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Trigonometric Proof of Steiner-Lehmus Theorem in Hyperbolic Geometry

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ABSTRACT

In this study, we give a trigonometric proof of the Steiner-Lehmus Theorem in hyperbolic geometry.

Key words: hyperbolic geometry, hyperbolic triangle

MSC 2000: 51K05, 51K99

Trigonometrijski dokaz Steiner-Lehmusovog teorema u hiperboličkoj geometriji

SAŽETAK

U ovom radu dajemo trigonometrijski dokaz Steiner-Lehmusovog teorema u hiperboličkoj geometriji.

Ključne riječi: hiperbolička geometrije, hiperbolički trokut

1 Introduction

Elementary hyperbolic geometry was born in 1903 when Hilbert provided, using the end-calculus to introduce coordinates, a first-order axiomatization for it by adding to the axioms for plane absolute geometry a *hyperbolic parallel axiom* stating that "Through any point P not lying on a line l there are two rays r_1 and r_2, not belonging to the same line, which do not intersect l, and such that every ray through P contained in the angle formed by r_1 and r_2 does intersect l" [2]. The hyperbolic geometry is a non-euclidean geometry. Here in this study, we give hyperbolic version of Steiner-Lehmus theorem. The well-known Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles [1].

Lemma 1 *(Sines Theorem) In the hyperbolic triangle ABC let α,β,γ denote at A,B,C and a,b,c denote the hyperbolic lengths of the sides opposite A,B,C, respectively, then*

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c} \tag{1}$$

[3, p.125].

2 Main results

Theorem 2 *If the internal angle bisectors of two angles of a triangle are equal, then the triangle isn't isosceles.*

Proof. Let BB' and CC' be the respective internal angle bisectors of angles B and C in triangle ABC, and let a,b and c denote the sidelengths in the standard order. As shown in Figure 1, we set

$B = 2\beta,\ C = 2\gamma,\ u = AB',\ \ U = B'C,\ v = AC',\ V = C'B.$

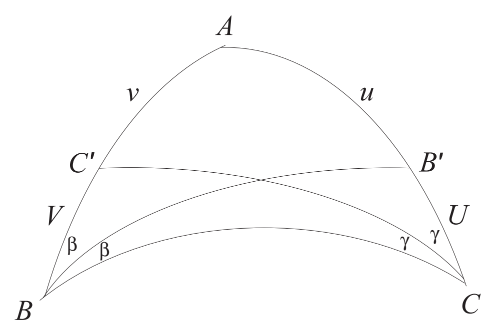

Figure 1

If we use the sines theorem in the triangles $ABC, BB'C, BB'A, BCC', ACC'$
respectively (See Lemma 1), then

$$\frac{\sin A}{\sinh a} = \frac{\sin 2\gamma}{\sinh c} = \frac{\sin 2\beta}{\sinh b} \tag{2}$$

$$\frac{\sinh U}{\sinh BB'} = \frac{\sin\beta}{\sin 2\gamma} \quad (3)$$

$$\frac{\sinh u}{\sinh BB'} = \frac{\sin\beta}{\sin A} \quad (4)$$

$$\frac{\sinh V}{\sinh CC'} = \frac{\sin\gamma}{\sin 2\beta} \quad (5)$$

$$\frac{\sinh v}{\sinh CC'} = \frac{\sin\gamma}{\sin A} \quad (6)$$

If ratios the equations (3) and (4) among themselves, respectively, then

$$\frac{\sinh U}{\sinh u} = \frac{\sin A}{\sin 2\gamma} \quad (7)$$

If ratios the equations (5) and (6) among themselves, respectively, then

$$\frac{\sinh V}{\sinh v} = \frac{\sin A}{\sin 2\beta} \quad (8)$$

Let BB'=CC' and $C > B$ (and $c > b$). Here, there are two cases.

$$\frac{\sinh b}{\sinh u} > \frac{\sinh c}{\sinh v}, \quad \frac{\sinh b}{\sinh u} < \frac{\sinh c}{\sinh v} \quad (9)$$

$$\frac{\sinh b}{\sinh u} : \frac{\sinh c}{\sinh v} = \frac{\sinh b}{\sinh c} \cdot \frac{\sinh v}{\sinh u}$$

If we put the values of $\frac{\sinh b}{\sinh c} = \frac{\sin 2\beta}{\sin 2\gamma}$ in the equation (2) and $\frac{\sinh v}{\sinh u} = \frac{\sin\gamma}{\sin\beta}$ (see (4), (6)) then

$$\begin{aligned}\frac{\sinh b}{\sinh u} : \frac{\sinh c}{\sinh v} &= \frac{\sin 2\beta}{\sin 2\gamma} \cdot \frac{\sin\gamma}{\sin\beta} \\ &= \frac{2\sin\beta\cos\beta}{2\sin\gamma\cos\gamma} \cdot \frac{\sin\gamma}{\sin\beta} \\ &= \frac{\cos\beta}{\cos\gamma} > 1 \quad (10)\end{aligned}$$

Clearly (11) lead to the contradiction $(C > B)$. On the other hand,

$$\begin{aligned}\frac{\sinh b}{\sinh u} - \frac{\sinh c}{\sinh v} &= \frac{\sinh(U+u)}{\sinh u} - \frac{\sinh(V+v)}{\sinh v} \\ &= \frac{\sinh U\cosh u + \cosh U\sinh u}{\sinh u} \\ &\quad - \frac{\sinh V\cosh v + \cosh V\sinh v}{\sinh v} \\ &= \frac{\sinh U}{\sinh u}\cosh u + \cosh U \\ &\quad - \frac{\sinh V}{\sinh v}\cosh v - \cosh V \quad (11)\end{aligned}$$

If we put the values of $\frac{\sinh U}{\sinh u} = \frac{\sin A}{\sin 2\gamma}, \frac{\sinh V}{\sinh v} = \frac{\sin A}{\sin 2\beta}$ in the equations (7) and (8), then

$$\begin{aligned}\frac{\sinh b}{\sinh u} - \frac{\sinh c}{\sinh v} &= \frac{\sin A}{\sin 2\gamma}\cosh u + \cosh U \\ &\quad - \frac{\sin A}{\sin 2\beta}\cosh v - \cosh V\end{aligned}$$

If we put the values of $\frac{\sinh a}{\sinh c} = \frac{\sin A}{\sin 2\gamma}, \frac{\sinh a}{\sinh b} = \frac{\sin A}{\sin 2\beta}$ in the equation (2), then

$$\begin{aligned}\frac{\sinh b}{\sinh u} - \frac{\sinh c}{\sinh v} &= \frac{\sinh a}{\sinh c}\cosh u + \cosh U \\ &\quad - \frac{\sinh a}{\sinh b}\cosh v - \cosh V < 0\end{aligned}$$

$$\frac{\sinh a}{\sinh c}\cosh u + \cosh U < \frac{\sinh a}{\sinh b}\cosh v - \cosh V$$

Because of $C > B$, $V > U$, $v > u$. Hence, $\sinh b < \sinh c$ (and $c > b$). Consequently, the case $C > B$ is satisfied while BB'=CC' . The triangle ABC can't isosceles.

□

References

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