of the students in each grade exhibited a relational understanding, there was a noticeable increase in this understanding with advancing grade levels. These findings were corroborated by longitudinal research by Alibali et al. (2007), which showed that the percentages of students with the relational view were approximately 20%, 37%, 44%, and 60% at the beginning of the sixth, seventh, and eighth grade and the end of the eighth grade, respectively.

It is generally expected that students will develop a relational view of the equals sign through extensive mathematical experiences by the time they reach high school and university. However, some studies lacking experimental evidence mentioned that many high school students continue to exhibit an operational view (Kieran, 1981). We found only two studies examining high school students' understanding

(Emre-Akdogan, 2023; Harrell, 2016; Simsek et al., 2019) and three on college students' understanding (Chesney et al., 2013; Fyfe et al., 2020; McNeil & Alibali, 2005a). These studies, using similar measurement tools (except for one study, Emre-Akdogan, 2023), indicated that some high school and college students still retain an operational understanding. Nevertheless, these limited studies had small sample sizes and may not have been representative, e.g., based on a specific region (the Midwestern United States) and school type (open-enrollment universities). Future research may benefit from increasing its focus on high school and college students.

As for kindergarteners, Blanton et al. (2018) conducted a study on 40 kindergarten students, finding that approximately 81% tended to hold the operational view. This finding has been consistently supported in other studies (Stephens et al., 2021). Blanton et al. attribute this to informal experiences, such as frequently calculating sums and differences of object sets, rarely decomposing objects, and emphasising unidirectional operations.

In summary, research findings indicated that most primary school students hold an operational view. Many middle school students have yet to develop a relational view of the equals sign. This operational view persists among some high school and college students. Several kindergarten students already exhibit tendencies toward an operational view. However, the understanding of the equals sign among high school and college students still requires further exploration.

The cognitive level of the equals sign

Researchers consistently categorise students' understanding of the equals sign into operational and relational views. As research has progressed, these categories have been further subdivided to capture more nuanced levels of understanding.

Matthews et al. (2012) developed a construct map (Table 1) to segment levels of students' understanding of the equals sign. They identified four levels: Level 1 and Level 3 represent the operational and relational views, respectively, with Level 2 serving as a transitional stage between these two levels. Level 4 involves advanced algebraic reasoning. The key distinction between Level 3 and Level 4 is that students at Level 3 verify equality by performing numerical calculations on both sides of the equation, while students at Level 4 understand that symbolic transformations can maintain equality without extensive numerical computations. Notably, students may hold different views of the equals sign depending on context and task, rather than being confined to a single level (Matthews et al., 2012).

However, the relational understanding here included only the sameness component. Jones et al. (2012) introduced the component of substitution as part of relational understanding, suggesting that the equals sign allows sides of an equation to be swapped. This is known as "the substitutional view," based on the principles of symmetry and transitivity in equivalence relations. For example, it involves replacing $5x + 8$ in $5x + 8 = 2x + 4$ with $x - 2$ in $x - 2 = 5x + 8$.

Table 1
Construct Map for knowledge of the equals sign

Level	Description	Core equation structure(s)
Level 4: Comparative Relational	Successfully solve and evaluate equations by comparing the expressions on the two sides of the equals sign, including using compensatory strategies and recognising transformations to maintain equality. Consistently generate a relational interpretation of the equals sign.	Equations that can be most efficiently solved by applying simplifying transformations: For example, without adding 67+86, can you tell if the number sentence "67 + 86 = 68 + 85" is true or false?
Level 3: Basic Relational	Successfully solve, evaluate, and encode equation structures with operations on both sides of the equals sign. Recognise the relational definition of the equals sign as correct.	Operations on both sides: a + b = c + d, a + b – c = d + e.
Level 2: Flexible Operational	Successfully solve, evaluate, and encode atypical equation structures that remain compatible with an operational view of the equals sign.	Operations on the right: c = a + b No operations: a = a
Level 1: Rigid Operational	Only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure. Define the equals sign operationally.	Operations on the left: a + b = c (including when blank is before the equals sign)

Note. Table adapted from Rittle-Johnson et al. (2011, p.87).

Jones (2008) found that students aged 9 to 12 could possess the substitutional view. To explore the distinctions between the sameness and substitution components, Jones et al. (2012) studied two 11-to 12-year-olds, demonstrating a clear difference between the substitutional and relational views, as supported by Sumpter and Löwenhielm (2024).

We found 10 articles that studied the substitutional view (Donovan et al., 2022a; Filloy et al., 2003; Jones, 2008; Jones et al., 2011; Jones et al., 2012; Jones et al., 2013; Jones & Pratt, 2012; Simsek et al., 2019; Sumpter & Löwenhielm, 2024). Given the substitutional view as a relatively new cognitive level and the progression from a rigid operational to a comparative relational level, as shown in Table 1, questions arise about the cognitive development of the equals sign: Does the substitutional view develop alongside or after the relational view, or do these levels follow distinct paths?

For instance, Jones et al. (2013) demonstrated a significant increase in both relational and substitutional views following the intervention. Specifically, 11 students aged 11-12 (n = 40) improved in accepting both views simultaneously, 10 leaned more towards the substitutional view, and 9 showed increased acceptance of the relational view. Conversely, using tasks adapted from Jones et al. (2013), Simsek et al. (2019) suggested that the relational view emerges earlier than the substitutional view. Their study, involving 57 students aged 14-16, found that 56.1% of the relational view endorsers rejected the substitutional view, while only 3.5% of the substitutional view acceptors rejected the relational view. Donovan et al. (2022b) supported this, arguing that the substitutional view logically follows relational understanding, as interchangeability in equations stems from equivalence. Thus, the developmental path of understanding the equals sign at different levels has not yet reached a consensus in light of current research findings, and further exploration is still necessary.

In summary, students exhibit varying levels of understanding of the equals sign: rigid operational (the operational view), flexible operational, basic relational (the relational view), comparative relational, and the substitutional view. Nonetheless, these levels do not follow a straightforward sequential pattern, and the developmental path of these levels remains uncertain.

Research on influencing factors and interventions in the understanding of the equals sign

Based on the above summary, research consistently reveals that students struggle to understand the equals sign, and they often adopt an operational understanding of it. In response, researchers have initiated investigations to uncover the factors influencing their understanding and have dedicated efforts to explore effective interventions for enhancing students' relational understanding of the equals sign.

The factors influencing the understanding of the equals sign

Initially, brain development was thought to limit students' ability to grasp the concept of the equals sign, with Kieran (1981) suggesting age 13 as a key point. However, the current consensus emphasises that students' maths experience is more crucial for understanding the equals sign.

For instance, Baroody and Ginsburg (1983) reported that a higher proportion of first-grade students achieved a relational view compared to second- and third-graders following the instruction, in which "equals" is defined as "the same as" and a variety of equation formats (1+_ = 3, _ =1+1, etc.) were introduced. This claim has been supported by the change-resistance account proposed by McNeil et al. (2006), and some subsequent studies have also supported the change-resistance account (Chesney & McNeil, 2014; McNeil et al., 2012; McNeil et al., 2015; McNeil et al., 2011). Applied to the domain of mathematics, the change resistance account suggests that difficulties with mathematical equivalence stem not from general conceptual or working memory limitations in childhood, but from children's representations of patterns routinely encountered in the first few years of formal arithmetic instruction (McNeil et al., 2011).

Moreover, McNeil and Alibali (2005) posited that the explicit teaching of the equals sign's meaning is less crucial than increasing exposure to nontraditional equation formats, while reducing exposure to conventional forms. Conversely, Simsek et al.

(2022) found that the format of arithmetic practice presented in students' current-year textbooks did not relate to students' understanding of the equals sign. On the other hand, Lee and Pang (2022) suggested that when teachers explain the meaning of the equals sign and connect it to a pan balance, students are more likely to transition from an operational to a relational view. This suggests that there are still open questions regarding which specific factors in mathematical experience affect students' understanding of the equals sign (Simsek et al., 2022).

In summary, on a macro level, there is greater support for the idea that students' mathematical experiences are a more significant factor affecting their understanding of the equals sign than their brain limitations. On a micro level, within the mathematical experiences in classroom instruction, how the equals sign is defined and presented—such as whether to provide a clear definition of the equals sign, the words interpreting it, and the expression format presenting it—may potentially influence students' understanding of the equals sign (Davenport et al., 2023; McNeil & Alibali, 2005b; McNeil et al., 2012; McNeil et al., 2015; McNeil et al., 2011; McNeil et al., 2019; Powell, 2012). However, determining which specific factors of these are more critical still necessitates further investigation. Additionally, concerning other factors, we found several studies indicating that teachers' understanding of the equal sign can influence students' understanding (e.g., Asquith et al., 2007; Simsek et al., 2022; Vermeulen & Meyer, 2017). Nevertheless, we did not find explicit inquiries into additional factors. Consequently, there is a shortage of research on extra factors that influence students' understanding of the equals sign; future research should explore whether there are other crucial factors.

The interventions improving the understanding of the equals sign

The earliest explorations into interventions aimed at promoting students' relational understanding of the equals sign can be traced back to the work of Weaver (1973). For over five decades, researchers have been striving to develop highly effective instructional methods to foster students' robust understanding of the equals sign as a relational concept.

Understanding the equals sign requires both conceptual knowledge of its meaning and procedural knowledge for solving problems involving operations on both sides of it (Qetrani et al., 2021). Students' conceptual knowledge about the equals sign was typically measured using three items: (a) definitions of the equals sign, (b) equation encoding, and (c) True/false number sentences, while procedural knowledge is often measured through open number sentences (DeCaro & Rittle-Johnson, 2012; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2016).

Recent research has revealed that students exhibit notably less improvement in the task "definitions of the equals sign" when juxtaposed with their advancements in "equation encoding," "True/false number sentences," or "open number sentences." Moreover, their absolute improvement in the task "definitions of the equals sign" also remains low (Davenport et al., 2023; McNeil et al., 2019). Bearing this in mind, we categorized the review of interventions into three sections: (a) interventions that significantly improve both procedural and conceptual knowledge, accompanied by a relatively significant improvement in the task "definitions of the equals sign", (b) interventions that have fostered students' procedural and conceptual knowledge, albeit with a modest improvement in the task "definitions of the equals sign", and (c) those did not assess students' definitions of the equals sign.

Before detailing these studies, we first focus on those from the past decade, summarising their measurement tools and theoretical frameworks in Table 2. As shown in Table 2, recent intervention studies commonly use four measurement items: True/False number sentences, definitions of the equals sign, open number sentences, and equation encoding. Their theoretical frameworks mainly include embodied cognition theory, change resistance, and physical or concrete representation.

Table 2
Overview of intervention experiments conducted in the past decade

Reference	Intervention	Participant	Theory guidance	Measurement
Studies with relatively high improvement in definitions of the equals sign (more than 50% of participants)				
Chow and Wehby (2019)	Nonsymbolic instruction	Second graders	Visual support	ONS, DES, TFNS
Donovan et al. (2022b)	Sameness+Substitutive instruction	Fourth- and fifth-graders	Both sameness and substitutive component	ONS, DES
Studies with relatively low improvement in definitions of the equals sign (less than 50% of participants)				
Bajwa and Perry (2021)	Pan-balance scale	Second- and Third-grade	Grounded and idealised representations (Belenky & Schalk, 2014)	ONS, DES, TFNS
McNeil et al. (2015)	Non-traditional arithmetic practice	Second graders	Change-resistance	ONS, DES, EE
McNeil et al. (2019)	(a) Introducing the equal sign before arithmetic, (b) Concreteness fading exercises, and (c) comparison and explanation	Second graders	Change-resistance	ONS, DES, EE
Davenport et al. (2023)	(a) Introducing the equal sign before arithmetic, (b) Non-traditional arithmetic practice, (c) Concreteness fading exercises, and (d) comparison and explanation	Second graders	Change-resistance	ONS, DES, EE

Reference	Intervention	Participant	Theory guidance	Measurement
Studies without testing students' definitions of the equals sign				
Stephens et al. (2015)	Comprehensive early algebra curriculum	Third graders	Curriculum Research Framework (Clements, 2007)	ONS, TFNS
Blanton et al. (2019)	comprehensive early algebra curriculum	3 to 5 graders	Curriculum Research Framework	ONS, TFNS
Fyfe et al. (2022)	Embedding reflective metacognitive questions	First- and second-graders	The importance of metacognition in mathematics	ONS
Fischer et al. (2019)	ACE (Arithmetic Comprehension at Elementary school)	Second graders	Change-resistance	ONS
Koumoutsakis et al. (2016)	Observing gestures, videotaped vs live instruction	Third- and fourth-graders	Embodied cognition theory	ONS
Valdiviejas et al. (2022)	Observing gestures	Second graders	Embodied cognition theory	ONS
Despina et al. (2024)	Observing representational gestures	Third graders	Embodied cognition theory	ONS, TFNS
Novack et al. (2014)	Producing gestures	Third graders	Embodied cognition theory	ONS
Kersey et al. (2024)	Producing gestures and action	Third- and fourth-graders	Embodied cognition theory	ONS

Abbreviations: ONS: open-number sentences; TFNS: true and false number sentences; DES: definitions of the equal sign; EE: equation encoding.

Studies with relatively high improvement in definitions of the equals sign

These studies revealed that students exhibited enhanced procedural and conceptual knowledge following the intervention, with the majority (over 50%) capable of producing relational definitions for the equals sign post-intervention. For example, Hattikudur and Alibali (2010) discovered that third- and fourth-grade students who received instruction comparing the equals sign with inequality symbols were more inclined to provide relational definitions (74%) compared to those who received only equal sign instruction (57%) or no instruction at all (22%). At the pretest, 38% of all students defined the equals sign relationally. The improvement in defining the equals sign (corresponding to the definitions of the equals sign task) varies by condition, but encoding and solving mathematical equivalence problems (corresponding to the equation encoding and open number sentences task, respectively) did not.

Word problem plus equal-sign instruction combined tutoring has also been found by Powell and Fuchs (2010) to exhibit similar effects for third-graders with mathematics difficulty. They reported that less than 10% of participants provided relational definitions at the pretest. However, by the posttest, nearly all students who had received combined tutoring provided relational definitions, whereas those who had not received equal-sign tutoring provided operational definitions. There was no significant difference in improvement for solving mathematical equivalence problems, but significant variation in defining the equals sign and judging equality sentences (corresponding to the True/false number sentences task) across tutoring conditions.

A study by Donovan et al. (2022a) considered the substitutional view and reported that fourth- and fifth-graders receiving lessons on dual sameness and substitutive conception, sameness alone, and non-conception exhibited change rates of 50%-68%, 58%-77%, and 62%-60%, respectively, in their definitions of sameness from pretest to posttest. Notably, the aforementioned three studies did not assess whether children's improvements were sustained over time.

In a retention test, DeCaro and Rittle-Johnson (2012) demonstrated that second- to fourth-graders who tackled unfamiliar equivalence problems prior to receiving conceptual instruction on mathematical equivalence were more likely to provide relational definitions (58%) compared to those who learned first (20%) in the Midwest. Additionally, conceptual knowledge scores (defining the equals sign and encoding equivalence problems) improved from the posttest to the retention test. There were no order effects on procedural knowledge, but significant effects on conceptual knowledge were observed.

Also, Chow and Wehby (2019) showed that at the follow-up assessment, second-graders who received symbolic intervention, nonsymbolic intervention, and business-as-usual instruction provided relational definitions with predicted probabilities of 75.3%, 58.4%, and 5%, respectively. However, at the pretest, these students were 10.5%, 6.2%, and 6.8%, respectively. Interventions demonstrated significant effect sizes versus the control group in defining the equals sign, judging equality sentences, and solving equivalence problems, but not for the interventions themselves. Nevertheless, this study, along with the aforementioned two studies (DeCaro & Rittle-Johnson, 2012; Hattikudur & Alibali, 2010), implemented the intervention in a one-on-one setting.

In summary, these interventions appear to be promising and effective methods for enhancing both procedural and conceptual knowledge about the equals sign, particularly in encouraging students to provide relational definitions. Nevertheless, future research is needed to investigate whether similar results can be obtained at the classroom level and examine whether these treatment effects are temporary or long-lasting. This is important because it pertains to the degree to which the intervention enhances students' understanding of the equals sign and its practical application in real classroom settings.

Studies with relatively low improvement in definitions of the equals sign

Such studies found that students showed greater procedural and conceptual knowledge post-intervention; however, improvement in defining the equals sign was typically lower than on other tasks, with less than half capable of producing relational definitions after the intervention. For example, McNeil and Alibali (2000) reported that third- and fourth-graders increased by 11%, 6%, and 0% in defining the equals sign after receiving learning, performance, or no goals, respectively. However, those with a learning goal for encoding equations and judging equality sentences showed improvements of 53% and 74%, respectively. Total score changes depended on the goals they received, with those who were given learning goals showing the most improvement.

Building on the change-resistance account and integrating four research-based strategies, McNeil and colleagues, through a series of studies, proposed a supplemental, affordable, and portable comprehensive intervention called Improving Children's Understanding of Equivalence (ICUE) aimed at achieving mastery-level math equivalence understanding for second graders (Davenport et al., 2023; Fyfe et al., 2015; McNeil et al., 2012; McNeil et al., 2015; McNeil et al., 2011; McNeil et al., 2019). The comprehensive intervention had three primary goals: (1) exceeding their "best-case" benchmarks (McNeil et al., 2015), (2) all children demonstrating basic understanding, and (3) at least 50% reaching mastery. Nonetheless, McNeil et al. (2019) acknowledged that they didn't reach their goal of 100% basic understanding and 50% mastery, mainly falling short in defining the relational aspect of the equals sign. While 45% of participants mastered solving and encoding mathematical equivalence problems, only 38% defined the equals sign relationally.

In summary, it suggests that these interventions are likely more effective in enhancing students' ability to judge equality sentences and encode and solve mathematical equivalence equations, but they seem to be less effective in improving the production of relational definitions. This review presents two possible explanations for this phenomenon.

First, these interventions included extensive procedural lessons, which may directly improve procedural skills without ensuring conceptual understanding (McNeil & Alibali, 2000). For instance, subjects in McNeil and Alibali (2000) learned a procedure for solving equivalence problems, and arithmetic practice was a key component of ICUE. In contrast, Hattikudur and Alibali's (2010) intervention focused on conceptual instruction, leading to greater improvement in defining the equals sign than in encoding or solving problems. Therefore, it can be inferred that interventions emphasising conceptual or procedural aspects may be more effective in enhancing corresponding knowledge. This argument is to some extent supported by Rittle-Johnson and Alibali (1999), who observed that although there are causal relations between conceptual and procedural knowledge, conceptual knowledge may have a greater influence on procedural knowledge than the reverse. We argue that procedural knowledge of the equals sign may be easier to improve, while conceptual knowledge is more challenging. Future work can explore these issues and consider the balance between conceptual and procedural approaches to teaching mathematical equivalence.

Second, consistent with Matthews et al. (2012), tasks that require students to define concepts may underestimate their actual conceptual knowledge, as providing verbal or written definitions is more challenging than solving problems, such as equivalence tasks. Therefore, it is unclear whether students failed to deepen their understanding of the equals sign after interventions or struggled to articulate their understanding.

Research without testing the definitions of the equals sign

Such studies found that students showed improvement in judging equality sentences, encoding and solving mathematical equivalence problems, or other tasks after interventions. Although the majority of these studies are successful in these tasks, they are not without limitations.

Some interventions often require highly trained and motivated teachers and researchers (Fyfe et al., 2022), substantial implementation time (Fischer et al., 2019), or both (Blanton et al., 2019; Saenz-Ludlow & Walgamuth, 1998; Stephens et al., 2015). For instance, Blanton et al. (2019) involved grade-level teachers across schools, providing three years of professional development (PD) to support implementation. Similarly, Fyfe et al. (2022) carried out the intervention and measurement by research assistants and researchers. Stephens et al. (2015) replaced regular math classes with 20 one-hour early algebra lessons over the school year.

Some studies, though not time-intensive, did not assess long-term effects (Alibali et al., 2009; Perry, 1991) or classroom-level impacts (Alibali, 1999; Alibali et al., 2009). For instance, Alibali et al. (2009) conducted posttest and transfer tests, but no follow-up tests and implemented the intervention in a one-on-one setting. Some studies utilised modern information technology, which may have limited the generalizability and replicability of their results. For instance, Bajwa and Perry (2021) created a digital pan-balance applet to manipulate instructional conditions and conducted the intervention on a computer in a game context. A chain of studies has demonstrated that incorporating gestures into spoken math equivalence instructions benefits students' relational understanding of the equals sign (Congdon et al., 2017; Kersey et al., 2024; Valdiviejas et al., 2022). Many of these studies, such as Kersey et al. (2024), focus on solving equivalence problems and do not assess conceptual knowledge, leaving it unclear whether gestures also enhance conceptual understanding.

Overall, it is important to note that the interventions discussed above are not universally beneficial for every participant, even one-on-one instruction is no exception (Hattikudur & Alibali, 2010). It may be necessary to continue exploring theoretical frameworks or improving experimental methods that can guide more genuinely effective interventions. This review presents three key considerations that future research could focus on.

First, we assume that preventing the formation and reducing the activation of children's operational views differ in difficulty. Decreasing reliance on the operational view may be more complicated than preventing its formation, as many students are deeply entrenched in it (McNeil et al., 2015). Thus, interventions must clarify whether they aim to prevent formation or reduce activation. According to the change-resistance account, operational views strengthen during early formal schooling, peaking around age 9 (McNeil, 2007). Many studies focus on children aged 6-9, while others include older children (Table 2). As noted earlier, many in these age groups already hold operational understanding, so results often reflect effectiveness in discouraging activation rather than formation. Future studies should pretest students' reliance on the operational view, design instructions introducing the equals sign before its formation, and assess their effectiveness.

Second, a relational understanding of the equals sign includes multiple components, like sameness and substitutive aspects (Jones et al., 2012). Donovan et al. (2022b) found that teaching both sameness and substitutive components together can be effective. This raises the question of whether understanding one component depends on the other or if both should be learned simultaneously due to their interdependence. Currently, this remains unclear, and there is a lack of theoretical frameworks explaining the learning mechanisms or cognitive processes behind the equals sign. Consequently, instructional designs grounded in the developmental path of understanding the equals sign are still limited.

Third, Table 2 shows that while researchers' measurement tools fall under four categories—definitions of the equals sign, open number sentences, equation encoding, and true/false number sentences—they are not uniform in their approach. Some studies include specific items, while others do not. Future research could adopt a unified measurement scale. For example, Matthews et al. (2012) developed a tool to assess levels of understanding of the equal sign, but omitted the substitutive view, a potential focus for future studies.

Cross-cultural research on the understanding of the equals sign

This section reviews international comparative literature on students' understanding of the equals sign to explore differences in how students from diverse cultural backgrounds understand the equals sign and identify factors contributing to these differences. The countries included were based on the retrieved articles.

Comparative studies reveal that students in certain countries struggle less with the equals sign than others (Simsek et al., 2022). For example, Capraro and colleagues reported that approximately 28% of U.S. sixth-graders (n=105) and 98% of Chinese sixth-graders (n=145) accurately answered open number sentences and True/False number sentences with conceptually accurate explanations (Capraro et al., 2007; Capraro et al., 2010; Li et al., 2008). Similar comparisons included South Korean (n = 193) and Turkish (n = 334) sixth-graders, with 59.6% of Korean and 28.4% of Turkish students answering correctly (Capraro et al., 2010). Eichhorn et al. (2018) included Jordan (Grades 2–3, n = 1486) and India (Grade 2, n = 185), finding that both groups struggled with open-number sentences.

Madej (2022) compared Swedish and South Korean students using the Matthews et al. (2012) assessment, noting that Swedish Grade 3 and 6 students outperformed South Korean Grade 3 students in understanding the non-operational meaning of the equals sign. Jones et al. (2012) compared Chinese (n = 150) and UK (n = 101) secondary students on their understanding of substitutive definitions, finding that the UK group rated operational definitions higher and substitutive-relational definitions lower, while the Chinese group had a higher rating for sameness-relational definitions.

Researchers have explored the factors behind cross-country differences in understanding the equals sign, with a focus on classroom materials. Analyses of U.S. and Chinese teacher guides and textbooks revealed that U.S. materials often lack a clear definition of the equals sign, using terms like “makes” and presenting mostly traditional arithmetic sentences. In contrast, Chinese materials emphasise equality as a relational concept, introduce the equals sign alongside inequality symbols (e.g., >, <), and include nontraditional equations early (Li et al., 2008). This strategy aligns with variation theory, which posits that juxtaposing a concept with its opposites (e.g., equality vs. inequality) deepens conceptual understanding by contrasting the differences between the two and identifying the key characteristics of the target concept (Kullberg et al., 2017). Similarly, textbooks in Jordan, India, and South Korea predominantly use traditional arithmetic formats, while Swedish textbooks for early grades favour open-number sentences (Eichhorn et al., 2018; Madej, 2022). These findings suggest that definitions, interpretations, and presentation formats of the equals sign may shape students’ understanding. However, these are not results based on experimental findings.

Simsek et al. (2022) conducted a large-scale study across six countries (China, England, New Zealand, South Korea, Turkey, and the United States), revealing that the format of arithmetic practice in students’ current textbooks did not correlate with their understanding of mathematical equivalence. This result was surprising, as it contradicted predictions from the change-resistance account, which has primarily been studied in the U.S. This suggests that factors in classroom materials influencing the equals sign's understanding require further investigation.

In summary, while the misunderstanding of the equals sign seems prevalent across multiple countries, research shows cultural differences in its understanding. However, questions remain about the role of classroom materials in shaping students' understanding (Simsek et al., 2022). This is crucial for designing effective interventions and explaining their varying success. Future research should explore more fundamental factors influencing this understanding. Additionally, we found no research on the distinction between enacted and intended curriculum for mathematical equivalence, an area that warrants future exploration.

This review highlights certain aspects of the equals sign in the primary school curriculum standards of the aforementioned nine countries, as presented in Table 3. We included the grade level at which the equals sign is first introduced, its description, the related context, and the stage at which arithmetic problems are introduced. We intended to provide insights for future research to explore factors influencing the equals sign's understanding in classroom materials, so we did not carry out a detailed, standardised comparison of these curriculum standards, as it falls outside the scope of our focus.

Table 3
Content about the equal sign in the curriculum standards of nine countries

Countries	Stage of the equal sign	Stage of arithmetic	descriptions	Related contexts
China	Grades 1 to 2	Grades 1 to 2	Understand the meanings of the symbols <, =, >, and compare the sizes of numbers within ten thousand; through comparing the sizes of numbers, gain an understanding of equality and inequality relationships	In concrete situations involving equal, greater, and lesser quantities, guide students to perceive the relationships of equality and inequality between numbers
England	Grade 1	Grade 1	Pupils should be taught to: read, write, and interpret mathematical statements involving addition (+), subtraction (-) and equals (=) signs	solve one-step problems that involve addition and subtraction, using concrete objects and pictorial representations, and missing number problems such as 7 = __– 9
India	Grades 6 to 8	Grades 3 to 5	Understands equality between numerical expressions and learns to check arithmetical equations	N/A
Jordan	Grade 1	Grade 1	Comparison using the symbols: =, >, <	Comparing numbers using the symbols <, >, =

Countries	Stage of the equal sign	Stage of arithmetic	descriptions	Related contexts
New Zealand	Grade 4	Grade 1	Record and interpret additive and simple multiplicative strategies, using words, diagrams, and symbols, with an understanding of equality	Equations and expressions
South Korea	Grades 1 to 2	Grades 1 to 2	Explore the relationship between two quantities and represent it using the equal sign	Help them understand that the quantities on both sides of the equal sign (=) are equal to each other in addition, subtraction, and multiplication.
Swedish	Grades 1 to 3	Grades 1 to 3	Mathematical similarities and the meaning of the equal sign	When students work with the meaning of the equal sign, it can be important to compare what is equal to what is not equal. The meaning of the equal sign can be made visible by structurally exploring what does not constitute a similarity, what constitutes a similarity, and how differences can be transformed into similarities. The symbols greater than and less than can be introduced at the same time as the equal sign. They can also be used to work with mathematical similarities and differences, as well as symbols for these concepts, in practical exercises.
Turkey	Grade 1	Grade 1	The symbol for the addition operation (+) and the equal sign (=) are introduced, and their meanings are emphasised	The addition operation is performed both horizontally and vertically. When performing vertical addition, the importance of the operation line having a similar meaning to the equal sign is emphasised.
United States	Grade 1	Kinder-garten	Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. For example, which of the following equations are true and which are false? 6 = 6, 7 = 8-1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2	Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8 + _= 11, 5 = _ - 3, 6 + 6 = _

Conclusions

This review aimed to (a) summarise students' understanding of the equals sign across age groups and the various levels of understanding they exhibit; (b) provide an overview of the factors influencing students' understanding of the equals sign and the effectiveness of interventions designed to improve their understanding; (c) outline the differences in students' understanding of the equals sign across various cultural contexts and the factors resulting in these differences; and (d) identify research gaps and suggest future directions. The review found that (a) Primary students often struggle to understand the equals sign. Their difficulty with this concept manifests as they view the equals sign operationally rather than relationally. As grade levels increase, although more students develop a relational view, some retain an operational view even into college. Many kindergarten students may already exhibit tendencies toward operational understanding; (b) Students can have different levels of understanding of the equals sign: rigid operational (the operational view), flexible operational, basic relational (the relational view), comparative relational, and the substitutional view; (c) Students' mathematical experiences play a major role in their understanding of the equals sign. The definition and presentation of the equals sign may be an influencing factor. (d) Most interventions can improve procedural knowledge of the equals sign, but conceptual knowledge, particularly the definitions of the equals sign, varies. Some studies enhanced the generation of the relational definition of the equals sign for most participants (over 50%), while some only for a few (below 50%), and some did not test defining this concept. Recent intervention experiments primarily utilise four measurement items: true/false number sentences, definitions of the equals sign, open-number sentences, and equation encoding. The theoretical frameworks guiding these interventions primarily include embodied cognition theory, change resistance, and physical or concrete representation; and (e) Students from different cultures understand the equals sign differently, likely due to variations in how curriculum standards and textbooks handle the equals sign.

Current research on the cognitive development of the equal sign is relatively comprehensive; however, several issues warrant further investigation. These include: (a) Focus on high school and college students' understanding should increase; (b) The development path of understanding of the equals sign remains uncertain; (c) Whether the definition or the presentation of the equals sign has a greater impact on students' understanding of the equals sign still requires further research. Whether there are other more critical factors is still unknown; (d) There is a lack of interventions that can intensely and robustly improve both students' conceptual and procedural knowledge of the equals sign, which are easy to implement at the classroom level. In particular, portable interventions that can strongly improve students' ability to produce a relational definition of the equals sign are lacking; and (e) Open questions remain about whether the expression format of the equals sign causes differences in children's understanding across countries. The reasons for these differences need further exploration, and research on the enacted versus intended curriculum for mathematical equivalence is currently lacking.

This review provides some suggestions for future research on instructional interventions: (a) consider distinguishing and appropriately allocating conceptual and procedural teaching of the equals sign; (b) develop theories on the developmental path of different levels of understanding of the equals sign, or theories on the cognitive processes or mechanisms involved in understanding the equals sign; (c) clarify whether interventions aim to reduce or prevent the operational view, investigating whether co-teaching equality and inequality signs in early instruction - as implemented in some not standard practice - move beyond the operational view. This approach, rooted in variation theory (Bussey et al., 2013), could be tested in contexts where inequality signs are traditionally delayed, offering insights into the generalizability of cross-cultural findings; and (d) develop a unified measure to assess understanding of the equals sign.

Implications

The theoretical implications of this study are (a) Deepening Cognitive Theory: By analysing students' understanding of the equals sign across different ages and varying levels of understanding, it enhances our understanding of cognitive development in learning mathematical symbols. It also suggests directions for new cognitive theories, aiding in explaining students' thinking processes and barriers in mathematics symbol learning; (b) Enriching Mathematics Education Theory: By reviewing four areas (students' understanding of the equals sign, influencing factors, interventions, and cross-cultural comparisons), it helps develop a more systematic research framework for mathematics education; (c) By reviewing international comparison research, it provides an overview of how students' understanding of the equals sign varies across different cultural backgrounds and the impact of cultural factors on mathematics learning. This enhances the development of cross-cultural mathematics education theory.

The practical implications of this study are (a) Personalized Teaching: By outlining understanding across age groups and different levels of understanding, it aids in meeting the diverse learning needs of different age groups; (b) Improving Teaching Methods: By discussing influencing factors and interventions, it helps teachers design more effective instructions, such as balancing procedural and conceptual teaching of the equals sign to enhance understanding its meaning and application; (c) Teacher Training: Review findings can be used for teacher training, enhancing their comprehension of students' cognitive development of the equal sign; (d) Curriculum Design: The review results offer insights for optimizing the content and structure of mathematics curricula to ensure students can fully understand the equals sign and its applications. For instance, curricula could integrate equality and inequality symbols from the earliest grades. By teaching these symbols contrastively, educators may help students disentangle the relational meaning of the equals sign from operational understanding, a hypothesis that warrants empirical testing in future interventions.

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Kognitivni razvoj znaka jednakosti: pregled

Sažetak

Kognitivni razvoj znaka jednakosti privlači pažnju znanstvenika već desetljećima, no sveobuhvatni pregledi nedostaju. Cilj je ove studije pregledati istraživanja u četiri područja: kako učenici razumiju znaka jednakosti, faktore koji utječu na njihovo razumijevanje, intervencije za poticanje razumijevanja i komparacije između kultura. Također, cilj je identificirati neriješena pitanja, praznine i razvojne trendove. Na temelju pregleda 85 radova od 1932. do 2024. godine iz šest baza podataka, ova studija doprinosi: (a) sažimanjem razumijevanja znaka jednakosti kod učenika različitih dobnih skupina, različitih razina razumijevanja, ključnih faktora koji utječu na razumijevanje učenika, teorijskih okvira i mjernih elemenata te rezultata koji su široko zastupljeni u istraživanjima intervencija i razumijevanja učenika iz različitih zemalja; (b) prepoznavanjem nedostatka istraživanja o razumijevanju učenika srednjih škola i studenata, razvojnih putova različitih razina razumijevanja, specifičnih faktora koji utječu na razumijevanje te prenosivih intervencija za poboljšanje proceduralnoga i konceptualnoga znanja znaka jednakosti i (c) sugeriranjem budućih istraživanja koja bi trebala istražiti kognitivne mehanizme razumijevanja znaka jednakosti, integrirati proceduralno i konceptualno poučavanje, razlikovati smanjenje ili sprječavanje nesporazuma, razviti ujedinjeni mjerni alat i ispitati faktore koji stoje iza razlika u razumijevanju između zemalja. Ovom studijom doprinosi se sustavnijoj teoriji i okvirima istraživanja u matematičkom obrazovanju.

Ključne riječi: *algebra; jednadžba; matematička ekvivalentnost; razumijevanje*

Uvod

Znak jednakosti („=”) je simbolički prikaz jednakosti ili matematičke ekvivalentnosti (Devlin i sur., 2023; Sumpter i Löwenhielm, 2024). Jednakost ili matematička ekvivalentnost označava da dvije strane jednadžbe imaju istu vrijednost, predstavljaju isti matematički objekt ili su zamjenjive (Jones i sur., 2012). Stoga, razumijevanje znaka jednakosti uključuje i jednakost i zamjenjivost (Jones i sur., 2013). Ukratko, znak jednakosti ima duboko i višestruko značenje.

Znak jednakosti je ključan u matematici i drugim disciplinama, poput znanosti (Matthews i Fuchs, 2020). To je temeljna komponenta algebarskoga razmišljanja i zaključivanja (Falkner i Levi, 1999; Molina i sur., 2009). Konkretno, prijelaz učenika s aritmetike na algebru usko je povezan s njihovim razumijevanjem znaka jednakosti (Matthews i sur., 2012). Štoviše, odgovarajuće razumijevanje znaka jednakosti u nižim razredima predviđa buduću matematičku kompetenciju u višim razredima, uključujući uspjeh na standardiziranim matematičkim ispitima (Davenport i sur., 2023; Knuth i sur., 2006; McNeil, Hornburg, Brletic-Shipley i sur., 2019), rješavanje jednadžbi (Knuth i sur., 2006) i rješavanje tekstualnih zadataka povezanih s matematičkom ekvivalentnošću (Matthews i Fuchs, 2020). Stoga, razvijanje fleksibilnijega razumijevanja ovoga simbola i izbjegavanje nesporazuma od ključne su važnosti za učenike.

Nažalost, djeca često imaju problema s razumijevanjem pojma znaka jednakosti (Florida State Univ i Behr, 1976; Lee i Pang, 2023). Kako bi se algebra učinila pristupačnom svim učenicima, ključno je razumjeti kako učenici percipiraju znak jednakosti i kako se to razumijevanje može poboljšati (Simsek i sur., 2022). Tijekom desetljeća, nastala su brojna istraživanja kako učenici razumiju znak jednakosti. Unatoč tome, vrlo je malo pokušaja da se sintetizira postojeće istraživanje. Pretražili smo EBSCO koristeći pojmove „math* equivalence," „equal* sign," „math* equality" i „equal symbol" kao termine za pretragu u sažetcima i pronašli samo dva povezana pregleda: jedan tradicionalni pregled iz 1981. (Kieran, 1981) i jednu metaanalizu koja se fokusira na intervencije povratne informacije (Fyfe i Brown, 2018).

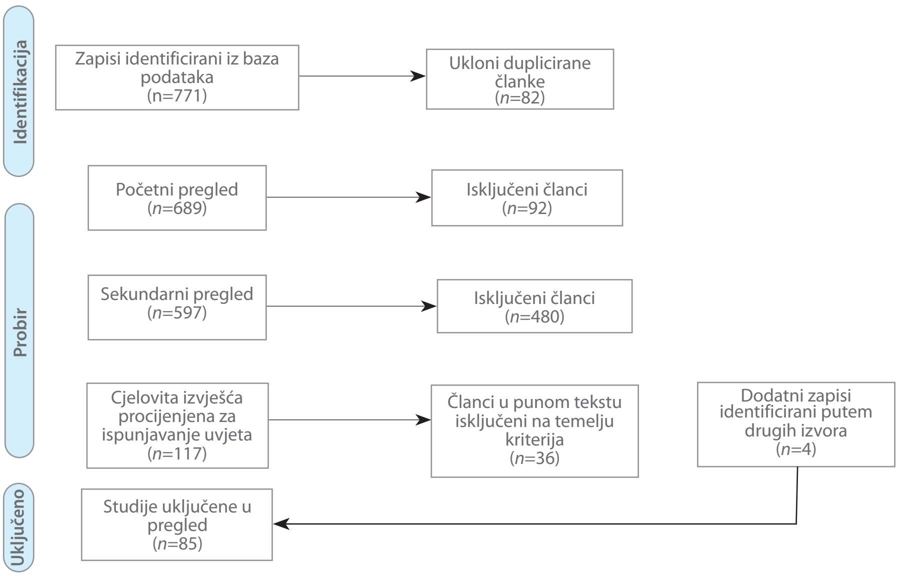
Ova studija ima za cilj sveobuhvatno pregledati literaturu o kognitivnom razvoju učenika u vezi sa znakom jednakosti kako bi se popunila ova praznina. Konkretno, ovaj pregled ima za cilj: (a) sažeti razumijevanje znaka jednakosti učenika različitih dobnih skupina i s obzirom na različite razine razumijevanja koje pokazuju; (b) pružiti pregled ključnih faktora koji utječu na to kako učenici razumiju znak jednakosti i učinkovitost intervencija dizajniranih za poboljšanje njihovoga razumijevanja; (c) opisati razlike u razumijevanju znaka jednakosti učenika koji žive u različitim kulturnim kontekstima i faktore koji uzrokuju te razlike te (d) identificirati neriješena pitanja ili praznine u tim područjima i pružiti potencijalne smjerove ili preporuke za buduća istraživanja.

Ovaj pregled započinje opisom strategije pretraživanja, zatim sažima kako učenici različitih dobnih skupina razumiju znak jednakosti i različite razine njihova razumijevanja. S obzirom na to da intervencije prilagođene temeljnim faktorima imaju veće šanse za uspjeh, analizira se istraživanje faktora koji utječu na razumijevanje prije nego što se raspravlja o intervencijama dizajniranim za njegovo poboljšanje. Sljedeće, pregledavaju se komparacije među kulturama, ističući razlike u tome kako učenici iz različitih zemalja razumiju znak jednakosti i faktore koji doprinose tim razlikama. Na kraju svake podsekcije pružaju se sažetci, raspravljaju neriješena pitanja ili praznine te se predlažu smjerovi za buduća istraživanja. Pregled završava zaključcima i implikacijama.

Metodologija

Za razliku od sustavnih pregleda sa strogim kriterijima i jačim dokazima, proveli smo tradicionalni pregled, naglašavajući širu, interpretativnu sintezu. Ovaj pristup omogućuje detaljnu analizu i kontekstualnu raspravu uključenih studija, pružajući dublje uvide u teme i trendove u istraživanjima kognitivnoga razvoja učenika u vezi sa znakom jednakosti (Rozas i Klein, 2010).

Pretraživanja su provedena u rujnu 2023. i ažurirana u listopadu 2024. korištenjem Web of Science i EBSCO (uključujući baze podataka APA PsycArticles, Psychology and Behavioral Science Collection, Teacher Reference Center, ERIC i Academic Search Premier). Korišteni su sljedeći pojmovi za pretragu: „equal* sign", „equal symbol", „math equivalence",„mathematical equivalence", „math equality"i „mathematical equality". U početku smo identificirali ukupno 771 rad o ovoj temi putem pretrage sažetaka. Dijagram tijeka na Slici 1 prikazuje proces pretraživanja i filtriranja literature.



Slika 1. Dijagram tijeka procesa odabira članaka

Kriteriji uključivanja za članke bili su: (a) akademski časopisi; (b) dostupni sažetci; (c) engleski jezik; (d) dostupni cijeli tekstovi i (e) studija je morala istraživati razumijevanje znaka jednakosti među učenicima. Kriteriji isključenja za literaturu bili su: studije koje nisu istraživale barem jedan od tri aspekta - faktore koji utječu, intervencije ili međunarodne komparacije razumijevanja znaka jednakosti među učenicima - bile su isključene. Kako bismo razumjeli izvore i temelje trenutačnih studija o razumijevanju znaka jednakosti među učenicima, pretraživali smo i pronašli prvi dostupni članak o ovoj temi, objavljen 1932. godine (Renwick, 1932). Stoga, vremenski okvir uključenih članaka obuhvaća razdoblje od 1932. do 2024. godine. U prvom krugu filtriranja uključeno je 597 članaka prema kriterijima (a), (b) i (c). Drugi krug filtriranja proveden je prema kriterijima (d) i kriterijima isključenja. Ovim je isključeno 480 članaka, a uključeno 117 radova. Nakon filtriranja prema kriteriju (e) i deduplikacije, uključeno je 81 članaka. Također smo pretraživali i uključili 4 izvještaja, čime je ukupno uključeno 85 radova. Među njima, 40 članaka bilo je označeno kao „razumijevanje znaka jednakosti među učenicima" (npr. Freiman i Lee, 2004; Jonesi& Pratt, 2006; Knuth i sur., 2008; Society for Research on Educational i sur., 2010), 33 članka označena su kao „pedagoške intervencije dizajnirane za poboljšanje razumijevanja učenika" (npr. Cook i sur., 2008; Jacobs i sur., 2007; Mann, 2004; Powell, 2015), 15 članaka označena su kao „faktori koji utječu na ovo razumijevanje" (npr. Asquith i sur., 2007; McNeil i Alibali, 2005; Powell, 2012), a 4 članka (Eichhorn i sur., 2018; Jones i sur., 2012; Li i sur., 2008; Madej, 2022) i 2 izvještaja (Capraro i sur., 2007; Capraro i sur., 2010) označena su kao „međunarodne komparacije".

Istraživanje razumijevanja znaka jednakosti

Među 40 članaka u kojima se govori o tome kako znak jednakosti razumiju učenici, značajan broj fokusirao se na učenike osnovnih škola (npr. Baroody i Ginsburg, 1983; Behr, 1980; Matthews i sur., 2012) i srednjih škola (npr. Kieran i Martínez-Hernández, 2022; Rittle-Johnson i sur., 2011; Sumpter i Löwenhielm, 2024), dok je samo mali broj istraživao razumijevanje djece predškolske dobi (Blanton i sur., 2018; Devlin i sur2023; Falkner i sur., 1999; Stephens i sur., 2021), srednjoškolaca (Emre-Akdogan, 2023; Harrell, 2016; Simsek i sur., 2019) i studenata (Chesney i sur., 2013; Fyfe i sur., 2020; McNeil i Alibali, 2005a). Obrazovni stupanj spomenut u ovom članku odnosi se na odgovarajuće nacionalne obrazovne sustave, pri čemu se specifične godine mogu razlikovati ovisno o zemlji. Sljedeće ćemo prvo pružiti pregled učeničkoga razumijevanja znaka jednakosti na svakom obrazovnom stupnju, odabirući neke studije kao primjere. Razlog odabira ovih studija jest taj što mogu odražavati evoluciju istraživanja ili imaju visoku citiranost. Zatim ćemo opisati različite razine razumijevanja znaka jednakosti među učenicima.

Razumijevanje znaka jednakosti u različitim dobnim skupinama

Najranije istraživanje o kognitivnom razvoju znaka jednakosti koje se može pronaći jest studija Renwicka iz 1932. godine, koja izvještava o nesporazumima učenika u vezi sa znakom jednakosti (Renwick, 1932). Od tada, značajna znanstvena pozornost posvećena je ovoj temi. Istraživači uglavnom koriste instrumente poput „točno/netačno brojčane rečenice" (npr. određivanje je li 3 = 1 + 2 ispravno ili netočno), „definicije znaka jednakosti" (npr. traženje od učenika da objasne znak jednakosti), „otvorene brojčane rečenice" (npr. popunjavanje praznina za 2 +_ = *3 + 5), i „kodiranje jednadžbi" (npr, rekonstrukcija jednadžbi poput 7 + 1* =_ + 6, tj. reproduciranje nekoliko sličnih problema iz sjećanja učenika) za testiranje razumijevanja znaka jednakosti među učenicima (Hornburg i sur., 2021; Matthews i sur., 2012).

Renwick (1932) kategorizirao je razumijevanje znaka jednakosti među učenicima u operativni i relacijski pogled. Operativni pogled odnosi se na percepciju znaka jednakosti kao jednostranoga simbola koji označava rezultat operacije i/ili kao signal „učini nešto", dok se relacijski pogled odnosi na tumačenje znaka jednakosti kao dvosmjernoga simbola koji označava da dvije strane znaka jednakosti imaju istu vrijednost ili količine. Renwick (1932) je otkrio da je većina sudionika u dobi od 8 do 12 godina i neki u dobi od 13 do 14 godina imali operativni pogled, što ukazuje na nedovoljno razumijevanje znaka jednakosti od osnovne do srednje škole. Ovo je procijenjeno putem točno/netačno brojčanih rečenica, pri čemu je ispravno smatrati 12 + 2 = 14 ÷ 2 = 7, ali pogrešno smatrati 17 × 2 = 30 + 4, što odražava operativni pogled.

U skladu s Renwickom (1932), Behr (1980) je otkrio da djeca u dobi od 6 do 12 godina većinom imaju operativni pogled. Koristili su otvorene brojčane rečenice, uključujući tradicionalni format (operacije na lijevoj strani, npr. 5 + 3 =_) i netradicionalni format (operacije na desnoj strani, s obje strane ili bez operacija, npr. 8 = +*6, 4* + = 5 + 9 ili 7 = 7), kao mjerno sredstvo. Rečenice u tradicionalnom formatu naglašavaju izvođenje operacija na jednoj strani znaka jednakosti kako bi se dobilo rješenje, dok rečenice u netradicionalnom formatu ističu jednakost između obje strane. S porastom dobi i razine obrazovanja, izloženost netradicionalnim formatima također raste. Međutim, Behr je primijetio da čak i učenici trećega razreda koji se susreću s izjavama poput a + b = c + d često prihvaćaju samo format a + b = c. Ovo jača mišljenje da godine i iskustvo ne utječu dovoljeno na razumijevanje znaka jednakosti kod učenika nije dovoljno. Ovo je otkriće ponovljeno u drugim studijama (npr. Matthews i Fuchs, 2020; McNeil, Hornburg, Devlin i sur., 2019; Society for Research on Educational i sur., 2010).

Što se tiče učenika srednjih škola, oni su već počeli učiti algebru. Knuth i sur. (2005) otkrili su da su postotci učenika 6. do 8. razreda (n = 373) koji su pokazivali relacijski pogled bili približno 29 %, 37 % i 46 % u različitim razredima. Iako manje od polovice učenika u svakom razredu pokazuje relacijsko razumijevanje, primijećen je značajan porast ovoga razumijevanja s napredovanjem u razredima. Ovi su rezultati potvrđeni longitudinalnim istraživanjem Alibali i sur. (2007), koje je pokazalo da su postotci učenika s relacijskim pogledom bili približno 20 %, 37 %, 44 % i 60 % na početku šestoga, sedmoga, osmoga razreda i na kraju osmoga razreda, redom.

Općenito se očekuje da će učenici razviti relacijski pogled na znak jednakosti tijekom opsežnih matematičkih iskustava u srednjoj školi i na fakultetu. Međutim, neka istraživanja koja nemaju eksperimentalne dokaze spominju da mnogi srednjoškolci i dalje pokazuju operativni pogled (Kieran, 1981). Pronašli smo samo 2 studije koje ispituju razumijevanje srednjoškolaca (Emre-Akdogan, 2023; Harrell, 2016; Simsek i sur., 2019) i 3 o razumijevanju studenata (Chesney i sur., 2013; Fyfe i sur.., 2020; McNeil i Alibali, 2005a). Ova istraživanja, koristeći slične mjerni instrumente (osim jedne studije, Emre-Akdogan, 2023), pokazala su da neki srednjoškolci i studenti još uvijek zadržavaju operativno razumijevanje. Ipak, ta ograničena istraživanja imala su male uzorke i možda nisu bila reprezentativna, npr. temeljena na specifičnoj regiji (srednji zapad SAD-a) i tipu škole (otvoreni univerziteti). Buduća istraživanja možda bi trebala povećati fokus na srednjoškolce i studente.

Što se tiče djece predškolske dobi Blanton i sur. (2018) proveli su istraživanje na 40 djece predškolske dobi, otkrivši da je približno 81 % njih imalo operativni pogled. Ovo otkriće dosljedno je podržano u drugim istraživanjima (Stephens i sur., 2021).

Blanton i sur. ovo pripisuju neformalnim iskustvima, poput čestoga računanja zbrojeva i razlika skupa objekata, rijetkoag rastavljanja objekata i naglašavanja jednostranih operacija.

Sažeto, istraživački nalazi ukazuju da većina učenika osnovnih škola ima operativni pogled. Mnogi učenici srednjih škola još uvijek nisu razvili relacijski pogled na znak jednakosti. Ovaj operativni pogled i dalje opstaje kod nekih srednjoškolaca i studenata. Nekoliko učenika predškolske dobi već pokazuje sklonost operativnom pogledu. Međutim, razumijevanje znaka jednakosti među srednjoškolcima i studentima još uvijek zahtijeva daljnje istraživanje.

Kognitivna razina znaka jednakosti

Istraživači dosljedno kategoriziraju razumijevanje znaka jednakosti među učenicima u operativni i relacijski pogled. Kako je istraživanje napredovalo, ove su kategorije dalje podijeljene kako bi se obuhvatile suptilnije razine razumijevanja.

Matthews i sur. (2012) razvili su konstrukt kartu (Tablica 1) za segmentiranje razina razumijevanja znaka jednakosti među učenicima. Identificirali su četiri razine: Razina 1 i Razina 3 predstavljaju operativni i relacijski pogled, redom, dok Razina 2 služi kao prijelazna faza između ove dvije razine. Razina 4 uključuje napredno algebarsko razmišljanje. Ključna razlika između Razine 3 i Razine 4 jest u tome što učenici na Razini 3 provjeravaju jednakost izvođenjem numeričkih izračuna na obje strane jednadžbe, dok učenici na Razini 4 razumiju da simboličke transformacije mogu održati jednakost bez opsežnih numeričkih izračuna. Važno je napomenuti da učenici mogu imati različite poglede na znak jednakosti, ovisno o kontekstu i zadatku, umjesto da budu ograničeni na jednu razinu (Matthews i sur., 2012).

Međutim, relacijsko razumijevanje ovdje je uključivalo samo komponentu jednakosti. Jones i sur. (2012) uveli su komponentu supstitucije kao dio relacijskoga razumijevanja, sugerirajući da znak jednakosti omogućava zamjenu strana jednadžbe. Ovo je poznato kao „supstitucijski pogled," temeljen na principima simetrije i tranzitivnosti u odnosima ekvivalencije. Na primjer, to uključuje zamjenu $5x + 8$ u $5x + 8 = 2x + 4$ s $x - 2$ u $x - 2 = 5x + 8$.

Jones (2008) otkrio je da učenici u dobi od 9 do 12 godina mogu posjedovati supstitucijski pogled. Kako bi istražili razlike između komponenti jednakosti i supstitucije, Jones i sur. (2012) proučavali su dvoje djece u dobi od 11 do 12 godina, pokazujući jasnu razliku između supstitucijskoga i relacijskoga pogleda, što su podržali Sumpter i Löwenhielma (2024).

Pronašli smo 10 članaka koji su proučavali supstitucijski pogled (Donovan et al., 2022b; Filloy et al., 2003; Jones, 2008; Jones et al., 2011; Jones et al., 2012; Jones et al., 2013; Jones & Pratt, 2012; Simsek et al., 2019; Sumpter & Löwenhielm, 2024). S obzirom na to da je supstitucijski pogled relativno nova kognitivna razina i na napredovanje s rigidnoga operativnog prema komparativnom relacijskom nivou u Tablici 1, postavljaju se pitanja o kognitivnom razvoju znaka jednakosti: Razvija li se supstitucijski pogled zajedno s relacijskim pogledom ili nakon njega, ili ove razine slijede različite putove?

Tablica 1
Konstrukt karta za znanje o znaku jednakosti

Razina	Opis	Osnovna struktura jednadžbe
Razina 4: Komparativni relacijski	Uspješno rješavanje i evaluacija jednadžbi usporedbom izraza s obje strane znaka jednakosti, uključujući korištenje kompenzacijskih strategija i prepoznavanje transformacija za održavanje jednakosti. Dosljedno generiranje relacijskoga tumačenja znaka jednakosti.	Jednadžbe koje se mogu najučinkovitije riješiti primjenom pojednostavljujućih transformacija: Na primjer, bez dodavanja 67+86, možete li reći je li brojčana rečenica „67 + 86 = 68 + 85" točna ili netočna?
Razina 3: Osnovni relacijski	Uspješno rješavanje, evaluacija i kodiranje struktura jednadžbi s operacijama na obje strane znaka jednakosti. Prepoznavanje relacijske definicije znaka jednakosti kao ispravne.	Operacije na obje strane: a + b = c + d, a + b – c = d + e.
Razina 2: Fleksibilni operativni	Uspješno rješavanje, evaluacija i kodiranje atipičnih struktura jednadžbi koje ostaju kompatibilne s operativnim pogledom na znak jednakosti.	Operacije na desnoj strani: c = a + b Bez operacija: a = a
Razina 1: Rigidni operativni	Uspješan samo s jednadžbama sa strukturom: operacija-jednako-odgovor, uključujući rješavanje, evaluaciju i kodiranje jednadžbi s ovom strukturom. Definiranje znaka jednakosti operativno.	Operacije na lijevoj strani: a + b = c (uključujući kada je prazno mjesto prije znaka jednakosti)

Napomena. Tablica prilagođena iz Rittle-Johnson i sur. (2011, str. 87).

Na primjer, Jones i sur. (2013) pokazali su značajan porast i u relacijskom i u supstitucijskom pogledu nakon intervencije. Konkretno, 11 učenika u dobi od 11 do 12 godina (n = 40) poboljšalo je prihvaćanje obaju pogleda istovremeno, 10 je više naginjalo supstitucijskom pogledu, a 9 je pokazalo povećano prihvaćanje relacijskoga pogleda. S druge strane, koristeći zadatke prilagođene iz Jones i sur. (2013), Simsek i sur. (2019) sugerirali su da se relacijski pogled pojavljuje prije supstitucijskoga pogleda. Njihovo istraživanje s 57 učenika u dobi od 14 do 16 godina pokazalo je da je 56,1 % onih koji podržavaju relacijski pogled odbacilo supstitucijski pogled, dok je samo 3,5 % onih koji prihvaćaju supstitucijski pogled odbacilo relacijski pogled. Donovan i sur. (2022a) podržavaju ovo, tvrdeći da supstitucijski pogled logički slijedi relacijsko razumijevanje jer zamjenjivost u jednadžbama proizlazi iz ekvivalencije. Dakle, razvojni put različitih razina razumijevanja znaka jednakosti još uvijek nije postigao konsenzus u svjetlu trenutačnih istraživačkih nalaza, a daljnje istraživanje i dalje je potrebno.

Ukratko, učenici pokazuju različite razine razumijevanja znaka jednakosti: rigidni operativni (operativni pogled), fleksibilni operativni, osnovni relacijski (relacijski pogled), komparativni relacijski i supstitucijski pogled. Ipak, ove razine ne slijede jednostavan sekvencijalni obrazac, a razvojni put tih razina ostaje neizvjestan.

Istraživanje faktora koji utječu i intervencija u razumijevanju znaka jednakosti

Na temelju prethodno rečenoga, istraživanja dosljedno otkrivaju izazove učenika u razumijevanju znaka jednakosti, te oni često usvajaju operativno razumijevanje znaka jednakosti. Kao odgovor, istraživači su započeli istraživanja kako bi otkrili faktore koji utječu na njihovo razumijevanje te su posvetili napore istraživanju učinkovitih intervencija za poboljšanje učeničkoga relacijskog razumijevanja znaka jednakosti.

Faktori koji utječu na razumijevanje znaka jednakosti

Isprva se smatralo da je razvoj mozga ograničavao sposobnost učenika da shvate pojam znaka jednakosti, a Kieran (1981) je sugerirao da je 13. godina ključna točka. Međutim, trenutačni konsenzus naglašava da je matematičko iskustvo učenika važnije za razumijevanje znaka jednakosti.

Na primjer, Baroody i Ginsburg (1983) izvijestili su da je veći postotak učenika prvih razreda postigao relacijski pogled u usporedbi s učenicima drugih i trećih razreda nakon nastave u kojoj je „jednako" definirano kao „isto kao" te su predstavljeni različiti formati jednadžbi (1+_=3, _=1+1 itd.). Ovu tvrdnju podržava račun otpornosti na promjene koji su predložili McNeil i sur. (2006), a neka kasnija istraživanja također su podržala račun otpornosti na promjene (Chesney i McNeil, 2014; McNeil i sur., 2012; McNeil i sur., 2015; McNeil i sur., 2011). Primijenjeno na područje matematike, račun otpornosti na promjene sugerira da poteškoće s matematičkom ekvivalentnošću ne proizlaze iz općih konceptualnih ili radnih ograničenja u dječjoj dobi, već iz dječjih prikaza obrazaca s kojima se redovito susreću u prvim godinama formalne nastave aritmetike (McNeil Ii sur., 2011).

Štoviše, McNeil i Alibali (2005) tvrde da je eksplicitno poučavanje značenja znaka jednakosti manje ključno od povećanja izloženosti netradicionalnim formatima jednadžbi uz smanjenje izloženosti konvencionalnim oblicima. Suprotno tome, Simsek i sur. (2022) otkrili su da format aritmetičke prakse prikazan u udžbenicima za tekuću školsku godinu nije bio povezan s učeničkim razumijevanjem znaka jednakosti. S druge strane, Lee i Pang (2022) sugerirali su da kada učitelji objašnjavaju značenje znaka jednakosti i povezuju ga s ravnotežom, učenici su skloniji prelasku s operativnoga na relacijski pogled. To sugerira da još uvijek postoje otvorena pitanja u vezi s tim koji specifični faktori u matematičkom iskustvu utječu na učeničko razumijevanje znaka jednakosti (Simsek i sur., 2022).

Ukratko, na makrorazini postoji veća podrška ideji da su matematička iskustva učenika važniji faktor koji utječe na njihovo razumijevanje znaka jednakosti nego ograničenja njihovoga mozga. Na mikrorazini, unutar matematičkih iskustava u učioničkoj nastavi, način na koji je znak jednakosti definiran i predstavljen — kao što je hoće li biti pružena jasna definicija znaka jednakosti, riječi koje ga interpretiraju i format izraza u kojem je predstavljen — može potencijalno utjecati na učeničko razumijevanje znaka jednakosti (Davenport i sur., 2023; McNeil i Alibali, 2005b; McNeil i sur., 2012; McNeil i sur., 2015; McNeil i sur., 2011; McNeil, Hornburg, Brletic-Shipley i sur., 2019; Powell, 2012). Međutim, određivanje koji su specifični faktori od ovih najvažniji još uvijek zahtijeva daljnje istraživanje. Dodatno, u vezi s drugim faktorima, pronašli smo nekoliko studija koje ukazuju na to da učiteljevo razumijevanje znaka jednakosti može utjecati na razumijevanje učenika (npr. Asquith i sur., 2007; Simsek i sur., 2022; Vermeulen i Meyer, 2017). Ipak, nismo našli eksplicitna istraživanja drugih faktora. Posljedično, postoji manjak istraživanja o dodatnim faktorima koji utječu na učeničko razumijevanje znaka jednakosti, stoga bi buduća istraživanja trebala istražiti postoje li drugi ključni faktori.

Intervencije koje poboljšavaju razumijevanje znaka jednakosti

Najranija istraživanja o intervencijama usmjerenima na promicanje relacijskoga razumijevanja znaka jednakosti mogu se pratiti unatrag do rada Weavera (1973). Više od pet desetljeća istraživači su nastojali razviti vrlo učinkovite upute za poticanje robusnoga relacijskog razumijevanja znaka jednakosti od strane učenika.

Razumijevanje znaka jednakosti zahtijeva konceptualno znanje o njegovom značenju, kao i proceduralno znanje za rješavanje problema s operacijama s obje strane znaka (Qetrani i sur., 2021). Konceptualno znanje učenika o znaku jednakosti obično se mjerilo pomoću tri stavke: (a) definicije znaka jednakosti, (b) kodiranje jednadžbi i (c) točno/netočno numeričke rečenice, dok se proceduralno znanje često mjeri pomoću otvorenih numeričkih rečenica (DeCaro i Rittle-Johnson, 2012; McNeil i Alibali, 2000; Rittle-Johnson i Alibali, 1999; Rittle-Johnson i sur., 2016).

Recentna istraživanja su pokazala da učenici postižu znatno manji napredak u zadatku „definicije znaka jednakosti“ u usporedbi s njihovim napretkom u zadatcima „kodiranje jednadžbi“, „točno/netočno numeričke rečenice“ ili „otvorene numeričke rečenice“. Štoviše, njihov apsolutni napredak u zadatku“‘definicije znaka jednakosti“ također ostaje nizak (Davenport i sur., 2023; McNeil, Hornburg, Brletic-Shipley i sur., 2019). Imajući ovo na umu, kategorizirali smo pregled intervencija u tri odjeljka: (a) intervencije koje značajno poboljšavaju i proceduralno i konceptualno znanje, uz relativno veliki napredak u zadatku „definicije znaka jednakosti“, (b) intervencije koje su potaknule proceduralno i konceptualno znanje učenika, iako s umjerenim napretkom u zadatku „definicije znaka jednakosti“ i (c) one koje nisu procjenjivale definicije znaka jednakosti učenika.

Prije nego što detaljno opišemo ove studije, prvo ćemo se fokusirati na one iz posljednjega desetljeća, sažimajući njihove mjerne instrumente i teorijske okvire u Tablici 2. Kao što je prikazano u Tablici 2, recentne intervencijske studije obično koriste četiri mjerna elementa: točno/netočno numeričke rečenice, definicije znaka jednakosti, otvorene numeričke rečenice i kodiranje jednadžbi. Njihovi teorijski okviri uglavnom uključuju teoriju utjelovljene spoznaje, teoriju otpornosti na promjene i fizičke ili konkretne reprezentacije.

Tablica 2
Pregled intervencija/eksperimenata provedenih u prošlom desestljeću

Referenca	Intervencija	Sudionik	Teorijsko usmjerenje	Mjerenje
Studije s relativno velikim poboljšanjem u definicijama znaka jednakosti (više od 50 % sudionika)				
Chow i Wehby (2019)	Nesimbolička nastava	Drugi razred	Vizualna podrška	ONS, DES, TFNS
Donovan i sur. (2022)	Nastava s naglaskom na istovjetnost i supstituciju	Četvrti i peti razred	I istovjetnost i zamjenska komponenta	ONS, DES
Studije s relativno niskim poboljšanjem definicije znakova jednakosti (manje od 50 % sudionika)				
Bajwa and Perry (2021)	*Pan-balance* vaga	Drugi i treći razred	Zasnovane i idealizirane reprezentacije (Belenky i Schalk, 2014)	ONS, DES, TFNS
McNeil i sur. (2015)	Netradicionalna aritmetička praksa	Učenici drugoga razreda	Otpornost na promjene	ONS, DES, EE
McNeil i sur. (2019)	(a) Uvođenje znaka jednakosti prije aritmetike, (b) Vježbe smanjenja konkretizacije i (c) usporedba i objašnjenje.	Učenici drugoga razreda	Otpornost na promjene	ONS, DES, EE
Davenport et al. (2023)	(a) Uvođenje jednakosti prije aritmetike, (b) Netradicionalna aritmetička praksa, (c) Vježbe smanjenja konkretizacije, i (d) usporedba i objašnjenje.	Učenici drugoga razreda	Otpornost na promjene	ONS, DES, EE
Studije koje nisu testirale definicije znakova jednakosti među učenicima				
Stephens i sur. (2015)	Sveobuhvatan kurikul ranoga algebarskog obrazovanja	Učenici trećega razreda	Okvir istraživanja kurikula (Clements, 2007)	ONS, TFNS
Blanton i sur. (2019)	Sveobuhvatan kurikul ranoga algebarskog obrazovanja	Učenici od 3. do 5. razreda	Okvir istraživanja kurikula	ONS, TFNS
Fyfe i sur. (2022)	Uključivanje refleksivnih metakognitivnih pitanja	Učenici prvoga i drugoga razreda	Važnost metakognicije u matematici	ONS
Fischer i sur. (2019)	ACE (Aritmetičko razumijevanje u osnovnoj školi)	Učenici drugog razreda	Otpornost na promjene	ONS

Referenca	Intervencija	Sudionik	Teorijsko usmjerenje	Mjerenje
Koumoutsakis i sur. (2016)	Promatranje geste, snimljena vs. uživo nastava	Učenici trećega i četvrtoga razreda	Teorija utemeljene spoznaje	ONS
Valdiviejas i sur. (2022)	Promatranje geste	Učenici drugoga razreda	Teorija utemeljene spoznaje	ONS
Despina i sur. (2024)	Promatranje reprezentativne geste	Učenici trećega razreda	Teorija utemeljene spoznaje	ONS, TFNS
Novack i sur. (2014)	Proizvodnja geste	Učenici trećega razreda	Teorija utemeljene spoznaje	ONS
Kersey i sur. (2024)	Proizvodnja gesta i akcije	Učenici trećega i četvrtoga razreda	Teorija utemeljene spoznaje	ONS

Skraćenice: ONS: otvorene brojčane rečenice; TFNS: istinite i lažne brojčane rečenice; DES: definicije znaka jednakosti; EE: kodiranje jednadžbi.

Studije s relativno visokim poboljšanjem u definicijama simbola jednakosti

U ovim studijama otkriveno je da su studenti pokazali poboljšanje u proceduralnom i konceptualnom znanju nakon intervencije, pri čemu je većina (preko 50 %) bila sposobna pružiti relacijsku definiciju za znak jednakosti nakon intervencije. Na primjer, Hattikudur i Alibali (2010) otkrili su da su učenici trećega i četvrtoga razreda koji su primili uputu uspoređujući znak jednakosti sa simbolima nejednakosti bili skloniji pružiti relacijsku definiciju (74 %) u usporedbi s onima koji su primili samo uputu o znaku jednakosti (57 %) ili uopće nisu imali uputu (22 %). Na predtestu, 38 % svih učenika definiralo je znak jednakosti relacijski. Poboljšanje u definiranju znaka jednakosti (što odgovara zadatku definicija znaka jednakosti) varira ovisno o uvjetima, ali kodiranje i rješavanje matematičkih problema ekvivalencije (što odgovara zadatku kodiranja jednadžbi i otvorenim brojevnim rečenicama) nije.

Učenje koje kombinira rješavanje verbalnih problema i uputu o znaku jednakosti također je pokazalo slične učinke za učenike trećega razreda s matematičkim teškoćama, prema istraživanju Powella i Fuchsa (2010). Izvijestili su da je manje od 10 % sudionika pružilo relacijsku definiciju na predtestu. Međutim, nakon intervencije, gotovo svi učenici koji su primili kombiniranu instrukciju dali su relacijsku definiciju, dok su oni bez instrukcija o znaku jednakosti dali operativne definicije. Nisu uočene značajne razlike u poboljšanju rješavanja matematičkih problema ekvivalencije, ali je bilo značajnih varijacija u definiranju znaka jednakosti i procjeni jednakosnih rečenica (što odgovara zadatku točnih i netočnih jednadžbi“) među različitim uvjetima instrukcije.

Studija koju su proveli Donovan i sur. (2022b) razmatrala je zamjenski pogled i izvijestila da su učenici četvrtih i petih razreda koji su primili lekcije o dvojnoj koncepciji jednakosti i zamjene, samo jednakost, te bez koncepcije pokazali stopu promjena od 50 % do 68 %, od 58 % do 77 % i od 62 % do 60 %, respektivno, od predtesta do posttesta u davanju definicija jednakosti. Važno je napomenuti da prethodno spomenute tri studije nisu pružile longitudinalne podatke o tome je li se spomenuto poboljšanje razumijevanje definicija jednakosti kod djece održalo tijekom vremena.

U testu zadržavanja koji su proveli s učenicima na Srednjem zapadu, DeCaro i Rittle-Johnson (2012) pokazali su da su učenici drugoga do četvrtoga razreda koji su se susreli s nepoznatim problemima ekvivalencije prije nego što su primili konceptualnu nastavu o matematičkoj ekvivalenciji bili skloniji davanju relacijskih definicija (58 %) od učenika koji su prvo pohađali takvu nastavu (20 %) Dodatno, rezultati konceptualnoga znanja (definiranje znaka jednakosti i kodiranje problema ekvivalencije) bili su bolji na testu zadržavanja nego na posttestu Nisu zabilježeni efekti reda na proceduralno znanje, ali su uočeni značajni efekti na konceptualno znanje.

Također, Chow i Wehby (2019) pokazali su da su na procjeni nakon intervencije učenici drugog razreda koji su primili simboličku intervenciju, nesimboličku intervenciju i uobičajenu nastavu dali relacijske definicije s predviđenim vjerojatnostima od 75,3 %, 58,4 % i 5 %, redom. Međutim, na početnom testu, ovi su učenici imali 10,5 %, 6,2 % i 6,8 %, redom. Intervencije su pokazale značajne veličine efekata u odnosu na kontrolnu skupinu u definiranju znaka jednakosti, ocjenjivanju rečenica o jednakosti i rješavanju problema ekvivalencije, ali ne i za same intervencije. Ipak, ova studija, zajedno s prethodno spomenutim dvjema studijama (DeCaro i Rittle-Johnson, 2012; Hattikudur i Alibali, 2010), provela je intervenciju s jednim učenikom.

Ukratko, čini se da su ove intervencije učinkovite metode za poboljšanje proceduralnoga i konceptualnoga znanja o znaku jednakosti, osobito u poticanju učenika da daju relacijske definicije. Ipak, potrebno je daljnje istraživanje kako bi se ispitalo mogu li se slični rezultati postići na razini učionice i proučilo jesu li ovi učinci intervencije privremeni ili dugotrajni. Ovo je važno jer se odnosi na stupanj u kojem intervencija poboljšava razumijevanje znaka jednakosti i njegovu praktičnu primjenu u stvarnim učioničkim uvjetima.

Studije s relativno malim poboljšanjem u definicijama znaka jednakosti

Takve studije pokazale su da su studenti pokazali veće proceduralno i konceptualno znanje nakon intervencije, ali poboljšanje u definiranju znaka jednakosti obično je bilo niže nego u drugim zadatcima, pri čemu je manje od polovice bilo sposobno proizvesti relacijske definicije nakon intervencija. Na primjer, McNeil i Alibali (2000) izvještavaju da su učenici trećih i četvrtih razreda povećali postotak za 1 1%, 6 % i 0 % u definiranju znaka jednakosti nakon što su upoznati s ciljevima učenja, izvedbe ili im ishodi učenja nisu bili poznati, tim redom. Međutim, oni koji su imali cilj učenja za kodiranje jednadžbi i prosuđivanje jednadžbi poboljšali su se za 53 % i 74 %. Promjene ukupnoga rezultata ovisile su o ciljevima koje su primili, pri čemu su oni kojima su ishodi učenja bili poznati pokazali najveće poboljšanje.

Izgrađujući na računalnoj teoriji otpora promjenama i integrirajući četiri istraživačke strategije, McNeil i kolege, kroz niz studija, predložili su dodatnu, pristupačnu i prijenosnu sveobuhvatnu intervenciju pod nazivom „Poboljšanje razumijevanja jednakosti kod djece" (Improving Children's Understanding of Equivalence, ICUE), s ciljem postizanja razumijevanja matematičke jednakosti na razini majstorstva za učenike drugog razreda (Davenport i sur., 2023; Fyfe i sur., 2015; McNeil i sur., 2012; McNeil i sur., 2015; McNeil i sur., 2011; McNeil, Hornburg, Brletic-Shipley i sur., 2019). Sveobuhvatna intervencija imala je tri osnovna cilja: (1) premašiti njihove „najbolje slučajeve" (McNeil i sur., 2015), (2) osigurati da svi učenici pokažu osnovno razumijevanje i (3) da najmanje 50 % postigne majstorstvo. Ipak, McNeil i sur. (2019) priznaju da nisu postigli cilj od 100 % osnovnoga razumijevanja i 50 % majstorstva, uglavnom su zaostali u definiranju znaka jednakosti relacijski. Iako je 45 % sudionika postiglo majstorstvo u rješavanju i kodiranju matematičkih problema jednakosti, samo 38 % definiralo je znak jednakosti relacijski.

Ukratko, to sugerira da su ove intervencije vjerojatno učinkovitije u poboljšanju sposobnosti učenika da procijene jednadžbe, kodiraju i rješavaju matematičke jednadžbe jednakosti, no čini se da su manje učinkovite u poboljšanju proizvodnje relacijskih definicija. U ovom pregledu daju se dva moguća objašnjenja ovoga fenomena.

Prvo, ove su intervencije uključivale opsežne proceduralne lekcije koje mogu izravno poboljšati proceduralne vještine bez osiguravanja konceptualnoga razumijevanja (McNeil i Alibali, 2000). Na primjer, sudionici u istraživanju koje su proveli McNeil i Alibali (2000) naučili su postupak za rješavanje problema ekvivalentnosti, a aritmetička praksa bila je ključna komponenta ICUE-a. S druge strane, intervencija Hattikudur i Alibali (2010) bila je usmjerena na konceptualno poučavanje, što je dovelo do većega poboljšanja u definiranju jednakosti nego u kodiranju ili rješavanju problema. Stoga se može zaključiti da intervencije koje naglašavaju konceptualne ili proceduralne aspekte mogu učinkovitije poboljšati odgovarajuće znanje. Ovaj argument djelomično podržava Rittle-Johnson i Alibali (1999), koji su primijetili da, iako postoje uzročno-posljedične veze između konceptualnoga i proceduralnog aznanja, konceptualno znanje može imati veći utjecaj na proceduralno znanje nego obrnuto. Smatramo da je proceduralno znanje o jednakosti lakše poboljšati, dok je konceptualno znanje izazovnije. U budućem radu mogu se istražiti ova pitanja i razmotriti ravnoteža između konceptualnoga i proceduralnoga poučavanja u matematičkoj ekvivalentnosti.

Drugi, u skladu s Matthews i sur. (2012), zadatci koji zahtijevaju od učenika da definiraju pojmove mogu podcijeniti njihovo stvarno konceptualno znanje jer je davanje verbalnih ili pisanih definicija teže nego rješavanje problema poput zadataka ekvivalentnosti. Stoga nije jasno je li učenici nakon intervencija nisu uspjeli produbiti svoje razumijevanje znaka jednakosti ili su jednostavno imali poteškoća u izražavanju svojega razumijevanja.

Istraživanja bez testiranja definicija znaka jednakosti

Takve su studije pokazale poboljšanje u ocjenjivanju jednadžbi, kodiranju i rješavanju matematičkih problema ekvivalentnosti ili drugih zadataka nakon intervencija. Iako je većina tih studija bila uspješna u tim zadatcima, nisu bile bez ograničenja.

Neke intervencije često zahtijevaju visoko osposbljene i motivirane učitelje i istraživače (Fyfe i sur., 2022), značajno vrijeme za implementaciju (Fischer i sur., 2019, ili oboje (Blanton i sur., 2019; Saenz-Ludlow i Walgamuth, 1998; Stephens i sur., 2015). Na primjer, Blanton i sur. (2019) uključili su učitelje iz različitih škola, pružajući tri godine profesionalnoga razvoja (PD) kako bi podržali implementaciju. Slično tome, Fyfe i sur. (2022) proveli su intervenciju i mjerenje pomoću istraživačkih pomoćnika i istraživača. Stephens i sur. (2015) zamijenili su redovite sate matematike s 20 sati nastave ranoga algebarskog obrazovanja tijekom školske godine.

Iako nisu bile vremenski opsežne, neke studije nisu procijenjivale dugoročne učinke (Alibali i sur., 2009; Perry, 1991) niti učinke u učioničkom okruženju (Alibali, 1999; Alibali i sur., 2009). Na primjer, Alibali i suradnici (2009) proveli su posttest i testove prijenosa te proveli intervenciju s jednim učenikom. Neke studije koristile su modernu informacijsku tehnologiju, što je ograničilo mogućnost uopćavanja i ponavljanja njihovih rezultata. Na primjer, Bajwa i Perry (2021) stvorili su digitalnu *Pan Balance* aplikaciju kako bi upravljali uvjetima poduke te proveli eksperiment u kontekstu računalne igre. Niz studija pokazao je da uključivanje gesti u govornu poduku o pojmovima matematičke jednakosti doprinosti učeničkom odnosnom razumijevanju znaka jednakosti (Congdon I sur., 2017; Kersey i sur., 2024; Valdiviejas i sur., 2022). Mnoge od tih studija, poput one Kerseya i suradnika (2024), fokusiraju se na rješavanje problema jednakosti bez provjeravanja konceptualnoga znanja, što ne daje odgovor na pitanje doprinose li geste konceptualnom razumijevanju.

Općenito, važno je napomenuti da intervencije o kojima se raspravljalo nisu univerzalno korisne za svakog sudionika, pa čak ni nastava jedan-na-jedan nije iznimka (Hattikudur i Alibali, 2010). Možda će biti potrebno nastaviti istraživati teorijske okvire ili poboljšavati eksperimentalne metode koje mogu voditi do učinkovitijih intervencija. Ovaj pregled prikazuje tri razmatranja na koja bi buduća istraživanja mogla obratiti pažnju.

Prvo, pretpostavljamo da prevencija formiranja i smanjenje aktivacije dječjih operativnih prikaza razlikuju se po težini. Smanjenje oslanjanja na operativni prikaz može biti teže od sprječavanja njegovoga formiranja jer su mnogi učenici duboko ukorijenjeni u njemu (McNeil i sur., 2015; Seidenberg, 2005). Stoga intervencije moraju razjasniti žele li spriječiti formiranje ili smanjiti aktivaciju dječjih operativnih prikaza. Prema teoriji otpornosti na promjenu, operativni prikazi se jačaju tijekom rane formalne škole, dosežući vrhunac oko devete godine (McNeil, 2007). Mnoge studije fokusiraju se na djecu u dobi od 6 do 9 godina, dok druge uključuju stariju djecu (Tablica 2). Kao što je ranije spomenuto, mnogi u tim dobnim skupinama već posjeduju operativno razumijevanje, pa rezultati često odražavaju učinkovitost u obeshrabrivanju aktivacije, a ne formiranja. Buduća istraživanja trebaju prethodno testirati oslanjanje učenika na operativni prikaz, dizajnirati upute koje uvode znak jednakosti prije njegova formiranja i procijeniti njihovu učinkovitost.

Drugo, relacijsko razumijevanje znaka jednakosti uključuje više komponenata, poput aspekata jednakosti i supstitutivnosti (Jones i sur., 2012). Donovan i sur. (2022a) otkrili su da učenje o objema komponentama – jednakosti i supstitutivnosti – zajedno može biti učinkovito. Ovo postavlja pitanje ovisi li razumijevanje jedne komponente o drugoj ili bi obje trebale biti učene istovremeno zbog njihove međuzavisnosti. Trenutačno, to ostaje nejasno, a nedostaju teorijski okviri koji objašnjavaju mehanizme učenja ili kognitivne procese iza znaka jednakosti. Kao rezultat, dizajni nastave temeljeni na razvojnom putu razumijevanja znaka jednakosti još su uvijek ograničeni.

Treće, Tablica 2 pokazuje da, iako alati za mjerenje koje koriste istraživači spadaju u četiri kategorije—definicije znaka jednakosti, jednadžbe s jednom nepoznanicom, kodiranje jednadžbi itočne i netočne jednadžbe—oni nisu uniformni. Neka istraživanja uključuju određene stavke, dok druga ne. Buduća istraživanja mogla bi usvojiti ujedinjenu mjernu skalu. Na primjer, Matthews i sur. (2012) razvili su alat za procjenu razina razumijevanja znaka jednakosti, ali su izostavili supstitutivni pogled, što bi mogla biti potencijalna tema za buduća istraživanja.

Kroskulturna istraživanja o razumijevanju znaka jednakosti

U ovom odjeljku daje se pregled međunarodne komparativne literature o učeničkom razumijevanju znaka jednakosti kako bi istražile razlike u načinu na koji učenici iz različitih kulturnih pozadina razumiju znak jednakosti te identificirali faktori koji doprinose tim razlikama. Zemlje uključene u pregled temelje se na prikupljenim člancima.

Komparativne studije otkrivaju da učenici u određenim zemljama imaju manje problema sa znakom jednakosti nego drugi (Simsek i sur., 2022). Na primjer, Capraro i kolege izvijestili su da je oko 28 % američkih učenika šestih razreda (n = 105) i 98 % kineskih učenika šestih razreda (n = 145) točno riješilo jednadžbe s jednom nepoznanicom i točne i netočne jednadžbe s konceptualno točnim objašnjenjima (Capraro i sur., 2007; Capraro i sur., 2010; Li i sur., 2008). Slična usporedba uključivala je učenike šestih razreda iz Južne Koreje (n = 193) i Turske (n = 334), pri čemu je 59,6 % korejskih i 28,4 % turskih učenika odgovorilo točno (Capraro i sur., 2010). Eichhorn i sur. (2018) uključili su Jordan (2.-3. razred, n = 1486) i Indiju (2. razred, n = 185), otkrivši da obje skupine imaju problema s otvorenim brojčanim rečenicama.

Madej (2022) je usporedio švedske i južnokorejske učenike koristeći ocjenu Matthews i sur. (2012), primjećujući da su švedski učenici 3. i 6. razreda bili bolji od južnokorejskih učenika 3. razreda u razumijevanju neoperacijskoga značenja znaka jednakosti. Jones i sur. (2012) usporedili su kineske (n = 150) i britanske (n = 101) srednjoškolce u pogledu substitutivnoga razumijevanja, otkrivši da je britanska skupina ocijenila operacijske definicije višim ocjenama i substitutivno-relacijske definicije nižim, dok je kineska skupina imala višu ocjenu za relacijske definicije jednakosti.

Istraživači su istraživali faktore koji stoje iza razlika u razumijevanju znaka jednakosti među zemljama, s fokusom na nastavne materijale. Analize učiteljskih vodiča i udžbenika u SAD-u i Kini otkrile su da materijali u SAD-u često nemaju jasno definirani znak

Tablica 3

Zemlje	Faza jednakosti	Faza aritmetike	opisi	povezani konteksti
Kina	1. do 2. razred	1. do 2. razred	Razumjeti značenja simbola <, =, >, i usporediti veličine brojeva unutar deset tisuća; uspoređivanjem veličina brojeva steći razumijevanje odnosa jednakosti i nejednakosti	U konkretnim situacijama koje uključuju jednake, veće i manje količine, vodite učenike da percipiraju odnose jednakosti i nejednakosti između brojeva
Engleska	1. razred	1. razred	Učenici bi trebali: čitati, pisati i interpretirati matematičke iskaze koji uključuju znakove za zbrajanje (+), oduzimanje (-) i znak jednakosti (=)	Riješite jednostavne probleme koji uključuju zbrajanje i oduzimanje, koristeći konkretne predmete i slikovne prikaze te probleme s nedostajućim brojem, poput 7 = __– 9
Indija	6. do 8. razred	3. do 5. razred	Razumije jednakost između numeričkih izraza i uči provjeravati aritmetičke jednadžbe	N/A
Jordan	1. razred	1. razred	Usporedba pomoću simbola: =, >, <	Usporedba brojeva korištenjem simbola <, >, =
Novi Zeland	4. razred	1. razred	Zabilježiti i interpretirati aditivne i jednostavne multiplicativne strategije, koristeći riječi, dijagrame i simbole, uz razumijevanje jednakosti	Jednadžbe i izrazi
Južna Koreja	1. do 2. razred	1. do 2. razred	Istražiti odnos između dvije količine i prikazati ga pomoću jednakosti	Pomozite im da razumiju da su količine s obje strane znaka jednakosti (=) jednake jedna drugoj u zbrajanju, oduzimanju i množenju
Švedski	1. do 3. razred	1. do 3. razred	Matematičke sličnosti i značenje znaka jednakosti	Kada učenici rade sa značenjem znaka jednakosti, važno je usporediti ono što je jednako s onim što nije jednako. Značenje znaka jednakosti može se učiniti vidljivim stručnim istraživanjem onoga što ne čini sličnost, što čini sličnost i kako se razlike mogu pretvoriti u sličnosti. Simboli veće od i manje od mogu se uvesti istovremeno s znakom jednakosti. Također je korisno raditi s matematičkim sličnostima i razlikama te simbolima za njih u praktičnim vježbama
Turska	1. razred	1. razred	Uvodi se simbol za operaciju zbrajanja (+) i znak jednakosti (=), a njihovo se značenje naglašava	Operacija zbrajanja izvodi se i horizontalno i vertikalno. Pri izvođenju vertikalnoga zbrajanja, naglašava se važnost toga da operacijska linija ima slično značenje kao znak jednakosti
Sjedinjene Države	1. razred	vrtić	Razumijevanje značenja znaka jednakosti i određivanje jesu li jednadžbe koje uključuju zbrajanje i oduzimanje točne ili netočne. Na primjer, koje su od sljedećih jednadžbi točne, a koje netočne? 6 = 6, 7 = 8-1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2	Odredi nepoznati cijeli broj u jednadžbi zbrajanja ili oduzimanja koja sadrži tri cijela broja. Na primjer, odredi nepoznati broj koji jednadžbu čini točnom u svakoj od sljedećih jednadžbi: 8 + _= 11, 5 = _ - 3, 6 + 6 = _

jednakosti, koristeći pojmove poput „čini" i predstavljajući uglavnom tradicionalne aritmetičke rečenice. S druge strane, kineski materijali naglašavaju jednakost kao relacijski pojam, uvode znak jednakosti uz simbole nejednakosti (npr. >, <) i uključuju netradicionalne jednadžbe rano (Li i sur., 2008). Ova strategija usklađena je s teorijom varijacije kojom se tvrdi da usporedba pojma s njegovim suprotnostima (npr. jednakost vs. nejednakost) produbljuje konceptualno razumijevanje kontrastiranjem razlika između njih i identificiranjem ključnih karakteristika ciljanoga koncepta (Kullberg i sur, 2017). Slično tome, udžbenici u Jordanu, Indiji i Južnoj Koreji uglavnom koriste tradicionalne aritmetičke formate, dok švedski udžbenici za rane razrede preferiraju jednadžbe s jednom nepoznanicom (Eichhorn i sur., 2018; Madej, 2022). Ovi nalazi sugeriraju da definicije, tumačenja i formati prezentacije znaka jednakosti mogu oblikovati razumijevanje učenika. Međutim, ovi nisu rezultati temeljeni na eksperimentalnim nalazima.

Simsek i sur. (2022) proveli su veliko istraživanje u šest zemalja (Kina, Engleska, Novi Zeland, Južna Koreja, Turska i Sjedinjene Američke Države), otkrivajući da format aritmetičke prakse u trenutačnim udžbenicima za učenike nije bio povezan s njihovim razumijevanjem matematičke ekvivalencije. Ovaj rezultat bio je iznenađujuć jer je proturječio predviđanjima iz teorije otpornosti na promjene, koja je prvenstveno proučavana u SAD-u. To sugerira da faktori u učioničkim materijalima koji utječu na razumijevanje znaka jednakosti zahtijevaju daljnje istraživanje.

Ukratko, iako nesporazum oko znaka jednakosti čini se rasprostranjenim u više zemalja, istraživanja pokazuju kulturne razlike u njegovom razumijevanju. Međutim, ostaju pitanja o ulozi učioničkih materijala u oblikovanju razumijevanja učenika (Simsek i sur., 2022). Ovo je ključno za dizajniranje učinkovitih intervencija i objašnjenje njihovoga različitog uspjeha. Buduća istraživanja trebala bi istražiti dublje faktore koji utječu na ovo razumijevanje. Dodatno, nismo pronašli istraživanja o provedenom u odnosu na planirani kurikul za matematičku ekvivalenciju, što je područje za buduće istraživanje.

Ovaj pregled prikazuje neke aspekte znaka jednakosti u trenutačnim kurikulskim standardima osnovnih škola devet navedenih zemalja, kako je prikazano u Tablici 3. Uključili smo razred u kojem se znak jednakosti prvi put uvodi, njegov opis, povezani kontekst i fazu u kojoj se uvode aritmetički problemi. Cilj nam je bio pružiti uvide za buduća istraživanja koja će istraživati faktore koji utječu na razumijevanje znaka jednakosti u nastavnim materijalima, pa nismo proveli detaljno, standardizirano uspoređivanje tih kurikula jer to izlazi iz okvira našega fokusa.

Zaključci

Cilj je ovoga pregleda: (a) sažeti razumijevanje simbola jednakosti među učenicima različitih dobnih skupina i različite razine razumijevanja koje oni pokazuju; (b) pružiti pregled čimbenika koji utječu na razumijevanje simbola jednakosti među učenicima i učinkovitosti intervencija dizajniranih za poboljšanje njihova razumijevanja; (c) prikazati razlike u razumijevanju simbola jednakosti među učenicima iz različitih kulturnih konteksta i čimbenika koji dovode do tih razlika te (d) identificirati istraživačke praznine i sugerirati smjerove za buduća istraživanja. Pregled je pokazao da: (a) Učenici u osnovnoj školi često imaju poteškoće u razumijevanju simbola jednakosti. Ova poteškoća očituje se u tome što učenici simbol jednakosti vide operativno, a ne relacijski. I u višim razredima, iako više učenika razvija relacijski pogled, neki zadržavaju operativni pogled čak i do fakulteta. Mnogi učenici u vrtiću već pokazuju sklonosti prema operativnom razumijevanju; (b) Učenici mogu imati različite razine razumijevanja simbola jednakosti: rigidno operativno (operativni pogled), fleksibilno operativno, osnovno relacijsko (relacijski pogled), komparativno relacijsko i substitutivni pogled; (c) Matematička iskustva učenika igraju glavnu ulogu u njihovom razumijevanju simbola jednakosti. Način na koji je simbol jednakosti definiran i predstavljen može biti obećavajući čimbenik utjecaja; (d) Većina intervencija može poboljšati proceduralno znanje simbola jednakosti, ali se konceptualno znanje, osobito definicije simbola jednakosti, razlikuje. Neka istraživanja su poboljšala generiranje relacijske definicije simbola jednakosti za većinu sudionika (više od 50 %), dok su neka samo za nekoliko (ispod 50 %), a neka nisu testirala definiranje ovoga pojma. Nedavne intervencijske studije uglavnom koriste četiri mjernih stavki: točne i netočne jednadžbe, definicije simbola jednakosti, jednadžbe s jednom nepoznanicom i kodiranje jednadžbi. Teorijski okviri koji vode ove intervencije uključuju teoriju utjelovljenog kognitivizma, otpor prema promjenama i fizičke ili konkretne reprezentacije te (e) Učenici iz različitih kultura različito razumiju simbol jednakosti, vjerojatno zbog razlika u tome kako kurikulski standardi i udžbenici obrađuju simbol jednakosti.

Trenutačna istraživanja o kognitivnom razvoju simbola jednakosti relativno su sveobuhvatna; međutim, nekoliko pitanja zaslužuje daljnje istraživanje. To uključuje: (a) više se fokusirati na razumijevanje simbola jednakosti među srednjoškolcima i studentima; (b) Razvojni put razumijevanja simbola jednakosti ostaje nesiguran; (c) Imaju li definicija ili prezentacija simbola jednakosti veći utjecaj na razumijevanje simbola jednakosti među učeniciima, još uvijek zahtijeva daljnje istraživanje. Također, nisu poznati drugi ključni čimbenici koji utječu na ovo razumijevanje; (d) Nedostaju intervencije koje mogu intenzivno i robusno poboljšati i konceptualno i proceduralno znanje učenika o simbolu jednakosti, a koje se mogu lako implementirati u nastavi. Osobito nedostaju prenosive intervencije koje mogu snažno poboljšati sposobnost učenika da generiraju relacijsku definiciju simbola jednakosti i (e) Otvorena su pitanja o tome uzrokuje li format izraza simbola jednakosti razlike u razumijevanju djece u različitim zemljama. Razloge za ove razlike potrebno je dalje istraživati, a istraživanje provedene naspram zamišljene kurikulske nastave za matematičku ekvivalenciju trenutačno nedostaje.

Ovaj pregled pruža nekoliko prijedloga za buduća istraživanja u vezi s nastavnim intervencijama: (a) razmotriti razlikovanje i odgovarajuće raspoređivanje konceptualnoga i proceduralnoga poučavanja simbola jednakosti; (b) razviti teorije o razvojnim putovima različitih razina razumijevanja simbola jednakosti ili teorije o kognitivnim procesima ili mehanizmima koji su uključeni u razumijevanje simbola jednakosti; (c) razjasniti imaju li intervencije cilj smanjiti ili spriječiti operativno razumijevanje, istražujući hoće li zajednička nastava simbola jednakosti i nejednakosti u ranom učenju

– kao što je provedeno u nekim nestandardnim praksama – pomaknuti učenike izvan operativnog stava. Ovaj pristup, zasnovan na teoriji varijacije (Bussey i sur., 2013), mogao bi se testirati u kontekstima u kojima su znakovi nejednakosti tradicionalno odgođeni, nudeći uvid u prenosivost međukulturnih nalaza i (d) razviti jedinstveni mjerni instrument za procjenu razumijevanja simbola jednakosti.

Implikacije

Teorijske implikacije ovoga istraživanja su: (a) Produbljivanje kognitivne teorije: Analizom razumijevanja znaka jednakosti među učenicima različite dobi i razina razumijevanja, poboljšava se naše razumijevanje kognitivnoga razvoja u učenju matematičkih simbola. Također sugerira smjerove za nove kognitivne teorije, pomažući u objašnjenju procesa mišljenja učenika i prepreka u učenju matematičkih simbola;(b) Obogaćivanje teorije matematičkoga obrazovanja: Pregledom četiriju područja (razumijevanje znaka jednakosti učenika, čimbenici koji utječu, intervencije i međukulturne usporedbe), pomaže u razvoju sustavnijega istraživačkog okvira za matematičko obrazovanje; (c) Pregledom međunarodnih istraživanja: pruža se pregled kako se razumijevanje znaka jednakosti učenika razlikuje u različitim kulturnim okvirima i utjecaj kulturnih čimbenika na učenje matematike. Time se doprinosi razvoju međukulturne teorije matematičkoga obrazovanja.

Praktične implikacije ove studije su (a) Personalizirano poučavanje: Razmatranjem razumijevanja različitih dobnih skupina i različitih razina razumijevanja, pomaže u zadovoljenju različitih obrazovnih potreba učenika; (b) Poboljšanje metoda poučavanja: Razmatranjem čimbenika koji utječu i intervencija, pomaže učiteljima u dizajniranju učinkovitijih instruktivnih metoda, poput uravnoteženja proceduralnoga i konceptualnoga poučavanja jednakosti kako bi se poboljšalo razumijevanje njezinoga značenja i primjene; (c) Obuka učitelja: Rezultati pregleda mogu se koristiti za obuku učitelja, poboljšavajući njihovo razumijevanje kognitivnoga razvoja učenika u vezi s razumijevanjem jednakosti; (d) Dizajn kurikula: Rezultati pregleda pružaju uvid u optimizaciju sadržaja i strukture matematičkih kurikula kako bi učenici mogli potpuno razumjeti jednakost i njezinu primjenu. Na primjer, kurikuli bi mogli integrirati simboličke znakove jednakosti i nejednakosti već u najranijim razredima. Poučavanjem ovih simbola u kontrastu, učitelji mogu pomoći učenicima da razjasne relacijsko značenje simbola jednakosti u odnosu na operativno razumijevanje, što je hipoteza koja zaslužuje empirijsko testiranje u budućim intervencijama.