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EVALUATION OF BERNOULLI PRODUCTION LINES WITH PRODUCT QUALITY INSPECTION AND REWORK STATIONS: ANALYTICAL SOLUTION AND THE FINITE STATE METHOD

Summary

Mathematical modelling of production systems is a central task of production system engineering, especially when considering the competitiveness and effectiveness of production companies aiming at quality management and customer-oriented production. Two different approaches to the modelling of a Bernoulli production line with product quality inspection and rework stations are presented in this study. In the first case, analytical modelling is introduced based on Markov chains and the formulation of constitutive transition matrices. In the second case, an approximative approach is presented by employing the finite state method and formulating new finite state elements. The validation of the approximative approach is performed based on theoretical production lines, and the main findings are employed in a shipyard's plate prefabrication line. The issues of computing requirements and quality-quantity coupling are also discussed, indicating the need for further research in the context of improvability analyses and the design of production systems.

Key words: *production system engineering; Bernoulli production line; steady-state response; analytical solution; finite state method*

1. Introduction

Product quality may be defined as a compliance measure concerning customer requirements, manufacturing standards, and product functionality. Thus, it directly impacts companies' profitability on one hand and productivity on the other. In the first case, a high-quality product policy may positively affect customer markets and lead to an increase in investment returns, revenue, and competitiveness. In the latter case, compliance with manufacturing and functionality standards may boost productivity and reduce rework requirements, resulting in lower production costs, improved due time performance, and an increase in finished goods inventory [1]. However, some published reports suggest that the implementation of quality management strategies yields poor or limited improvements compared to the no-quality management approach [2]. This indicates a pressing need to research the relationship between quality and productivity more profoundly by employing a systematic modelling approach.

From a historical perspective, product quality management has evolved remarkably, starting with productivity-oriented mass production and ending with new concepts balancing quality and productivity requirements in the context of sustainable and customer-oriented production [3]. Systematic quality management emerged during the 1950s in Japan and has gradually expanded to the rest of the world to become one of the most important aspects in the management of production systems, improving organisational effectiveness, competitiveness, and innovativeness. Thus, many quality management principles, practices, and techniques were discussed in a relevant research body, amounting to a surprising 224 different approaches to the problem. However, process management stands out as a key part of the quality management strategy as it directly increases the quality of outputs and improves the efficiency of resource consumption. Hence, improved performance of a production company can be achieved through the evaluation and continuous improvement of the production process [4] which, again, presupposes the application of suitable mathematical models.

The modelling of production systems with product quality inspection concerning productivity and potential improvements is an established research topic that was previously addressed by several research groups [5]. Usually, two different types of quality inspection setups are considered, one involving the scrapping of defective parts and the other including rework stations which are employed when defective parts are detected. In both cases, it is important to evaluate the key performance indicators as well as to identify bottlenecks to enable subsequent improvements of the existing production systems or the design of new ones. In some cases, such an approach includes the problem of machines with quality-quantity coupling as well. Upon setup, the issue is usually evaluated by employing a modified aggregation procedure, including validation of the obtained key performance indicators concerning the results of numerical simulations [6]. The main motivation to employ numerical simulations as a validation ground is the lack of a rigorous analytical solution to the problem, as well as the nonexistence of formulations of transition matrices taking into account product quality inspections and rework stations. Although such a validation approach is understandable and rational, it can still lead to erroneous or biased conclusions. The same issue was pointed out earlier in the case of Bernoulli production lines when a consistent algorithm to formulate transition matrices was introduced [7]. This was followed by the introduction of a new and analytically based finite state method as an efficient procedure for evaluating key performance indicators and for the design of production systems [8].

Hence, the main purpose of this paper is to develop and present an analytical solution for evaluating key performance indicators in Bernoulli production lines incorporating quality checking and rework stations. In addition, new finite state elements will be introduced, validated, and employed to cope with well-known issues of the state-space scale system and the significant computing requirements associated with realistic industrial applications. It is expected that the results of this study will enable further validation and development of approximative approaches to the problem, namely the aggregation procedure and the finite state method, the introduction of more reliable improvements to existing production systems, and further development of the design procedures targeting optimal facility setups. Besides, these approaches will enable the implementation of conventional and new product quality policies based on quantified effects concerning key performance indicators, including the production rate, scrapping rate, repair rate, work-in-process, and the probabilities of blockage and starvation, among others.

The remainder of the paper is structured as follows: a brief literature review is presented in the next subsection. The mathematical modelling of the problem is presented in Section 2, including an analytical solution and the formulation of finite state elements, both in systems with the scrapping of defective products and rework stations. The developed framework is applied and validated in Section 2.4 in the case of theoretical production lines, while the main findings are discussed in Section 3. The application of the developed methodologies in a shipyard's prefabrication line is outlined in Section 4, and Section 5 summarises the main conclusions of the study and introduces prospects for further research.

1.1 Brief literature review

The modelling, analysis, and design of manufacturing systems are central topics in the body of production system engineering literature. In particular, two approaches to the problem, namely analytical and approximative ones, stand out. The analytical solution to the problem of the Bernoulli production line has been debated since 1962 when Sevast'yanov presented its integral form for a line composed of two machines and one intermediate buffer [9]. However, the analytical solution in its general form was not presented in the literature until recently when a transition matrix formulation employing constitutive matrices was developed for Bernoulli production lines of an arbitrary number of machines and buffers of arbitrary capacity. This enabled the analytical evaluation of different Bernoulli production systems, including serial [7, 10], splitting [11], and assembly [12] arrangements.

The long-term absence of an analytical solution prompted the introduction of approximative approaches to the problem. Thus, decomposition [13] and aggregation [14] procedures were presented and employed in a wide variety of system arrangements and industrial applications [5]. Similarly, the rather significant computing demands and the so-called curse-of-dimensionality issue associated with the analytical approach motivated the formulation of the finite state method as the third approximative approach [8, 15, 16]. Of these, the aggregation procedure has found the widest application both in academic research and industrial cases.

Production systems with product quality inspection features were previously considered in the relevant production system engineering literature, mainly within the framework of the decomposition or aggregation procedure. The most frequent issues addressed include performance evaluation, optimisation problems, production control, and quality and maintenance planning [17]. The performance of serial lines was first considered in the case of a two-machine line with a quality control device [18]. This was followed by the introduction of aggregation and decomposition procedures in long serial lines [19 - 22] as well as by considering associated bottleneck and design issues [6]. Further extensions of the developed theory included application in lines with rework stations [23, 24, 25], integrated failure-risk-quality analysis [26], flexible manufacturing systems with batch production [27], adaptive assembly systems [28, 29], economic optimisation of inspection policies for multi-station manufacturing systems [30], transient analysis with perishable products [31], joint optimisation and economic models of quality and maintenance policies [32, 33], and others.

In addition, the coupling of product quantity and quality has been addressed in recent publications. This research topic was first introduced by Inman in [34], addressing the possibility of a quality increase through production system design. This concept was further developed in several directions, including statistical process control and offline inspections combined with the Markov process [35, 36], system design concerning quality improvements [37], product sequencing in flexible manufacturing systems geared to product quality [38], quality improvability and the bottleneck sequence in flexible manufacturing [39], lead-time evaluation for multi-stage manufacturing of deteriorating products with target quality levels [40], and the Markov modelling of quality propagation [41]. A more profound literature review regarding the Markov modelling of manufacturing systems until 2019 is presented in [17].

Some of the most recent publications addressing product quality and the system productivity relationship include an analysis of the quality screening process by taking into account cost, operation time, and detection rates [42], optimisation of joint product quality and quantity decisions conditioned by consumer utility change [43], an integrated production-maintenance policy considering production time and cost and the quality levels of the output product [44], Quality 4.0 debates [45], and the introduction of new concepts such as Industry 4.0, Machine Learning and Artificial Intelligence [46 - 51].

Despite the critical role that mathematical modelling plays in performance evaluations of Bernoulli production lines coupled with quality inspection and rework stations, an associated analytical solution to the problem in its general form has not been presented. Its formulation is therefore the main contribution of this study. Besides this, the study includes the formulation and validation of new finite state elements employed within the finite state method to cope with quite significant computing requirements associated with realistic industrial applications. We expect that this contribution will enable further validation of the existing approximative approaches as well as the efficient integration of reliable mathematical models within Machine Learning and Artificial Intelligence frameworks.

2. Mathematical modelling

As previously outlined, the analytical and semi-analytical approaches to the evaluation of Bernoulli production lines with quality checking will be considered in this section. This will first be done in the case of serial lines with quality inspection and the scrapping of defective parts, and then in the more complex case of serial lines with rework stations. In both cases, we introduce the mathematical background, formulation of transition matrices, finite state elements, and finally theoretical examples enabling validation of both approaches and discussion of the main findings.

2.1 Bernoulli lines with quality inspection and scrapping of defective parts

Consider a serial production line composed of M machines m_i, i=1, 2, …, M, and M-1 buffers b_i, i=1, 2, …, M-1, Figure 1. The reliability of each machine equals p_i~*Bernoulli(pi),* while the capacity of the i-th buffer amounts to N_i, $N_i \in \mathbb{N}_0$. Apart from material processing, each machine performs a quality-checking procedure, identifying and scrapping defective parts at a rate q_i, i=1, 2, …, M. Additionally, assume that

- the production system is synchronous (all machines have identical cycle times and simultaneously start operations at the beginning of each cycle);
- the production system is homogeneous ($p_i \neq f(t)$ and $q_i \neq f(t)$);
- the status of each machine (up or down) is determined at the beginning of each cycle according to the reliability p_i;
- the status of each buffer is determined at the beginning of each cycle;
- the first machine is never starved (infinite capacity of the material source) and the last machine is never blocked (infinite capacity of the finished goods inventory).

The steady-state response of such a system can be mathematically represented as a discrete Markov chain over a d-dimensional system state space S given by the Cartesian product of buffer-levelled subspaces [52, 5, 53],

$$S = S_1 \times S_2 \times \ldots \times S_{M-1}, \tag{1}$$

where $S_i = \{0,1,2,\ldots,N_i\}$ is the i-th subspace and $d = \prod_{i=1}^{M-1} (N_i + 1)$. Given this, the associated steady-state probability distribution vector, $\{\pi\}$, equals

$$\begin{aligned} &\{\pi\} = [P]\{\pi\}, \\ &\sum_{i=1}^{d} \pi_i = 1, \end{aligned} \tag{2}$$

where $[P]$ is the transition matrix.

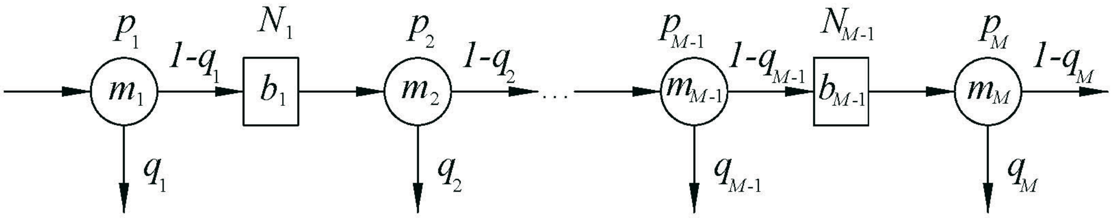


Fig. 1 Mathematical model of a serial Bernoulli production line with quality inspection and scrapping of defective parts

In the considered case, the transition matrix, $[P]$, can be formulated employing the separation of variables and the constitutive matrices approach [7], yielding

$$[P]=\prod_{i=1}^{M}[P_i]. \tag{3}$$

Thus, the first constitutive matrix, $[P_1]$, equals

$$[P_1]=\begin{cases}1-p_1+p_1q_1 \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,...,h_{M-1}^{n+1}=h_{M-1}^n, h_1^n<N_1,\\ p_1(1-q_1) \text{ if } h_1^{n+1}=h_1^n+1, h_2^{n+1}=h_2^n,...,h_{M-1}^{n+1}=h_{M-1}^n, h_1^n<N_1,\\ 1 \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,...,h_{M-1}^{n+1}=h_{M-1}^n, h_1^n=N_1,\\ 0, \text{ else,}\end{cases} \tag{4}$$

where h_i^n is the state of the i-th buffer in the cycle n and h_i^{n+1} is its state in the cycle $n+1$. Similarly, for $[P_i]$, i=2, 3, …, M-1 it follows that

$$[P_i]=\begin{cases}1-p_i \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,...,h_{M-1}^{n+1}=h_{M-1}^n, h_{i-1}^n>0, h_i^n<N_i,\\ p_i(1-q_i) \text{ if } ...h_{i-2}^{n+1}=h_{i-2}^n, h_{i-1}^{n+1}=h_{i-1}^n-1, h_i^{n+1}=h_i^n+1, h_{i+1}^{n+1}=h_{i+1}^n,...,h_{i-1}^n>0, h_i^n<N_i,\\ p_iq_i \text{ if } ...h_{i-2}^{n+1}=h_{i-2}^n, h_{i-1}^{n+1}=h_{i-1}^n-1, h_i^{n+1}=h_i^n, h_{i+1}^{n+1}=h_{i+1}^n,...,h_{i-1}^n>0, h_i^n<N_i,\\ 1 \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,...,h_{M-1}^{n+1}=h_{M-1}^n, h_{i-1}^n=0 \vee h_i^n=N_i,\\ 0, \text{ else.}\end{cases} \tag{5}$$

Finally, the constitutive matrix $[P_M]$ can be formulated in a somewhat simpler form as

$$[P_M]=\begin{cases}1-p_M \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,...,h_{M-1}^{n+1}=h_{M-1}^n, h_{M-1}^n>0,\\ p_M \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,...,h_{M-1}^{n+1}=h_{M-1}^n-1, h_{M-1}^n>0,\\ 1 \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,...,h_{M-1}^{n+1}=h_{M-1}^n, h_{M-1}^n=0,\\ 0, \text{ else.}\end{cases} \tag{6}$$

It can easily be seen that when q_i is set to 0 value, Eqs. (4)-(6) reduce to algorithms employed to formulate constitutive matrices of a serial Bernoulli production line without a quality checking feature [7]. Further application and validation of Eqs. (3)-(6) are presented in more detail in Appendix A in a serial Bernoulli line composed of three machines and two buffers with the capacity N_1=N_2=2.

Once the transition matrix is formulated, the analytical solution to the problem can be obtained by solving Eq. (2) as an eigenvalue problem. In that case, the first eigenvector of the transition matrix, associated with the first and also the largest eigenvalue $\Omega_1 = 1$, represents the steady-state probability distribution vector, $\{\pi\}$, with elements stating the probability of the occurrence of a particular system state, $\pi_{h_1 h_2 \dots h_{M-1}}$. In the simplest case of a production system composed of two machines and an intermediate buffer of capacity N, the steady-state probability distribution vector equals

$$\{\pi\} = \pi_0 \begin{Bmatrix} 1 \\ \dfrac{\pi_1}{\pi_0} = \dfrac{1}{1-p_2}\alpha \\ \dfrac{\pi_2}{\pi_0} = \dfrac{1}{1-p_2}\alpha^2 \\ \vdots \\ \dfrac{\pi_N}{\pi_0} = \dfrac{1}{1-p_2}\alpha^N \end{Bmatrix}, \quad \pi_0 = \frac{1-p_2}{1-p_2+\sum_{i=1}^{N}\alpha^i}, \quad \alpha = \frac{p_1(1-q_1)(1-p_2)}{p_2(1-p_1+p_1 q_1)}. \tag{7}$$

Note that, for shortness, $\pi_0 = \pi_{h_1=0 h_2=0 \dots h_{M-1}=0}$, $\pi_1 = \pi_{h_1=0 h_2=0 \dots h_{M-1}=1}$, $\pi_2 = \pi_{h_1=0 h_2=0 \dots h_{M-1}=2}$, …, $\pi_N = \pi_{h_1=N_1 h_2=N_2 \dots h_{M-1}=N_{M-1}}$. Again, when q_1 is set to $q_1 = 0$, Eq. (7) reduces to a well-known analytical solution in the case of a two-machine and one-buffer line without a quality-checking feature (e.g. [5]). Unfortunately, the closed-form analytical solution for $M > 2$ cannot be presented symbolically due to its complexity. However, as presented in the case of theoretical examples, it can be determined numerically.

Based on the steady-state probability distribution vector, $\{\pi\}$, a set of key performance indicators like the production rate, *PR*, scrapping rate, *SR*, work-in-process, *WIP*, and the probabilities of blockage and starvation, *BL* and *ST*, can be determined (e.g. [5]). Their quantification is of great importance because it allows for the reliable and measurable determination of the properties of the considered production systems. Thus, the production rate, *PR*, as the expected number of good parts produced by the last machine, m_M, can be defined as

$$\begin{aligned} PR &= P\left[\{m_M \text{ up}\} \cap \{\text{good product}\} \cap \{b_{M-1} \text{ not empty}\}\right] = \\ &= p_M (1-q_M)\left(1 - \sum_{h_1=0}^{N_1}\sum_{h_2=0}^{N_2}\dots\sum_{h_{M-2}=0}^{N_{M-2}} \pi_{h_1 h_2 \dots h_{M-2} h_{M-1}=0}\right). \end{aligned} \tag{8}$$

Similarly to the production rate, the scrapping rate, *SR*, as the expected number of defective parts scrapped at the machine m_M, equals

$$\begin{aligned} SR &= P\left[\{m_M \text{ up}\} \cap \{\text{defective product}\} \cap \{b_{M-1} \text{ not empty}\}\right] = \\ &= p_M q_M \left(1 - \sum_{h_1=0}^{N_1}\sum_{h_2=0}^{N_2}\dots\sum_{h_{M-2}=0}^{N_{M-2}} \pi_{h_1 h_2 \dots h_{M-2} h_{M-1}=0}\right). \end{aligned} \tag{9}$$

The rest of the performance measures, i.e. the work-in-process at the *i*-th buffer, WIP_i, the probability of blockage, BL_i, and the probability of starvation, ST_i, of the *i*-th machine remains the same as those developed for the serial Bernoulli production line without quality-checking procedures [7]. Thus, they will not be specified here.

2.2 Bernoulli lines with rework stations

Let us now consider a production line with a rework station, machines m_Q and m_R, according to Figure 2. The same assumptions defined in the case of a production line with quality inspection and scrapping of defective parts, Section 2.1, apply here as well. In this case, the constitutive matrices associated with machines $m_1, m_2, \ldots m_{Q-1}$ and machines $m_{Q+1}, m_{Q+2}, \ldots, m_{R-1}$ remain the same as those valid for ordinary serial Bernoulli production lines [7]. However, the constitutive matrices for the quality inspection and rework machines, m_Q and m_R, equal

$$\left[P_Q\right]=\begin{cases}1-p_Q \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,\ldots,h_{R-1}^{n+1}=h_{R-1}^n, h_{Q-1}^n>0, h_Q^n<N_Q, h_{R-1}^n<N_{R-1},\\ p_Q\left(1-q_Q\right) \text{ if } \ldots h_{Q-2}^{n+1}=h_{Q-2}^n, h_{Q-1}^{n+1}=h_{Q-1}^n-1, h_Q^{n+1}=h_Q^n+1, h_{Q+1}^{n+1}=h_{Q+1}^n,\ldots,h_{Q-1}^n>0, h_Q^n<N_Q,\\ p_Q q_Q \text{ if } \ldots h_{Q-2}^{n+1}=h_{Q-2}^n, h_{Q-1}^{n+1}=h_{Q-1}^n-1, h_{R-1}^{n+1}=h_{R-1}^n+1, h_Q^{n+1}=h_Q^n,\ldots,h_{Q-1}^n>0, h_{R-1}^n<N_{R-1},\\ 1-p_Q+p_Q q_Q \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,\ldots,h_{R-1}^{n+1}=h_{R-1}^n, h_{Q-1}^n>0, h_{R-1}^n=N_{R-1},\\ 1 \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,\ldots,h_{R-1}^{n+1}=h_{R-1}^n, h_{Q-1}^n=0 \vee h_Q^n=N_Q \vee h_{R-1}^n=N_{R-1},\\ 0, \text{ else,}\end{cases} \quad (10)$$

$$\left[P_R\right]=\begin{cases}1-p_R \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,\ldots,h_{R-1}^{n+1}=h_{R-1}^n, h_{Q-1}^n<N_{Q-1}, h_{R-1}^n>0,\\ p_R \text{ if } \ldots h_{Q-2}^{n+1}=h_{Q-2}^n, h_{Q-1}^{n+1}=h_{Q-1}^n+1, h_{R-1}^{n+1}=h_{R-1}^n-1, h_Q^{n+1}=h_Q^n,\ldots,h_{Q-1}^n<N_{Q-1}, h_{R-1}^n>0,\\ 1 \text{ if } h_1^{n+1}=h_1^n, h_2^{n+1}=h_2^n,\ldots,h_{R-1}^{n+1}=h_{R-1}^n, h_{Q-1}^n=N_{Q-1} \vee h_{R-1}^n=0,\\ 0, \text{ else.}\end{cases} \quad (11)$$

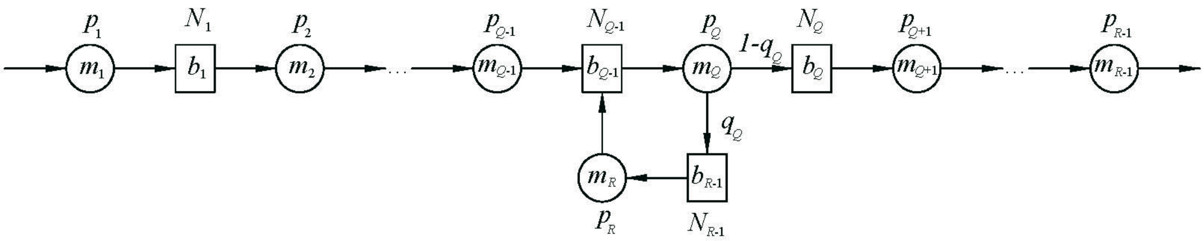


Fig. 2 Production line with rework station

Upon the formulation of the constitutive matrices, the transition matrix of the considered system follows from a relationship similar to that given in Eq. (3). Yet, the introduction of an additional machine m_R requires some modifications reflecting its presence and impact on relationships between system states. Thus, we differentiate between the two system subspaces $\tilde{S}$ and $\tilde{S}^C = S\setminus\tilde{S}$. Let $\tilde{S}$ be such that $\tilde{S}=S_1\times S_2\times\ldots S_{R-2}\times\left(h_{R-1}=0\right)$. In other words, $\tilde{S}$ is a subspace of all states for which a relationship $\tilde{S}_i=\left\{h_1,h_2,\ldots,h_{R-1},h_{R-1}=0\right\}$ holds. Hence, the column of the transition matrix, $\left\{P_{*,i}\right\}$, associated with the state $i\in\tilde{S}$, equals

$$\left\{P_{*,i}\right\}=\left\{\left[P\right]_{*,i}\right\}, \quad (12)$$

where $\left[P\right]$ is given in Eq. (3). In the case where $i\notin\tilde{S}$, the i-th column of the transition matrix, $\left\{P_{*,i}\right\}$, equals

$$\left\{P_{*,i}\right\}=\left\{\left[\hat{P}\right]_{*,i}\right\}, \quad (13)$$

where

$$\left[\hat{P}\right]=\left(\prod_{j=1}^{Q-1}\left[P_j\right]\right)\left[P_R\right]\left(\prod_{j=Q}^{R-1}\left[P_j\right]\right). \quad (14)$$

Similar to the case of lines with quality inspection and scrapping of defective parts, a detailed validation of constitutive and transition matrices associated with lines including rework stations, Eqs. (10)-(13), is presented in Appendix B.

Upon the formulation of the transition matrix, an analytical solution to the problem can be retrieved in numerical form by employing Eq. (2) and the eigenvalue problem. However, for complexity reasons, its symbolic form cannot be presented. In both cases, Eq. (2) yields the steady-state probability distribution vector, $\{\pi\}$, which can be employed to evaluate the key performance indicators. Thus, in production lines with rework stations, the production rate, *PR*, can be determined in the same way as in the case of Bernoulli lines with quality inspection and scrapping of defective parts using Eq. (8). Similarly, the fault rate, *FR*, corresponds to the scrapping rate, *SR*, Eq. (9). Additionally, the setup of the considered system allows us to introduce a new key performance indicator stating the rate of faulty parts being repaired by machine m_R and returned to the main production line. Thus, the repair rate, *RR*, can be determined as

$$RR=P\left[\left\{m_R\ \text{up}\right\}\cap\left\{b_{R-1}\ \text{not empty}\right\}\right]=p_R\left(1-\sum_{h_1=0}^{N_1}\sum_{h_2=0}^{N_2}\cdots\sum_{h_{M-2}=0}^{N_{M-2}}\pi_{h_1h_2\ldots h_{M-2}h_{R-1}=0}\right). \quad (15)$$

While the work-in-process, WIP_i, and probabilities of starvation, ST_i, remain the same as those presented in the case of ordinary serial Bernoulli production lines [7], the probability of blockage of the machine with a quality checking feature, BL_Q, and the rework machine, BL_R, require some modifications due to the system setup. Hence, the probabilities of blockage, BL_Q and BL_R, equal

$$\begin{aligned}BL_Q=P\Big[&\left(\left\{m_Q\ \text{up}\right\}\cap\left\{\text{good product}\right\}\cap\left\{b_Q\ \text{full}\right\}\cap\left\{m_{Q+1}\ \text{down}\right\}\right)\cup\\&\left(\left\{m_Q\ \text{up}\right\}\cap\left\{\text{good product}\right\}\cap\left\{b_Q\ \text{full}\right\}\cap\left\{m_{Q+1}\ \text{blocked}\right\}\right)\cup\\&\left(\left\{m_Q\ \text{up}\right\}\cap\left\{\text{fault product}\right\}\cap\left\{b_{R-1}\ \text{full}\right\}\cap\left\{m_R\ \text{down}\right\}\right)\cup\\&\left(\left\{m_Q\ \text{up}\right\}\cap\left\{\text{fault product}\right\}\cap\left\{b_{R-1}\ \text{full}\right\}\cap\left\{m_R\ \text{blocked}\right\}\right)\Big],\\BL_R=P\Big[&\left(\left\{m_R\ \text{up}\right\}\cap\left\{b_{Q-1}\ \text{full}\right\}\cap\left\{m_Q\ \text{down}\right\}\right)\cup\left(\left\{m_R\ \text{up}\right\}\cap\left\{b_{Q-1}\ \text{full}\right\}\cap\left\{m_Q\ \text{blocked}\right\}\right)\Big],\end{aligned} \quad (16)$$

and thus follow from the system of equations

$$\begin{aligned}BL_Q&=p_Q\left(1-q_Q\right)\sum_{h_1=0}^{N_1}\sum_{h_2=0}^{N_2}\cdots\sum_{h_{R-1}}^{N_{R-1}}\pi_{h_1h_2\ldots h_Q=N_Q\ldots h_{R-1}}\left(1-p_{Q+1}+BL_{Q+1}\right)+\\&\qquad p_Qq_Q\sum_{h_1=0}^{N_1}\sum_{h_2=0}^{N_2}\cdots\sum_{h_{R-2}}^{N_{R-2}}\pi_{h_1h_2\ldots h_{R-2}h_{R-1}=N_{R-1}}\left(1-p_R+BL_R\right),\\BL_R&=p_R\sum_{h_1=0}^{N_1}\sum_{h_2=0}^{N_2}\cdots\sum_{h_{R-1}}^{N_{R-1}}\pi_{h_1h_2\ldots h_{Q-1}=N_{Q-1}\ldots h_{R-1}}\left(1-p_Q+BL_Q\right).\end{aligned} \quad (17)$$

The probabilities of blockage of all other machines are determined following the usual expressions valid for ordinary serial Bernoulli lines [7].

2.3 The finite state method

The developed analytical approach to the problem is unfortunately tied to rather significant CPU and memory storage issues associated with the transition matrix formulation and the direct solution of Eq. (2) in the case of large-scale production systems possibly involving myriads of different system states. Thus, a new set of finite state elements will be developed and presented here in the case of Bernoulli lines with quality inspection and scrapping of defective parts as well as Bernoulli lines with rework stations to cope with the requirements of realistic production systems. Similar to previous studies (e.g. [8, 11]), the structure of finite state elements is given by the system itself and is defined concerning the position of the least reliable machine $m_m \equiv \min\{p_i\}$, i=1, 2, …, M. In this way, a set of upstream and downstream elements can be formulated according to Figures 3 and 4.

The steady-state probability distribution, $\{\pi\}_e$, at the level of each finite state element, e, obeys a well-known distribution for a two-machine-one-buffer system [5]

$$\{\pi\}_e = \pi_0^e \begin{Bmatrix} 1 \\ \frac{1}{1-p_2^e}\alpha \\ \frac{1}{1-p_2^e}\alpha^2 \\ \vdots \\ \frac{1}{1-p_2^e}\alpha^{N_e} \end{Bmatrix}, \quad \pi_0^e = \frac{1-p_2^e}{1-p_2^e+\sum_{i=1}^{N_e}\alpha^i}, \quad \alpha = \frac{p_1^e\left(1-p_2^e\right)}{p_2^e\left(1-p_1^e\right)}, \tag{18}$$

where, in a Bernoulli line with quality inspection and scrapping of defective parts,

$$p_1^e = \begin{cases} p_e\prod_{i=e}^{m-1}\left(1-q_i\right), \text{ for upstream elements, } e=1,2,...,m-1, \\ p_m\prod_{i=m}^{e}\left(1-q_i\right), \text{ for downstream elements, } e=m,m+1,...,M-1, \end{cases}$$
$$p_2^e = \begin{cases} p_m, \text{ for upstream elements, } e=1,2,...,m-1, \\ p_{e+1}, \text{ for downstream elements, } e=m,m+1,...,M-1. \end{cases} \tag{19}$$

Similarly, in a Bernoulli line with rework stations, the probabilities p_1^e and p_2^e equal

$$p_1^e = \begin{cases} p_e, \text{ for upstream elements, } e=1,2,...,m-1, \\ p_m, \text{ for downstream elements, } e=m,m+1,Q-2, \\ p_m + p_R q_Q, \ e=Q-1, \\ p_m\left(1-q_Q\right), \text{ for downstream elements, } e=Q,Q+1,R-1, \\ p_m q_Q, \ e=R-1, \end{cases}$$
$$p_2^e = \begin{cases} p_m, \text{ for upstream elements, } e=1,2,...,m-1, \\ p_{e+1}, \text{ for downstream elements, } e=Q-2,Q-1,Q,...,R-1. \end{cases} \tag{20}$$

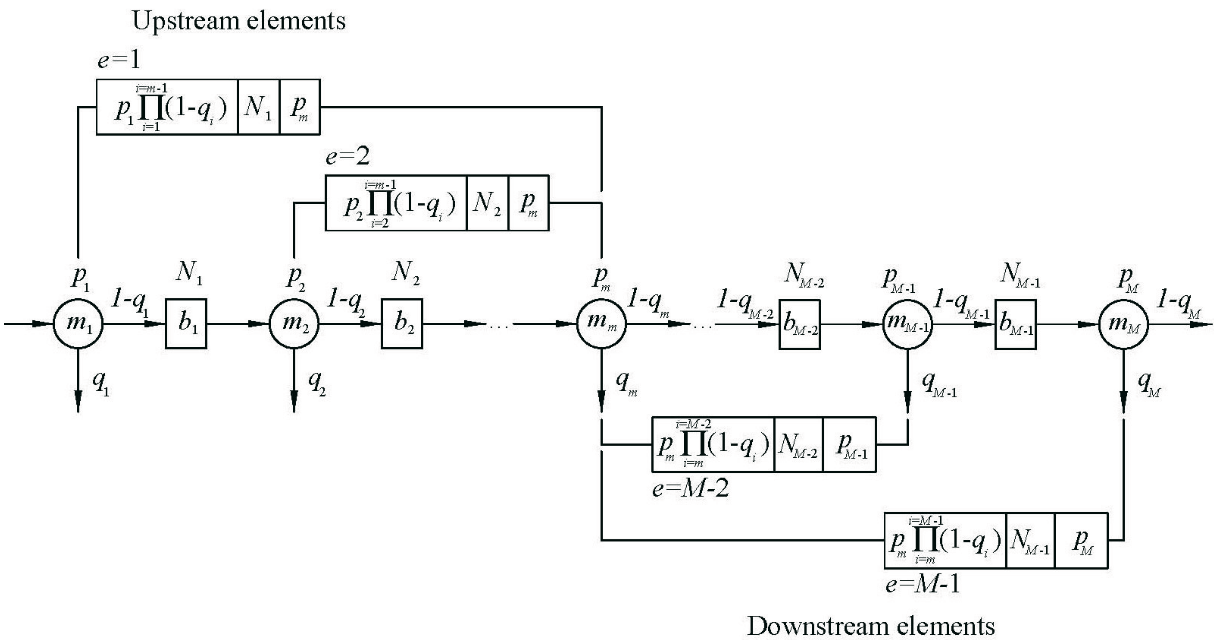


Fig. 3 Formulation of finite state elements in a Bernoulli line with quality inspection and scrapping of defective parts

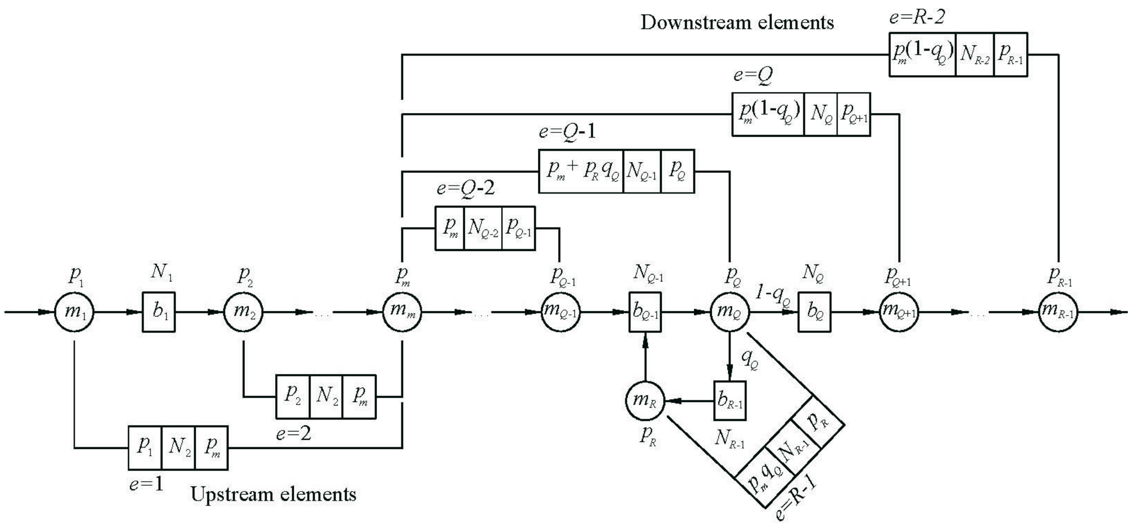


Fig. 4 Formulation of finite state elements in the case of a Bernoulli line with rework stations

Once the element-level steady-state distributions, $\{\pi\}_e$, are determined, a corresponding system-level probability distribution, $\{\pi\}$, can be evaluated by employing a well-known proportionality property of the system states at hand [15]. Thus, by assuming the independence of events at each buffer, it follows that

$$\pi_i = \pi_{h_1 h_2 \ldots h_{M-1}} = \pi_{h_1}^{e=1} \pi_{h_2}^{e=2} \ldots \pi_{h_{M-1}}^{e=M-1}, \; i = 1,2,\ldots d \tag{21}$$

which can be employed further on to evaluate the key performance indicators according to the procedure elaborated in Sections 2.1 and 2.2.

2.4 Theoretical examples

Application of the developed analytical and approximative frameworks is first presented in theoretical examples to enable validation of the introduced finite state elements. Four different production systems, two including quality inspection and the scrapping of defective parts (L5-A and L5-B) and two involving the rework station (L4+R-A and L4+R-B), are considered, Figure 5. The properties of each system are summarised in Table 1. The key performance indicators, including production rate, scrapping rate, fault rate, repair rate, work-in-process, and probabilities of blockage and starvation, were first evaluated employing the analytical approach (AN), Eqs. (2)-(17). The same indicators were then calculated using the finite state method (FSM), Eqs. (18)-(19), and a comparison of the obtained results is presented in Figures 6-9 concerning the buffer capacity N ranging between 1 and 30. Thus, the largest of the considered system state spaces was composed of 31^4 different states.

Table 1 Properties of theoretical production systems

Line	p_1	p_2	p_3	p_4	p_5	N_1	N_2	N_3	N_4	q_1	q_2	q_3	q_4	q_5
L5-A	0.4	0.5	0.6	0.7	0.8	N	N	N	N	0.05				
L5-B						N	¾N	½N	¼N					
L4+R-A						N	N	N	1	/	/	/	0.05	/
L4+R-B						N	¾N	½N	1	/	/	/		/

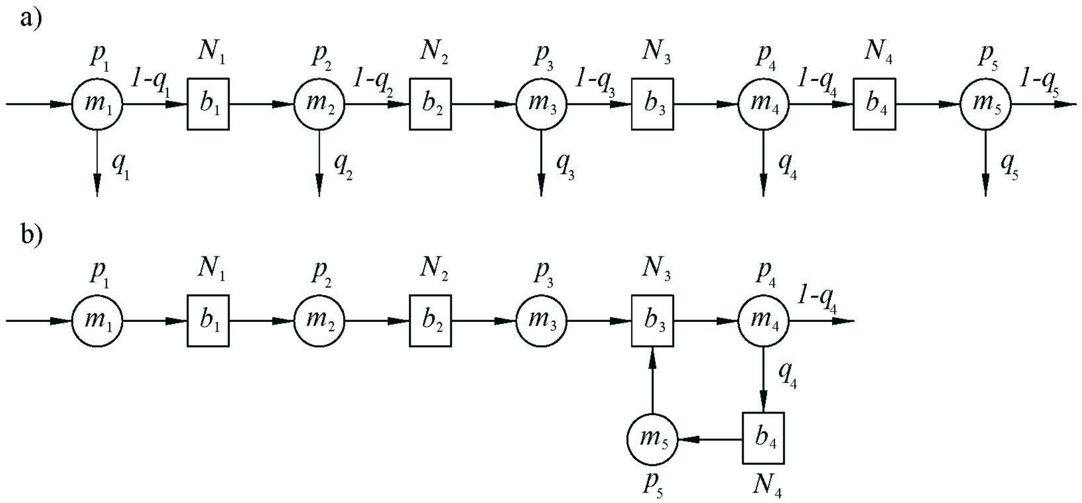


Fig. 5 Theoretical production systems: a) Bernoulli line with quality checking and scrapping of defective parts and b) Bernoulli line with rework station

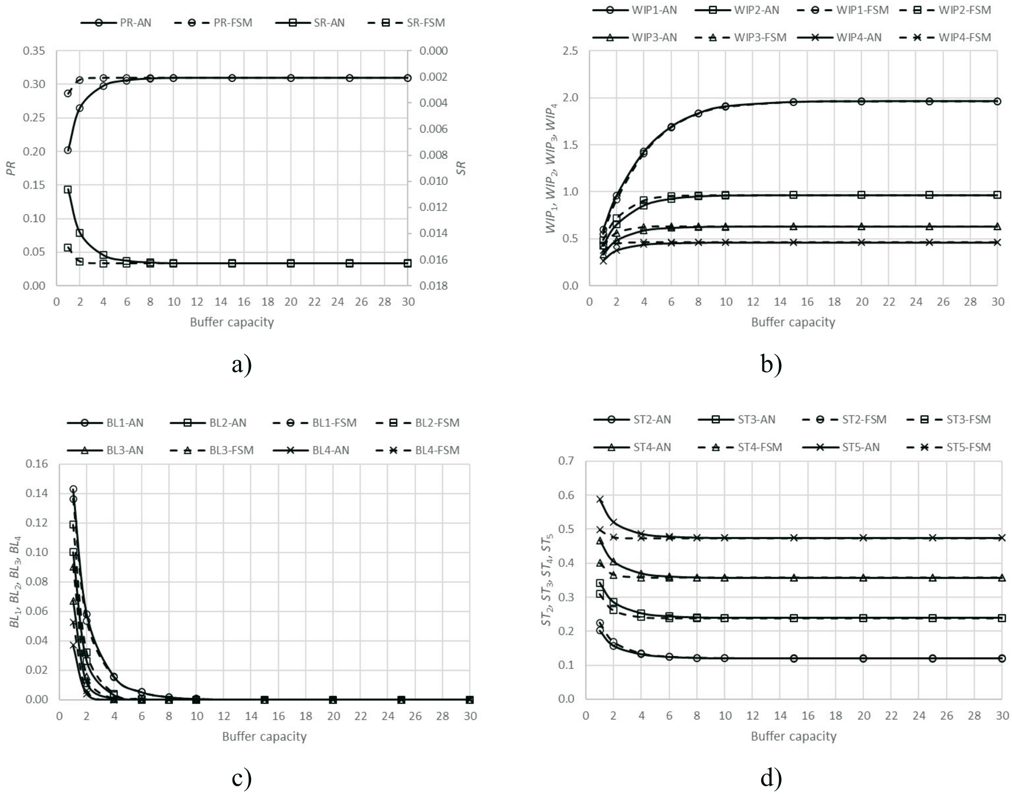


Fig. 6 L5-A, comparison of the performance measures concerning the buffer capacity, *N*

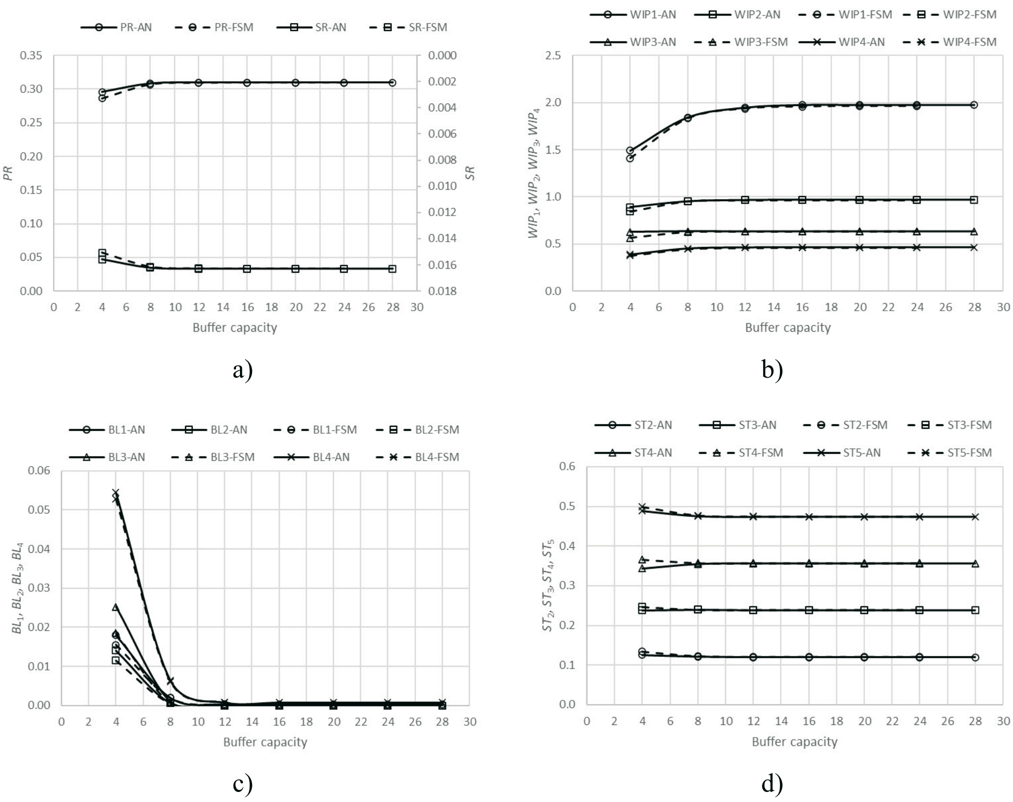


Fig. 7 L5-B, comparison of the performance measures concerning the buffer capacity, *N*

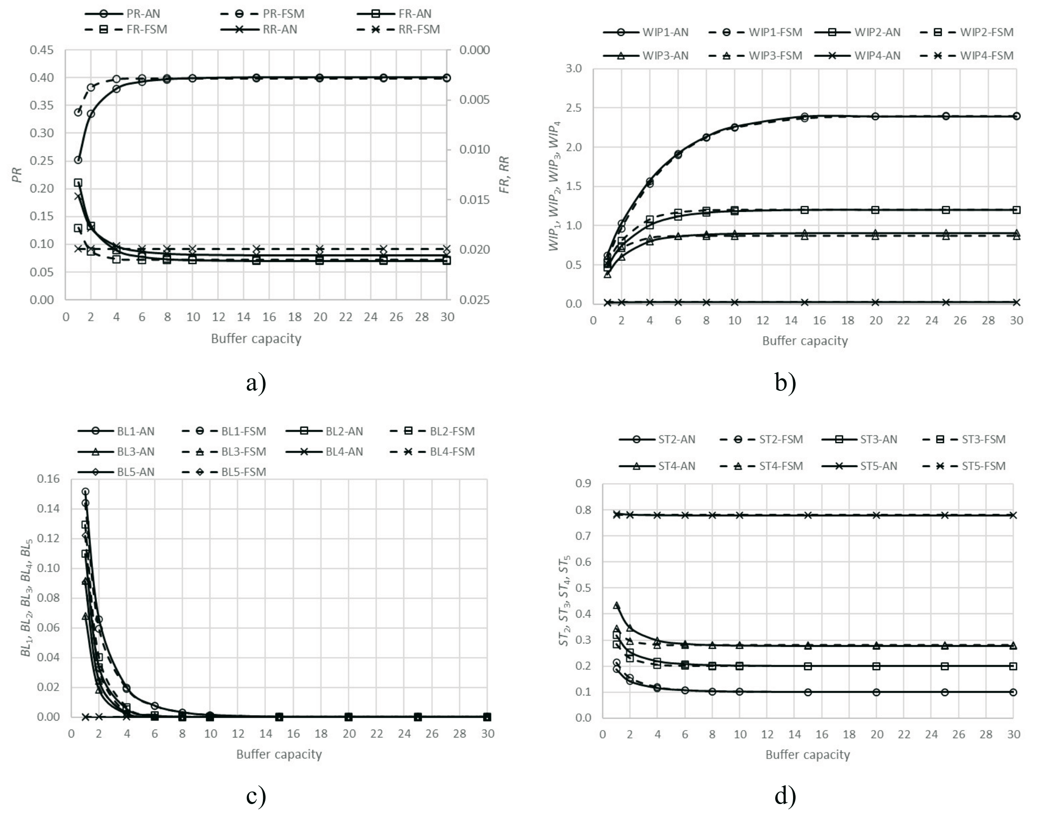


Fig. 8 L4+R-B, comparison of the performance measures concerning the buffer capacity, N

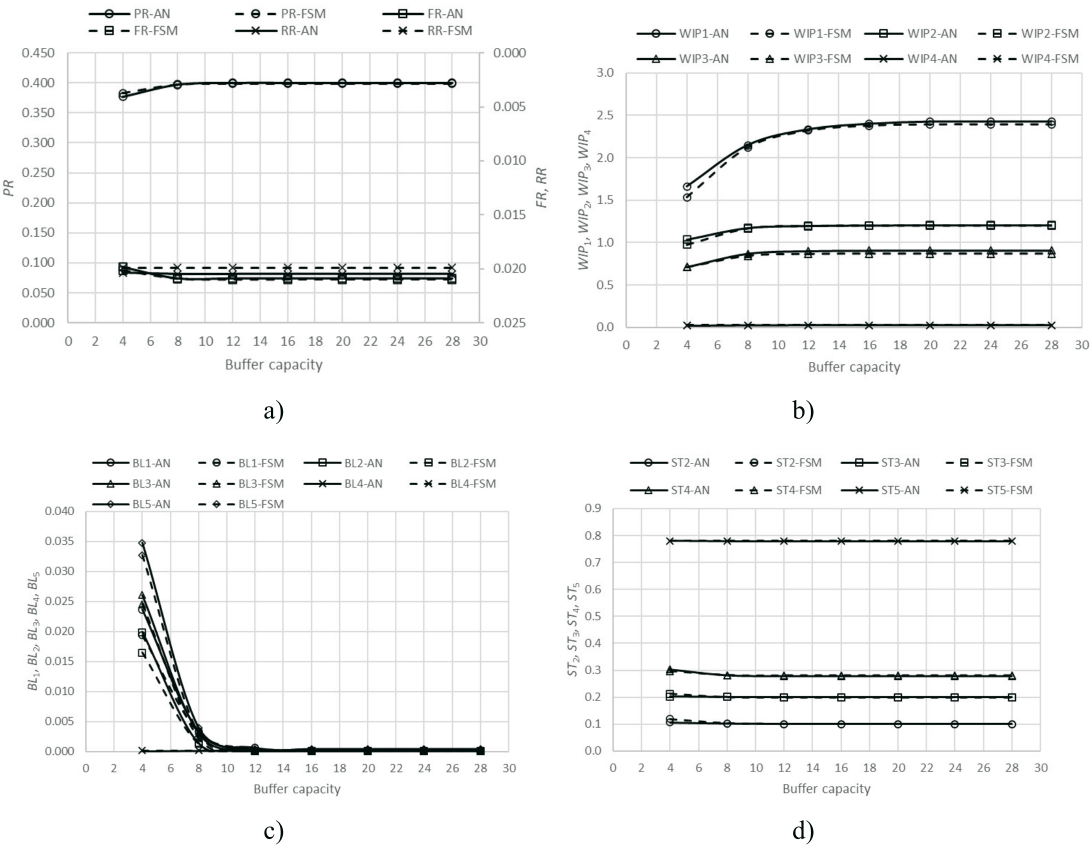


Fig. 9 L4+R-B, comparison of the performance measures concerning the buffer capacity, N

By observing Figures 6-9, it can be noted that the results obtained employing the finite state method agree very well with those retrieved through the analytical approach to the problem. However, some discrepancies may be identified for N<8 in the case of the production rate, scrapping rate, fault rate, and probability of starvation. This issue is related to the rather rough discretisation of the system state space associated with the finite state method and it diminishes quickly with the increase of the state space scale as detailed in [7, 8]. Furthermore, both methods converge to the same asymptotic values which, in the case of lines L5-A and L5-B, equal

$$\underset{N\to\infty}{PR} = p_m \prod_{i=1}^{5}\left(1-q_i\right),\ \underset{N\to\infty}{SR} = p_m \prod_{i=1}^{4}\left(1-q_i\right) - \underset{N\to\infty}{PR},$$
$$\underset{N\to\infty}{BL_i} = 0,\ \underset{N\to\infty}{ST_i} = p_i - p_m \prod_{j=2}^{i}\left(1-q_{i-1}\right), \tag{22}$$

while in lines L4+R-A and L4+R-B they amount to

$$\underset{N\to\infty}{PR} = p_m,\ \underset{N\to\infty}{FR} = \underset{N\to\infty}{RR} = p_m q_Q,\ \underset{N\to\infty}{BL_i} = 0,\ \underset{N\to\infty}{ST_i} = p_i - \underset{N\to\infty}{PR}. \tag{23}$$

Additionally, the work-in-process contained at all buffers gradually approaches the constant value with an increase in buffer capacity in all the considered cases as a direct consequence of the fact that the first machine is also the least reliable one.

3. Discussion

The presented comparison and validation of the obtained results prove that the finite state method is a sufficiently accurate and reliable computational approach. However, it cannot be recommended when the system state space contains only a few states as its rather rough discretisation leads to less reliable key performance indicators. In this case, it is necessary to apply an analytical approach, which is also appropriate given the significantly lower computational requirements associated with smaller state spaces. Still, the formulation of the transition matrix remains the main obstacle to the application of the analytical approach as its computational complexity is $O(d^{M-1})$ and thus grows exponentially with the number of machines and buffers involved. This negatively impacts both the required evaluation time and memory storage requirements and quickly leads to its unusability for practical problems. For example, the analytical evaluation of a problem including more than $5\cdot10^5$ states becomes very demanding in terms of evaluation time, amounting to days and weeks. This is where the finite state method steps in as a sufficiently reliable and computationally efficient approach requiring only a few seconds of evaluation time. Additionally, as compared to other approximative approaches (the decomposition method and the aggregation procedure), the formulated finite state elements retain the functional relationships between all variables involved in the problem. This fact enables its reliable application in the case of bottleneck evaluation and various production system design procedures relying on partial derivatives of the key performance indicators with respect to system variables [8].

Further, the presented approach may be readily extended to other production system setups as well, including serial lines with the scrapping of defective parts combined with rework stations or splitting and assembly systems with quality checking and/or rework stations. In these cases, the extension would require a combination of suitable finite state elements reflecting the setup properties and constitutive matrices of the system at hand. The developed approach also enables efficient evaluation of the transient response of systems involving quality checking and/or rework stations, assuming the nonhomogeneous properties of the associated Markov chains reflecting the possible deterioration of the machines' efficiency and system maintenance policies [10].

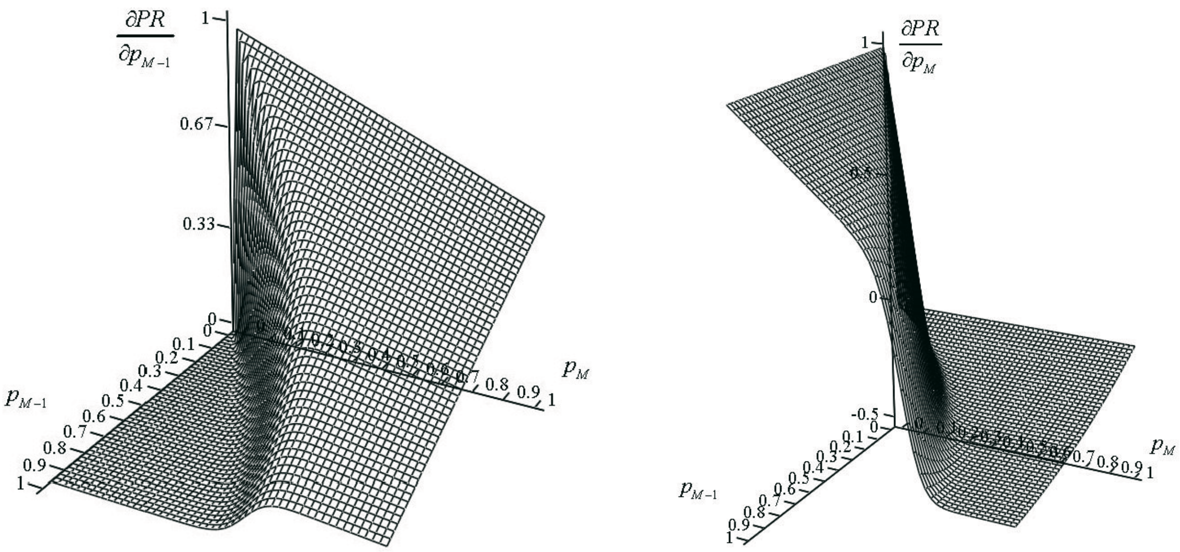
Finally, the quality-quantity relationship is of particular interest when considering production systems with quality-checking features. This issue has been considered in several publications including numerical facts pointing to the existence of $PR = f\left(p_i\right)$ as a monotonically increasing and monotonically decreasing function in the range $0 \le p_i \le 1$ and conditioned by $q_i = f\left(p_i\right)$, [5]. Since the existence of such an optimal point is of great importance in improvability evaluation and in the implementation of quality management, we will consider it here for Bernoulli lines with quality checking and the scrapping of defective parts by employing the finite state method. In this case, the *PR* can be formulated analytically as [8]

$$PR = p_M\left(1-q_M\right)\left(1-\pi_0^{M-1}\right) \tag{24}$$

which, using Eqs. (18) and (19) and by assuming a linear relationship between *pi* and q_i, yields

$$\begin{gathered} PR = p_M\left(1-kp_M\right)\left(1-\frac{\left(1-p_{M-1}\left(1-kp_{M-1}\right)\right)\left(1-\alpha^{M-1}\right)}{1-\frac{p_{M-1}\left(1-kp_{M-1}\right)}{p_M}\left(\alpha^{M-1}\right)^{N_{M-1}}}\right), \\ \alpha^{M-1} = \frac{p_{M-1}\left(1-kp_{M-1}\right)\left(1-p_M\right)}{p_M\left(1-p_{M-1}\left(1-kp_{M-1}\right)\right)}, \end{gathered} \tag{25}$$

where $0 < k < 1$. Partial derivatives of Eq. (25) with respect to p_{M-1} and p_M (assuming $k = 0.4$ and $N_{M-1} = 10$) are presented in Figure 10. It can easily be seen that the function $\frac{\partial PR}{\partial p_{M-1}}$ is strictly positive, pointing to the fact that the *PR* is a strictly increasing function concerning p_{M-1}. On the other hand, the function $\frac{\partial PR}{\partial p_M}$ demonstrates both positive and negative values, leading to the conclusion that in this case the *PR* has an optimum point in the considered range. Hence, we cannot always guarantee that the optimum point of the *PR* function exists for $0 \le p_{M-1} \le 1$ and $0 \le p_M \le 1$. Thus, this issue still requires further research which, due to its complexity, must be considered in a dedicated paper.



a) b)

Fig. 10 Partial derivatives $\frac{\partial PR}{\partial p_{M-1}}$ and $\frac{\partial PR}{\partial p_M}$ of Eq. (25) in the range $0 \le p_{M-1} \le 1$ and $0 \le p_M \le 1$, $k = 0.4$ and $N_{M-1} = 10$

4. Application of the developed theory

The developed theory is applied in the case of a prefabrication line typical of most large-scale shipyards, Figure 11. The prefabrication line is placed at the beginning of the shipbuilding process, and it involves operations like plate straightening using a heavy-duty roller flattening machine, hot air drying, blasting, anticorrosive protection (preserving), and finally marking. All work stations are connected by a heavy-duty roller conveyor. Upon completion of the prefabrication, each plate is transferred to a plate fabrication workshop for operations like oxy-acetylene cutting, plasma cutting, and bending if required. Hence, the dynamics of all subsequent operations significantly depend on the efficiency of the prefabrication line. It is therefore important to inspect its key performance indicators given that a part of the material flow can be scrapped as defective or sent back to reinitiate its prefabrication. This will be done by modelling the prefabrication line as a Bernoulli line with quality checking and scrapping of defective parts. The considered input data, acquired on the shipyard floor, are presented in Table 2. Three quality-checking points at machines m_1, m_3, and m_4 are included. In all cases, the checks are based on compliance with criteria provided by the supervising ship classification society and newbuilding contracts on plate flatness, quality of the material surface, and anticorrosive coating.

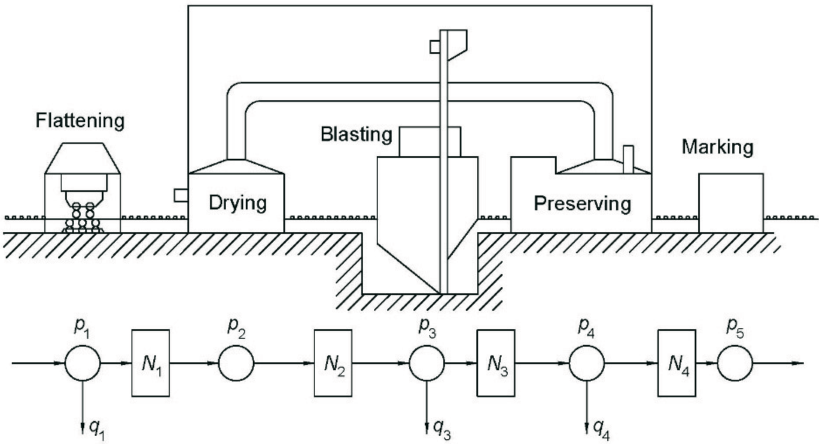


Fig. 11 The prefabrication line and the mathematical model [7]

Table 2 Prefabrication line, input data

Operation	Flattening, p_1	Drying, p_2	Blasting, p_3	Preserving, p_4	Marking p_5
Probability, p_i	0.9	0.912	0.885	0.801	0.955
Scrapping rate, q_i	0.2	/	0.05	0.05	/
Buffer	N_1	N_2	N_3	N_4	
Capacity of buffer, N_i	2	1	1	1	

The key performance indicators of the considered production line are determined based on the mathematical model of a Bernoulli line with quality checking and scrapping of defective parts, Figure 11, and by employing the developed finite state method framework. The obtained results are summarised in Table 3. It can be seen that the considered system has a potential for improvement, especially regarding the increase in production rate and decrease in the probabilities of the starvation and blockage of machines m_2, m_3, m_4 and m_5. Moreover, the work-in-process at the first buffer remains below its capacity, pointing to the possibility for it to be reduced. In addition, we are interested in whether increasing the reliability of the machine m_4 affects the production rate positively, given that it can be considered as a quality-quantity coupling machine with a polynomial relationship between reliability and quality set as $q_4 = 2p_4^2 - 2.9p_4 + 1.09$. The obtained results are presented in Figure 12, demonstrating that, under the considered conditions, the production rate is a monotonically decreasing function. Hence, improvement of the reliability p_4 is not recommended given that the quality of the product is significantly reduced, resulting in a higher scrapping rate of the machine m_4 and a lower overall production rate.

Table 3 The key performance indicators in the case of a prefabrication line

Key performance indicators	PR	SR_1	SR_3	SR_4	WIP_1	WIP_2	WIP_3	WIP_4
	0.631	0.180	0.032	0.029	0.93	0.74	0.73	0.66
	BL_1	BL_2	BL_3	BL_4	ST_2	ST_3	ST_4	ST_5
	0.023	0.144	0.176	0.037	0.202	0.227	0.216	0.324

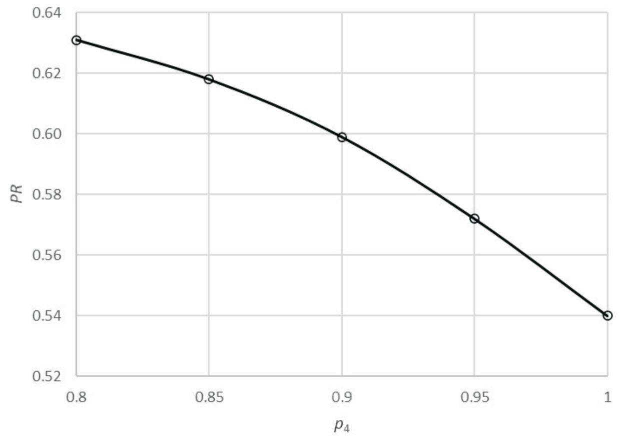


Fig. 12 Impact of the quality-quantity relationship on the production rate

As already pointed out, the shipbuilding process starts with material processing on the prefabrication line. Hence, its impact on the dynamics of all subsequent operations is significant. Thus, the evaluation of the associated key performance indicators represents an important component of establishing both long- and short-term production plans. In addition, a thorough analysis of production processes enables the identification of potential bottlenecks that must be resolved and improvements that must be made to avoid possible negative effects associated with eventual quality-quantity coupling issues. Finally, the introduced modelling approach enables the establishment of a rational techno-economic evaluation of production systems in terms of investments and system maintenance.

5. Conclusion

The management of modern production systems focuses on improvements in organisational effectiveness, competitiveness, and innovativeness conditioned by quality management principles and sustainable and customer-oriented production. Its main goal is to create a positive impact on the company's profitability through competitive positioning in the global customer market. The complexity of this task requires the introduction of sophisticated mathematical models suitable for the continuous evaluation and improvement of production processes.

Two approaches to the mathematical modelling of production systems have been presented in this study, namely the analytical and the approximative one. In the first case, dedicated transition matrices were formulated for the first time for Bernoulli lines with quality checking and the scrapping of defective parts, and for Bernoulli lines with rework stations. In the second case, the finite state method was employed, including the formulation of new finite state elements reflecting the setup of the considered systems. The validation of the introduced elements was presented for four different theoretical production lines, and the main findings were discussed, including evaluation time and memory storage issues, possible extensions to the problems of splitting lines, assembly systems, and evaluation of the transient response. The quality-quantity issue was also addressed, pointing to the need for further research. Finally, the developed approach was applied to a shipyard's plate prefabrication line, including evaluation of the associated key performance indicators and potential improvements in quality-quantity coupling.

The research methodology used represents a sound ground for its further application in future research, including the due-time performance of production systems, evaluation of asynchronous and/or nonhomogeneous production systems, and systems composed of machines with non-Bernoulli reliability distribution. As pointed out in the Discussion section, the issue of quality-quantity coupling must be considered thoroughly in the context of the improvability analysis and the design of production systems. Such a systematic approach to the mathematical modelling of production systems will significantly contribute to its extension and reliable implementation in the context of the constantly growing fields of Machine Learning and Artificial Intelligence, especially when developing digital twins of production systems.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix A

Formulation and validation of the transition matrix for a Bernoulli serial line with three machines

Consider a serial Bernoulli line composed of machines m_1, m_2, and m_3 with the associated reliabilities p_1, p_2, and p_3. For simplicity, let us also assume that the capacities of buffers b_1 and b_2 equal N_1=N_2=2. In this case, the dimension of the system state space takes a value d=9 and is composed of the states

$$S = S_1 \times S_2 = \{0,1,2\} \times \{0,1,2\} = \\ = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}. \quad \text{(A1)}$$

Thus, the constitutive matrices, Eqs. (4)-(6), are equal to

$$[P_1] = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(A2)}$$

$$[P_2] = \begin{bmatrix} 1 & 0 & 0 & p_2q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma & p_2q_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-p_2 & 0 & 0 & p_2q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-p_2 & 0 & \gamma & p_2q_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(A3)}$$

$$\left[P_3\right]=\begin{bmatrix}1 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1-p_3 & p_3 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1-p_3 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & p_3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1-p_3 & p_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1-p_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p_3 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p_3 & p_3\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p_3\end{bmatrix} \tag{A4}$$

where $\alpha=\left(1-p_1\right)+p_1q_1$, $\beta=p_1\left(1-q_1\right)$, and $\gamma=p_2\left(1-q_2\right)$. The transition matrix of the considered systems can be obtained by introducing Eqs. (A2) - (A4) to Eq. (3). Thus, it follows that

$$\left[P\right]=\begin{bmatrix}\alpha & \delta & 0 & \vartheta & \vartheta p_3 & 0 & 0 & 0 & 0\\ 0 & \varepsilon & \delta & \iota & \vartheta\left(1-p_3\right)+\iota p_3 & \vartheta p_3 & 0 & 0 & 0\\ 0 & 0 & \varepsilon & 0 & \iota\left(1-p_3\right) & \iota p_3 & 0 & 0 & 0\\ \beta & \zeta & 0 & \kappa & \kappa p_3 & 0 & \vartheta & \vartheta p_3 & 0\\ 0 & \eta & \zeta & \lambda & \kappa\left(1-p_3\right)+\lambda p_3 & \kappa p_3 & \iota & \vartheta\left(1-p_3\right)+\iota p_3 & \vartheta p_3\\ 0 & 0 & \eta & 0 & \lambda\left(1-p_3\right) & \lambda p_3+\varepsilon & 0 & \iota\left(1-p_3\right) & \iota p_3\\ 0 & 0 & 0 & \mu & \mu p_3 & 0 & \nu & \nu p_3 & 0\\ 0 & 0 & 0 & 0 & \mu\left(1-p_3\right) & \mu p_3 & \lambda & \nu\left(1-p_3\right)+\lambda p_3 & \nu p_3\\ 0 & 0 & 0 & 0 & 0 & \eta & 0 & \lambda\left(1-p_3\right) & \lambda p_3+\left(1-p_3\right)\end{bmatrix} \tag{A5}$$

where

$$\begin{aligned}\delta&=\left(1-p_1\right)p_3+p_1q_1p_3,\\ \varepsilon&=\left(1-p_1\right)\left(1-p_3\right)+p_1q_1\left(1-p_3\right),\\ \varsigma&=p_1\left(1-q_1\right)p_3,\\ \eta&=p_1\left(1-q_1\right)\left(1-p_3\right),\\ \vartheta&=\left(1-p_1\right)p_2q_2+p_1q_1p_2q_2,\\ \iota&=\left(1-p_1\right)p_2\left(1-q_2\right)+p_1q_1p_2\left(1-q_2\right),\\ \kappa&=\left(1-p_1\right)\left(1-p_2\right)+p_1\left(1-q_1\right)p_2q_2+p_1q_1\left(1-p_2\right),\\ \lambda&=p_1\left(1-q_1\right)p_2\left(1-q_2\right),\\ \mu&=p_1\left(1-q_1\right)\left(1-p_2\right),\\ \nu&=\left(1-p_2\right)+p_1\left(1-q_1\right)p_2q_2.\end{aligned} \tag{A6}$$

By setting $q_1=q_2=0$, Eqs. (A2) - (A6) reduce to a well-known form of constitutive matrices and a transition matrix associated with an ordinary serial Bernoulli line without a quality-checking feature [7].

Appendix B

Formulation and validation of the transition matrix for a Bernoulli line with a rework station

Consider a serial Bernoulli line composed of machines m_1, $m_2 \equiv m_Q$, and $m_3 \equiv m_R$ with the associated reliabilities p1, pQ, and pR, Figure B1. For simplicity, let us also assume that the capacities of buffers b_1 and b_2 equal $N_1=N_2=2$.

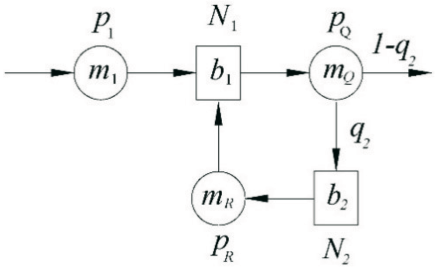

Fig. B1 A simple Bernoulli production line with a rework station

The constitutive matrices follow directly from [7] and Eqs. (10) and (11) as

$$\left[P_1\right]=\begin{bmatrix} 1-p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-p_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 1-p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 1-p_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_1 & 0 & 0 & 1-p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 1 \end{bmatrix} \tag{B1}$$

$$\left[P_Q\right]=\begin{bmatrix} 1 & 0 & 0 & p_Q\left(1-q_Q\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & p_Q q_Q & p_Q\left(1-q_Q\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & p_Q q_Q & p_Q\left(1-q_Q\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-p_Q & 0 & 0 & p_Q\left(1-q_Q\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-p_Q & 0 & p_Q q_Q & p_Q\left(1-q_Q\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-p_Q+p_Q q_Q & 0 & p_Q q_Q & p_Q\left(1-q_Q\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-p_Q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p_Q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p_Q+p_Q q_Q \end{bmatrix} \tag{B2}$$

$$[P_R]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-p_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-p_R & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_R & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_R & 0 & 1-p_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-p_R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_R & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_R & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{B3}$$

Further, employing Eqs. (12) and (13) yields the transition matrix as

$$[P_R]=\begin{bmatrix} \alpha & 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & \zeta & \varepsilon(1-p_R) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & \zeta p_R & \varepsilon(1-p_R) & 0 & 0 & 0 \\ p_1 & \gamma & 0 & \eta & \varepsilon p_R & 0 & \varepsilon & 0 & 0 \\ 0 & \delta & \gamma & \vartheta & \eta(1-p_R)+\zeta p_R & \varepsilon p_R & \zeta & \varepsilon(1-p_R) & 0 \\ 0 & 0 & \delta & 0 & \vartheta(1-p_R) & \eta(1-p_R)+\zeta p_R & 0 & \zeta(1-p_R) & \varepsilon(1-p_R) \\ 0 & p_1 p_R & 0 & \iota & \kappa & 0 & \mu & \nu & 0 \\ 0 & 0 & p_1 p_R & 0 & \iota(1-p_R)+\vartheta p_R & \kappa+\lambda & \vartheta & \xi & \nu \\ 0 & 0 & 0 & 0 & 0 & (\vartheta+\iota)(1-p_R) & 0 & \vartheta(1-p_R) & o \end{bmatrix} \tag{B4}$$

where

$$\begin{gathered} \alpha=1-p_1,\ \beta=(1-p_1)(1-p_R),\ \gamma=(1-p_1)p_R, \\ \delta=p_1(1-p_R),\ \varepsilon=(1-p_1)p_Q(1-q_Q),\ \zeta=(1-p_1)p_Q q_Q, \\ \eta=(1-p_1)(1-p_Q)+p_1 p_Q(1-q_Q),\ \vartheta=p_1 p_Q q_Q,\ \iota=p_1(1-p_Q), \\ \kappa=(1-p_Q)p_R+p_1 p_Q(1-q_Q)p_R,\ \lambda=p_Q q_Q p_R, \\ \mu=(1-p_Q)+p_1 p_Q(1-q_Q),\ \nu=p_Q(1-q_Q)p_R, \\ \xi=(1-p_Q)+p_1 p_Q(1-q_Q)(1-p_R)+p_Q q_Q p_R, \\ o=(1-p_Q)+p_1 p_Q(1-q_Q)(1-p_R)+p_Q q_Q. \end{gathered} \tag{B5}$$

Setting q_Q=0 and neglecting the states of the system including occupancy of the buffer b_2 equal to h_2=1 and h_2=2 yields a well-known transition matrix

$$[P]=\begin{bmatrix} (1-p_1) & (1-p_1)p_2 & 0 \\ p_1 & (1-p_1)(1-p_2)+p_1 p_2 & (1-p_1)p_2 \\ 0 & p_1(1-p_2) & (1-p_2)+p_1 p_2 \end{bmatrix} \tag{B6}$$

associated with a serial Bernoulli production line composed of two machines and one intermediate buffer of a capacity equal to 2 [5].