Original scientific paper
Accepted 20. 11. 2008.

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Automorphic Inversion and Circular Quartics in Isotropic Plane

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ABSTRACT

In this paper circular quartics are constructed by automorphic inversion (inversion that keeps the absolute figure fixed) as the images of conics. They are classified depending on their position with respect to the absolute figure. It is shown that only 1, 2 and 4-circular quartics can be obtained by automorphic inversion.

Key words: isotropic plane, circular quartic, automorphic inversion

MSC 2000: 51M15, 51N25

Automorfna inverzija i cirkularne kvartike u izotropnoj ravnini

SAŽETAK

U ovom su radu cirkularne kvartike konstruirate pomoću automorfne inverzije (inverzija koja apsolutnu figuru ostavlja fiksnom) kao slike konika. Klasificirane su s obzirom na položaj prema apsolutnoj figuri. Pokazano je da se automorfnom inverzijom konike mogu dobiti samo 1, 2 i 4 cirkularne kvartike.

Ključne riječi: izotropna ravnina, cirkularna kvartika, automorfna inverzija

1 Introduction

An *isotropic plane* I_2 is a real projective plane where the metric is induced by a real line f and a real point F incidental with it, [4]. The ordered pair (f,F) is called the *absolute figure* of the isotropic plane.

In the affine model of the isotropic plane where the coordinates of the points are defined by

$$x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0},$$

the absolute line f is determined by the equation $x_0 = 0$ and the absolute point F by the coordinates $(0,0,1)$.

The projective transformations that map the absolute figure onto itself form a 5-parametric group $\mathcal{G}_5$. They have equations of the form

$$\overline{x} = a + dx, \quad \overline{y} = b + cx + ey.$$

$\mathcal{G}_5$ is called the *group of similarities* of the isotropic plane, [1], [4]. Its subgroup $\mathcal{G}_3$ $\mathcal{G}_5$, consisting of the transformations of the form

$$\overline{x} = a + x, \quad \overline{y} = b + cx + y,$$

is called the *group of motions* of the isotropic plane. It preserves the quantities such as the distance between two points, snap between two parallel points or the angle between two lines. Thus, it has been selected for the fundamental group of transformations.
The ordered pair $(I_2, \mathcal{G}_3)$ is called the *isotropic geometry*.

All straight lines through the absolute point F are called *isotropic lines* and all points incidental with f are called *isotropic points*.
Two points $A(a_1,a_2)$ and $B(b_1,b_2)$ are called *parallel* if they lie on the same isotropic line. In that case, the *span* is defined by $s(A,B) = b_2 - a_2$. For two non-parallel points the *distance* is defined by $d(A,B) = b_1 - a_1$.

There are seven types of regular conics classified according to their position with respect to the absolute figure, [1], [4]. An *ellipse* (*imaginary* or *real*) is a conic that intersects the absolute line in a pair of conjugate imaginary points. If a conic intersects the absolute line in two different real points, it is called a *hyperbola* (of *1st* or *2nd type*, depending on whether the absolute point is outside or inside the conic). A conic passing through the absolute point is called a special *hyperbola* and a conic touching the absolute line is called *a parabola*. If a conic touches the absolute line at the absolute point, it is said to be a *circle*.

A curve in the isotropic plane is *circular* if it passes through the absolute point, [5]. Its *degree of circularity* is defined as the number of its intersection points with the

absolute line f falling into the absolute point. If it does not share any common point with the absolute line except the absolute point, it is *entirely circular*, [3].
The circular curve of order four can be 1, 2, 3 or 4-circular. The absolute line can intersect it, touch it, osculate it or hyperosculate it at the absolute point. The absolute point can be simple, double or triple point of the curve.

2 Automorphic inversion in isotropic plane

Definition 1 *An inversion with respect to the pole P and the fundamental conic q is a mapping where any point and its image are conjugate with respect to the conic q and their connecting line passes through the point P.*

The inversion is an involution, [2], [6]. Any point of the fundamental conic is mapped into itself. The lines joining a point to its image are called the *rays*. They are fixed lines as entities, but their points are not fixed. Let p be the polar line of the point P with respect to the fundamental conic q. The intersection points of the line p with the conic q are denoted by P_1 and P_2 and their polar lines by p_1 and p_2. These three points and three lines are said to be the *fundamental elements* of the given mapping. The fundamental points are the singular points of the inversion ($P \mapsto p, P_1 \mapsto p_1, P_2 \mapsto p_2$), and any point of the fundamental line is mapped into the corresponding fundamental point.

The inversion maps the curve k of order n into the curve $\overline{k}$ of order $2n$. Since k intersects any fundamental line in n points, $\overline{k}$ has three multiple points of order n in the fundamental points. If k passes through some of the fundamental points, $\overline{k}$ splits into the corresponding polar line and a curve of order $n-1$. The curve $\overline{k}$ passes through common points of the curve k and the fundamental conic.

Since we are interested in the property of circularity, we will restrict our interest on the inversions that keep the absolute figure fixed.

Definition 2 *An inversion which maps absolute figure into itself is called the automorphic inversion.*

According to [5] the following theorem is valid:

Theorem 1 *There are five types of the automorfic inversion:*

(1) The fundamental conic q is a special hyperbola. The pole P is a point of the absolute line, different from its intersections with the fundamental conic.

(2) The fundamental conic q is a special hyperbola. The pole P is an intersection point of the absolute line and the fundamental conic different from the absolute point.

(3) The fundamental conic q is a special hyperbola. The pole P is the absolute point.

(4) The fundamental conic q is a circle. The pole P is a point of the absolute line different from the absolute one.

(5) The fundamental conic q is a circle. The pole P is the absolute point.

Proof. The absolute point F has to be mapped into itself. Therefore, F has to be a point of the fundamental conic. Accordingly, the fundamental conic is either a special hyperbole or a circle. Since the absolute line f has to be mapped into itself, it has to be a ray of inversion. Consequently, the pole P is a point of the absolute line. □

A curve of order four can be obtained by inversion of a conic, but also by inversion of a curve of higher order passing through the fundamental points. For example, the inverse image of a cubic passing through two fundamental points is a curve of order six that splits into two lines and a curve of order four. We will study the quartics $\overline{k}$ obtained as an inverse images of a conic k that does not pass through the fundamental points. The conditions that the conic k has to fulfill in order to obtain a circular quartic of certain type will be determined for each of the types of the inversion.

2.1 Equation of fundamental conic

Every curve of second order is given by the equation of the form

$$a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2 = 0, \tag{1}$$

or in the affine coordinates

$$a_{00} + a_{11}x^2 + a_{22}y^2 + 2a_{01}x + 2a_{02}y + 2a_{12}xy = 0.$$

Considering the isotropic motion

$$\overline{x} = x - \frac{a_{01}a_{22} - a_{02}a_{12}}{a_{12}^2 - a_{11}a_{22}}, \quad \overline{y} = y - \frac{a_{02}a_{11} - a_{01}a_{12}}{a_{12}^2 - a_{11}a_{22}}$$

we get a simpler form of the equation

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + a = 0.$$

If $a_{12}^2 - a_{11}a_{22} = 0$, the conic is either a parabola or a circle.

The equation of parabola touching the absolute line $[1,0,0]$ at the point $(0,1,0)$ due to

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

is of the form

$$a_{00} + y^2 + 2a_{01}x + 2a_{02}y = 0.$$

Now, by the isotropic motion

$$\overline{x} = x + \frac{a_{00} - a_{02}^2}{2a_{01}}, \quad \overline{y} = y + a_{02}$$

it can be transformed into

$$y^2 + 2a_{01}x = 0, \quad a_{01} \neq 0.$$

1-circular conic (a special hyperbola) due to $a_{22} = 0$ has the equation of the form

$$a_{11}x^2 + 2a_{12}xy + a = 0, \quad a_{12} \neq 0.$$

The assumption that the other intersection point of the special hyperbola and the absolute line is the point $(0,1,0)$ leads to even simpler form

$$xy + a = 0. \tag{2}$$

If the conic given by (1) is 2-circular, the line $f[1,0,0]$ is the tangent of the conic at the point $F(0,0,1)$, consequently $a_{12} = a_{22} = 0, a_{02} \neq 0$. Therefore, the equation

$$a_{11}x^2 + 2a_{01}x + 2a_{02}y + a_{00} = 0,$$

is obtained. Since a_{11} should not equal zero, we can chose $a_{11} = 1$. After applying the isotropic motion

$$\overline{x} = x, \quad \overline{y} = \frac{a_{00}}{2a_{02}} + \frac{a_{01}}{a_{02}}x + y$$

the equation becomes

$$x^2 + 2a_{02}y = 0. \tag{3}$$

2.2 Equations of inversion

Let the fundamental conic q be a special hyperbola and let the pole of the inversion $P(0,p_1,p_2)$ be a point of the absolute line. Without loss of generality we can assume that q has the equation $xy - 1 = 0$. Let us determine the coordinates of the image $\overline{T}(1,\overline{\alpha},\overline{\beta})$ of a given point $T(1,\alpha,\beta)$. The polar line t of the point T is determined by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2}\beta \\ \frac{1}{2}\alpha \end{bmatrix},$$

Or, in other words, the line with the equation

$$-2 + \beta x + \alpha y = 0. \tag{4}$$

Connecting line TP is given by

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ 0 & p_1 & p_2 \\ 1 & \alpha & \beta \end{vmatrix} = 0,$$

which is equivalent to

$$p_1\beta - p_2\alpha + p_2x - p_1y = 0. \tag{5}$$

The point $\overline{T}$ is the intersection of the lines t and TP so its coordinates can be determined by solving the system of the linear equations (4) and (5) and equal

$$\overline{\alpha} = \frac{p_2\alpha^2 - p_1\alpha\beta + 2p_1}{p_2\alpha + p_1\beta}, \quad \overline{\beta} = \frac{-p_2\alpha\beta + p_1\beta^2 + 2p_2}{p_2\alpha + p_1\beta}.$$

Thus, the inversion is given by

$$\overline{x} = \frac{p_2x^2 - p_1xy + 2p_1}{p_2x + p_1y}, \quad \overline{y} = \frac{-p_2xy + p_1y^2 + 2p_2}{p_2x + p_1y}. \tag{6}$$

If the pole is the absolute point $F(0,0,1)$ the previous expressions are turned into

$$\overline{x} = x, \quad \overline{y} = -y + \frac{2}{x}. \tag{7}$$

If the pole is the other intersection of the fundamental conic with the absolute line, i.e. the point $P(0,1,0)$, (6) becomes

$$\overline{x} = -x + \frac{2}{y}, \quad \overline{y} = y. \tag{8}$$

In the general case when the pole $P(0,1,p), p \neq 0$, is the point of the absolute line not belonging to the fundamental conic, the inversion is determined by

$$\overline{x} = \frac{px^2 - xy + 2}{px + y}, \quad \overline{y} = \frac{-pxy + y^2 + 2p}{px + y}. \tag{9}$$

Therefore, we conclude that inversion, fundamental conic of which is a circle $x^2 - y = 0$ (without loss of generality we can assume $a_{02} = -\frac{1}{2}$) and pole $P(0,p_1,p_2)$ is a point of the absolute line, has the equations

$$\overline{x} = \frac{-p_2x + 2p_1y}{2p_1x - p_2}, \quad \overline{y} = \frac{-2p_2x^2 + 2p_1xy + p_2y}{2p_1x - p_2}. \tag{10}$$

In the case when the pole is the absolute point $F(0,0,1)$ equalities (10) are turned into

$$\overline{x} = x, \quad \overline{y} = x^2 - y. \tag{11}$$

The general case, when the pole is different from the absolute point, may be simplified by choosing $(0,1,0)$ for its coordinates. Then, the inversion is given by

$$\overline{x} = \frac{y}{x}, \quad \overline{y} = y. \tag{12}$$

2.3 Circular quartics obtained by automorphic inversion of conic

Theorem 2 *An automorphic inversion in the isotropic plane maps the* 2^{nd}*-order curve* k *not passing through the fundamental points of the inversion into the* 4^{nd}*-order curve* $\overline{k}$*. The degree of circularity of the curve* $\overline{k}$ *depends on the type of the inversion and on the position of the conic* k *with respect to the fundamental and absolute elements as follows:*

- *If the fundamental conic* q *is a special hyperbola and the pole* P *is an isotropic point,* $P \neq F$*,* $\overline{k}$ *is 1-circular (if* k *is a special hyperbola) or 2-circular (if* k *is a circle) quartic.*
- *If the fundamental conic* q *is a special hyperbola and the pole* P *is the absolute point,* $\overline{k}$ *is 2-circular quartic.*
- *If the fundamental conic* q *is a circle and the pole* P *is an isotropic point,* $P \neq F$*,* $\overline{k}$ *is 2-circular quartic*
- *If the fundamental conic* q *is a circle and the pole* P *is the absolute point,* $\overline{k}$ *is 4-circular quartic.*

Proof. Detailed proofs of all facts will be given only in the case of the inversion of type (1). Since the approach is similar in all the other cases for the inversions of types (2)-(5), only the facts will be stated in those cases.

Type (1)

Let us consider the inversion of type (1) with the fundamental conic q given by the equation $xy-1=0$ and the pole $P(0,1,p)$.

The other two fundamental points are $P_1(p,\sqrt{-p},-p\sqrt{-p})$ and $P_2(p,-\sqrt{-p},p\sqrt{-p})$. Three fundamental lines p,p_1,p_2 are given by equations $y=-px$, $y=px-2\sqrt{-p}$, $y=px+2\sqrt{-p}$, respectively.

The inversion

$$\overline{x}=\frac{px^2-xy+2}{px+y}, \quad \overline{y}=\frac{-pxy+y^2+2p}{px+y}$$

maps the conic k

$$a_{00}+a_{11}x^2+a_{22}y^2+2a_{12}xy+2a_{01}x+2a_{02}y=0 \quad (13)$$

into the quatric $\overline{k}$

$$\begin{aligned}&a_{11}p^2x^4-2p(a_{12}p+a_{11})x^3y+(a_{22}p^2+4a_{12}p+a_{11})x^2y^2-\\&-2(a_{22}p+a_{12})xy^3+a_{22}y^4+2a_{01}p^2x^3-2a_{02}p^2x^2y-\\&-2a_{01}xy^2+2a_{02}y^3+((a_{00}+4a_{12})p^2+4a_{11}p)x^2-\\&-2(2a_{22}p^2+(4a_{12}-a_{00})p+2a_{11})xy+(4a_{22}p+a_{00}+4a_{12})y^2+\\&+4p(a_{02}p+a_{01})x+4(a_{02}p+a_{01})y+4(a_{22}p^2+2a_{12}p+a_{11})=0.\end{aligned}$$

The coefficient a_{11} must not equal $-a_{22}p^2-2a_{12}p$, since otherwise conic k would pass through the pole P and constructed quartic $\overline{k}$ would split into the line $px+y=0$ and a cubic.

The intersections of the absolute line $x_0=0$ with the quartic $\overline{k}$ are the points coordinates of which satisfy the equation

$$\begin{aligned}&a_{11}p^2x_1^4-2p(a_{12}p+a_{11})x_1^3x_2+(a_{22}p^2+4a_{12}p+a_{11})x_1^2x_2^2-\\&-2(a_{22}p+a_{12})x_1x_2^3+a_{22}x_2^4=0.\end{aligned}$$

Some short calculation turns it into

$$(x_2-px_1)^2(a_{11}x_1^2-2a_{12}x_1x_2+a_{22}x_2^2)=0.$$

It is obvious from here that the pole $P(0,1,p)$ is two times counted common point of the absolute line and the quartic. The absolute point is one of the intersections if and only if $a_{22}=0$, and it is two times counted common point if and only if $a_{12}=0$, too.

If $a_{22}=0$, the conic k is a special hyperbola with the equation

$$k \quad \dots \quad a_{00}+a_{11}x^2+2a_{12}xy+2a_{01}x+2a_{02}y=0, \quad (14)$$

and the quartic $\overline{k}$ is 1-circular, Figure 1.
If $a_{12}=a_{22}=0$, the conic k is a circle

$$k \quad \dots \quad a_{00}+a_{11}x^2+2a_{01}x+2a_{02}y=0, \quad (15)$$

and the quartic $\overline{k}$ is 2-circular.
Therefore, the degree of circularity of the constructed quartic $\overline{k}$ equals the degree of circularity of the conic k.

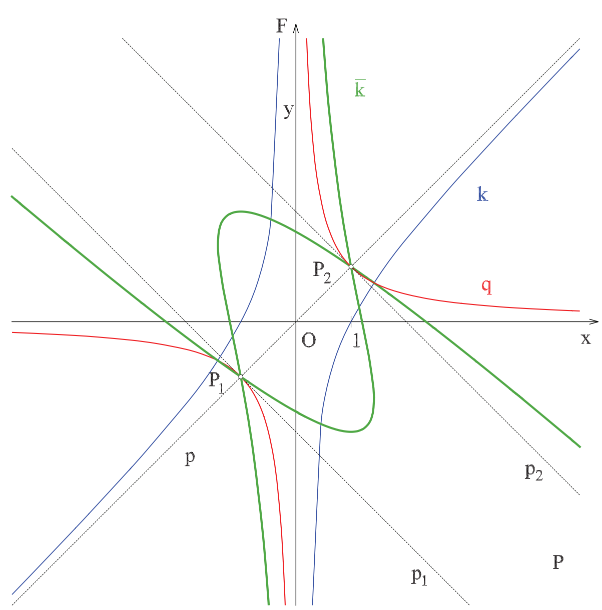

Figure 1

We should determine the tangent of the quartic $\overline{k}$ at the absolute point. Any line through the point $F(0,0,1)$, except the absolute one, has the equation of the form $x = m$, i.e. $x_1 = mx_0$. Its intersections with the quatric $\overline{k}$ are the points coordinates of which satisfy the equation

$$(4a_{01}mp+4a_{11}m^2p+4a_{02}mp^2+(a_{00}+4a_{12})m^2p^2+$$
$$+2a_{01}m^3p^2+a_{11}m^4p^2+4a_{11}+8a_{12}p)x_0^4+$$
$$+2(2a_{01}-2a_{11}m+2a_{02}p+(a_{00}-4a_{12})mp-a_{11}m^3p-$$
$$-a_{02}m^2p^2-a_{12}m^3p^2)x_0^3x_2+$$
$$+(a_{00}+4a_{12}-2a_{01}m+a_{11}m^2+4a_{12}m^2p)x_0^2x_2^2+$$
$$+2(a_{02}-a_{12}m)x_0x_2^3=0.$$

The line is a tangent at the point F if $x_0 = 0$ is double root of the equation above, and that is if and only if $a_{02} - a_{12}m = 0$.

Hence in the case of k being the special hyperbola given by (14), the line $x = \frac{a_{02}}{a_{12}}$ is a tangent of the quartic.

If k is the circle ($a_{12} = 0$) given by the equation (15), there is no line different from the absolute one that touches the quartic at the absolute point.

Any line passing through the pole $P(0,1,p)$ and different from the absolute one has the equation of the form $mx_0 - px_1 + x_2 = 0$, i.e. $y = px + m$. We need to determine m corresponding with a tangent. Its intersections with the quartic satisfy the equation

$$(2a_{02}m^3+(a_{00}-4a_{12})m^2+4(a_{01}+a_{02}p)m+4(a_{11}+2a_{12}p))x_0^4+$$
$$+(-2a_{12}m^3+2(3a_{02}p-a_{01})m^2+4(-a_{11}+a_{00}p)m+$$
$$+8p(a_{01}+a_{02}p))x_0^3x_1+$$
$$+((a_{11}-2a_{12}p)m^2+4p(-a_{01}+a_{02}p)m+4a_{00}p^2)x_0^2x_1^2=0.$$

Obviously, $x_0 = 0$ is a double solution for each m. Therefore, P is a double point of the curve $\overline{k}$. $x_0 = 0$ is a triple solution if $(a_{11} - 2a_{12}p)m^2 + 4p(-a_{01} + a_{02}p)m + 4a_{00}p^2$ equals zero.

It follows that the lines

$$y = px+2p\frac{a_{01}-a_{02}p\pm\sqrt{(a_{01}-a_{02}p)^2-a_{00}(a_{11}-2a_{12}p)}}{a_{11}-2a_{12}p}$$

are the tangents of the quartic at the pole P. The reality of these lines depends on the sign of the expression under the root, and P is a cusp if the expression equals zero. The reality of the points $T_{1,2}(1,t_{1,2},-pt_{1,2})$, at which the conic k intersects the fundamental line p, depends on the same expression, where

$$t_{1,2} = \frac{a_{02}p-a_{01}\pm\sqrt{(a_{02}p-a_{01})^2-a_{00}(a_{11}-2a_{12}p)}}{a_{11}-2a_{12}p}.$$

In the special case when $a_{11} = 2a_{12}p$ and the special hyperbola k intersects the absolute line at the absolute point and the point $A(0,1,-p)$ at which the line is intersected with the polar p of the pole P, one of the tangents of the quartic at the pole P coincides with the absolute line, while the other tangent has the equation $y = px + \frac{a_{00}p}{a_{01}-a_{02}p}$.

If $a_{01} = a_{02}p$, too, the conic k touches the fundamental line p at the point A. Therefore, the point P is a cusp at which both tangents coincide with the absolute line.

Figure 1 displays 1-circular quatric $\overline{k}...x^4 + 3x^3y + 3x^2y^2 + xy^3 - 7x^2 - 6xy - 3y^2 - 8 = 0$ obtained as the inverse image of the special hyperbola $k...1 - x^2 + xy = 0$ by the inversion $\overline{x} = \frac{-x^2-xy+2}{-x+y}$, $\overline{y} = \frac{xy+y^2-2}{-x+y}$ with the fundamental conic $q...xy - 1 = 0$ and the pole $P(0,1,-1)$. The tangent of the quartic at the absolute point is the line $x = 0$, while the absolute line is the tangent at the cusp P. Each of the fundamental lines $p_1...y = -x - 2$, $p_2...y = -x + 2$ intersects k at two different real points. Therefore, fundamental points $P_1(-1,-1)$, $P_2(1,1)$ are the nodes at which tangents have the equations $y = \frac{-9\pm4\sqrt{3}}{3}x + \frac{12\pm4\sqrt{3}}{3}$, $y = \frac{-9\pm4\sqrt{3}}{3}x + \frac{12\mp4\sqrt{3}}{3}$, respectively.

Type (2)

An inversion with the fundamental conic $q...xy - 1 = 0$ and the pole $P(0,1,0)$ is given by

$$\overline{x} = -x+\frac{2}{y}, \quad \overline{y} = y.$$

The image of the point $T(x,y)$ is the point $\overline{T}(-x+\frac{2}{y},y)$. Obviously, $d(T,Q) = d(Q,\overline{T})$, where Q is the intersection of the ray PT with the fundamental conic q, Figure 2.

The image of the conic k with the equation (13) is the quartic $\overline{k}$:

$$a_{11}x^2y^2-2a_{12}xy^3+a_{22}y^4-2a_{01}xy^2+2a_{02}y^3-$$
$$-4a_{11}xy+(a_{00}+4a_{12})y^2-4a_{01}y+4a_{11}=0.$$

The quartic $\overline{k}$ is circular if and only if $a_{22} = 0$, i.e. if and only if k is circular.

The conic k intersects the absolute line $x_0 = 0$ at the points $(0,-2a_{12},a_{11})$ and $(0,0,1)$, while the quartic $\overline{k}$ intersects it at the points coordinates of which satisfy the equation

$$x_1x_2^2(a_{11}x_1-2a_{12}x_2)=0.$$

It is obvious that $x_1 = 0$ is a solution, hence $F(0,0,1)$ is an intersection. Since $x_2 = 0$ is a double solution, $(0,1,0)$ is double joint point of the quartic with the absolute line. If $a_{12} \neq 0$, the fourth intersection is the point $(0,2a_{12},a_{11})$. If $a_{12} = 0$, then $a_{11} \neq 0$ (since otherwise the conic k is a line) and the point $(0,0,1)$ is the fourth intersection. In that case

k is a circle and $\overline{k}$ touches the absolute line at the absolute point.

It is easy to prove that $P(0,1,0)$ is a double point of the quartic at which both tangents are identical with the fundamental line $p\,[0,0,1]$.

In order to make the quartic $\overline{k}$ circular, it is necessary for conic k to be circular. More precisely, $\overline{k}$ is 1-circular if k is 1- circular, and $\overline{k}$ is 2-circular if k is 2- circular. The tangent of the conic k at the point F is the line $a_{02}x_0 + a_{12}x_1 = 0$, while the tangent of the quatric $\overline{k}$ at that point is the line $a_{02}x_0 - a_{12}x_1 = 0$.

The inversion $\overline{x} = -x + \dfrac{2}{y}, \overline{y} = y,$ maps the circle k with the equation $x^2 - y = 0$ into the 2-circular quartic $\overline{k}...x^2y^2 - y^3 + -4xy + 4 = 0$ touching the absolute line at the absolute point. The quartic has a cusp with the tangent $y = 0$ at the pole of the inversion, Figure 2.

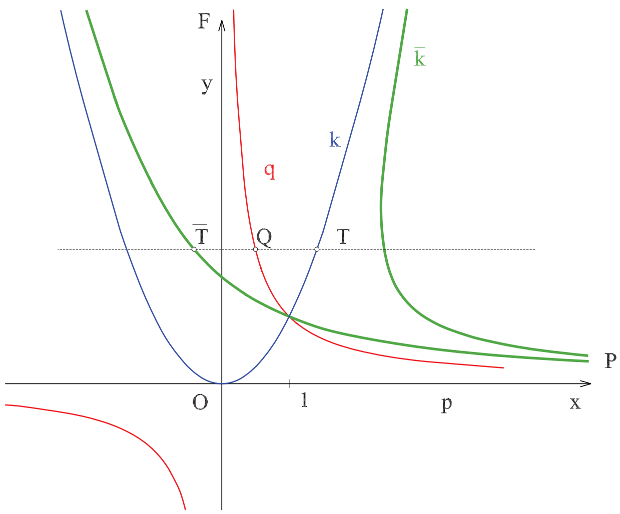

Figure 2

Type (3)

Let us now consider the inversion

$$\overline{x} = x, \quad \overline{y} = -y + \frac{2}{x}$$

with the fundamental conic $q...xy - 1 = 0$. The absolute point $F(0,0,1)$ is the pole and its polar line $x = 0$ is the fundamental line of the inversion. The point $(0,1,0)$ is another intersection of the fundamental conic with the absolute line.

The point $\overline{T}(x, -y - \dfrac{2a}{x})$ is the inverse image of the point $T(x,y)$. Clearly, $s(T,Q) = s(Q,\overline{T})$, where Q is the intersection of the ray FT with the fundamental conic q, Figure 3.

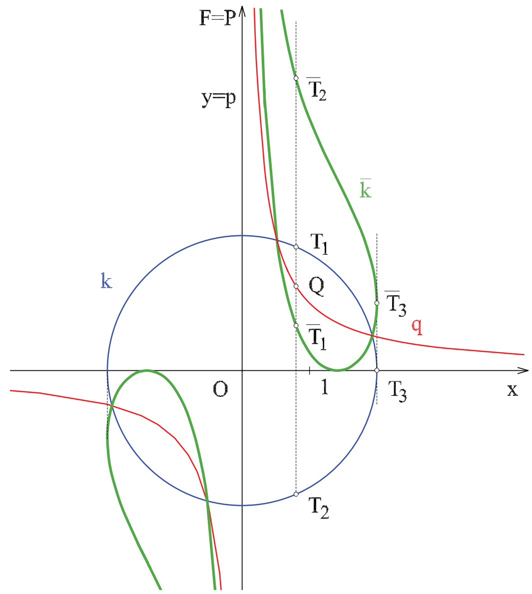

Figure 3

Let the conic k be given with the equation (13). The equation of the constructed quartic $\overline{k}$ is then

$$a_{11}x^4 - 2a_{12}x^3y + a_{22}x^2y^2 + 2a_{01}x^3 - 2a_{02}x^2y + \\ + (a_{00} + 4a_{12})x^2 - 4a_{22}xy + 4a_{02}x + 4a_{22} = 0.$$

The coefficient a_{22} should not equal zero since in that case the conic k would pass through the absolute point (the pole) and the quartic $\overline{k}$ would split into the line with the equation $x = 0$ and a cubic.

The intersections of the conic k with the absolute line are points $\left(0, -a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}, a_{11}\right)$ if $a_{11} \neq 0$ and $(0,1,0)$, $(0, a_{22}, -2a_{12})$ if $a_{11} = 0$. If $a_{12}^2 = a_{11}a_{22}$, the conic k touches the absolute line.

Let us now determine the intersections of the quartic $\overline{k}$ and the absolute line $x_0 = 0$. Their projective coordinates should satisfy the equation

$$a_{11}x_1^4 - 2a_{12}x_1^3x_2 + a_{22}x_1^2x_2^2 = 0.$$

It is evident that $x_1 = 0$ is a double solution of the equation. Therefore, the point $F\,(0,0,1)$ is their two times counted common point. Since $a_{22} \neq 0$, $x_1 = 0$ cannot be a triple solution.

If $a_{11} \neq 0$, the other two intersections are points $\left(0, a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}, a_{11}\right)$.

If $a_{11} = 0$, the intersections of the quartic with the absolute line are point $(0,0,1)$ counted twice and points $(0,1,0)$, $(0, a_{22}, 2a_{12})$. If a_{12} equals zero, $\overline{k}$ touches the absolute line f.

Our next goal is to determine the equation of the tangent of the quartic at the absolute point.
Any line t through F (except the absolute one) is given by equation $x_1 = mx_0$, i.e. $x = m$. The coordinates of its intersections with the quartic satisfy the equation

$$x_0^2[(a_{11}m^4 + 2a_{01}m^3 + (a_{11} + 4a_{12})m^2 + 4a_{02}m + 4a_{22}^2)x_0^2 + \\ + (-2a_{12}m^3 - 2a_{02}m^2 - 4a_{22}m)x_0x_2 + a_{22}m^2x_2^2] = 0.$$

Due to the fact that $x_0 = 0$ is a double solution for each m we conclude that $(0,0,1)$ is a double point of the curve.
$x_0 = 0$ is a triple solution if m equals zero. In that case the solutions are given by

$$x_0^2\left[4a^2a_{22}^2x_0^2\right] = 0.$$

Thus $x_0 = 0$ is a quadruple solution.
Consequently, the line p $(x_1 = 0)$ is two times counted tangent of the quartic $\overline{k}$ at the absolute point F.

Figure 3 displays an inverse image $\overline{k}$ of the conic $k...x^2 + y^2 - 4 = 0$ with respect to the fundamental conic $q...xy - 1 = 0$. The inversion is given by the equations $\overline{x} = x$, $\overline{y} = -y + \frac{2}{x}$, and the quartic $\overline{k}$ by the equation $x^4 + x^2y^2 - 4x^2 - 4xy + 4 = 0$. The quartic intersects the absolute line at the double point F at which both tangents coincide with the y-axis and the pair of conjugate imaginary points $(0,1,i)$ and $(0,1,-i)$.

Type (4)

If the circle $q...x^2 - y = 0$ is the fundamental conic and the point $P(0,1,0)$ is the pole of inversion

$$\overline{x} = \frac{y}{x}, \quad \overline{y} = y,$$

the conic k given by the equation (13) is mapped by this transformation into the quartic

$$a_{22}x^2y^2 + 2a_{02}x^2y + 2a_{12}xy^2 + a_{00}x^2 + 2a_{01}xy + a_{11}y^2 = 0. \tag{16}$$

Coefficients a_{00}, a_{11}, a_{22} must not equal zero, since otherwise k would pass through some of the fundamental points.
The intersection points of the conic k with the fundamental lines $x_0 = 0$, $x_1 = 0$, $x_2 = 0$ are the points $(0, a_{22}, -a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}})$, $(a_{22}, 0, -a_{02} \pm \sqrt{a_{02}^2 - a_{00}a_{22}})$, $(a_{11}, -a_{01} \pm \sqrt{a_{01}^2 - a_{00}a_{11}}, 0)$, respectively.
Since $F(0,0,1)$, $P(0,1,0)$ are the intersection points of the quartic (16) with the absolute line and each of them two times counted, an inversion of the type (4) produces 2-circular quartics, Figure 4.

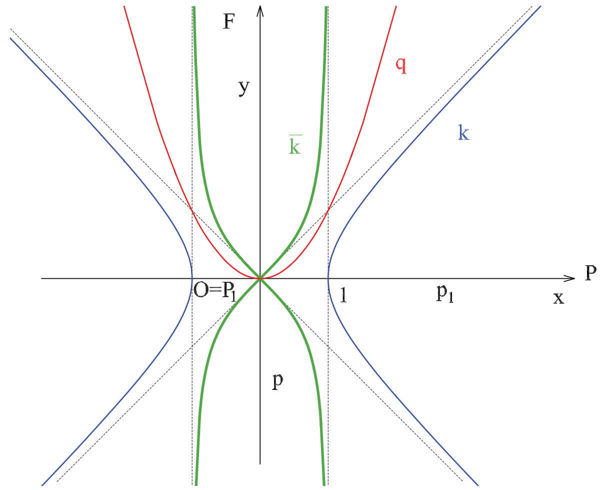

Figure 4

Its tangents at the double point $F(0,0,1)$ are given by

$$x = \frac{-2a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{2a_{22}},$$

at the pole $P(0,1,0)$ by

$$y = \frac{-a_{02} \pm \sqrt{a_{02}^2 - a_{00}a_{22}}}{a_{22}},$$

and at the third fundamental point $P_1(1,0,0)$ by

$$y = \frac{-a_{01} \pm \sqrt{a_{01}^2 - a_{00}a_{11}}}{a_{11}}x.$$

Obviously, the reality of the tangents and the type of the double point depend on the reality of the intersections of the conic k with the corresponding fundamental line.

The conic $k...1 - x^2 + y^2 = 0$ and its inverse image $\overline{k}...x^2y^2 + x^2 - y^2 = 0$ obtained by the inversion $\overline{x} = \frac{y}{x}$, $\overline{y} = y$ are presented in Figure 4. The quartic possesses a node at the absolute point and an isolated double point at $(0,1,0)$. Its tangents at the point F are lines $x = \pm 1$, at the pole P the lines $y = \pm i$ and at the fundamental point P_1 the lines $y = \pm x$.

Type (5)

Let us now suppose that the circle $q...x^2 - y = 0$ is the fundamental conic and the absolute point $F(0,0,1)$ is the pole of an inversion. The inversion is given by

$$\overline{x} = x, \quad \overline{y} = 2x^2 - y.$$

This type of inversion is analogous with the ordinary inversion in the Euclidean plane.

The inverse image of the point $T(x,y)$ is the point $\overline{T}(x, 2x^2-y)$. So, $s(T,Q) = s(Q,\overline{T})$, where Q is the intersection of the ray FT with the fundamental conic, Figure 5.

The image of the conic (13) is the quartic $\overline{k}$

$$4a_{22}x^4 + 4a_{12}x^3 - 4a_{22}x^2y + (a_{11}+4a_{02})x^2 - 2a_{12}xy +$$

$$+a_{22}y^2 + 2a_{01}x + 4a_{02}y + a_{00} = 0.$$

Obviously, $a_{22} = 0$ is not allowed. In that case the conic (13) would pass through the pole and the constructed quartic $\overline{k}$ would split into the absolute line and a cubic.

An easy computation shows that the absolute point is a double point of the curve at which both tangents coincide with the absolute line. Therefore, $\overline{k}$ is an entirely circular quartic.

The quartics $4x^4 - 4x^2y - x^2 + y^2 + 1 = 0$, $4x^4 - 4x^2y + x^2 + y^2 - 4 = 0$, $4x^4 - 4x^2y + y^2 - x = 0$ in Figure 5 possess a node, an isolated double point, or a cusp at the absolute point, respectively. They are obtained as images of the conics $x^2 - y^2 = 1$, $x^2 + y^2 = 4$, $y^2 = x$ by inversion $\overline{x} = x, \quad \overline{y} = 2x^2 - y.$ □

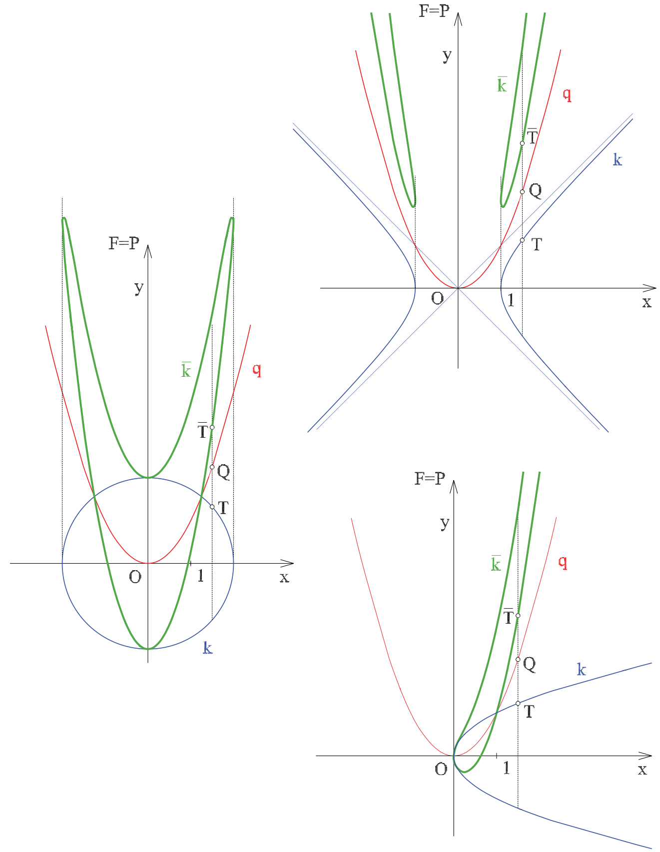

Figure 5

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