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Cycloidal Cyclical Surfaces

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ABSTRACT

The paper presents a family of cycloidal cyclical surfaces, which are created by a movement of a circle alongside the special spatial cycloidal curve, where the circle is located in the curve normal plane and its centre is on this curve. The spatial cycloidal curve can be created by simultaneous revolution of a point about three different axes ${}^3o, {}^2o$ and ${}^1o$ in the space. The form of the cycloidal curve and also of the cycloidal cyclical surface depends on the relative position of the three axes of revolutions, on multiples of angular velocities and orientations of separate revolutions. The analytic representation, the classification of surfaces and some of their geometric properties are derived.

Key words: revolution, translation, angular velocity, cycloidal curve, cyclical surface

MSC 2000: 51A04, 52A05, 14J26

Cikloidne kružne plohe

SAŽETAK

Članak predstavlja porodicu cikloidnih kružnih ploha koje nastaju gibanjem kružnice duž posebne prostorne cikloide, pri čemu kružnica leži u normalnoj ravnini te krivulje, a središte joj je na krivulji. Prostorna cikloida može nastati istodobnom rotacijom točke oko tri različite osi ${}^1o$, ${}^2o$, ${}^3o$, u prostoru. Oblik prostorne cikloide, isto kao i oblik cikloidne kružne plohe, ovisi o međusobnom položaju rotacijskih osi, kratnosti kutnih brzina i orijentacijama pojedinih rotacija. Izvedena je analitička reprezentacija tih ploha, njihova klasifikacija te neka od geometrijskih svojstava.

Ključne riječi: rotacija, translacija, kutna brzina, prostorna cikloida, cikloidna ploha

1 Introduction

A cycloidal cyclical surface can be created by movement of a circle alongside a spatial cycloidal curve. The circle is located in the normal plane of the curve and its centre is on this curve.

The spatial cycloidal curve can be created by simultaneous revolutions of a point about three different lines, axis ${}^3o, {}^2o$ and ${}^1o$. Trajectories of the point P, which revolves about single axes of revolutions are circles ${}^1k$, ${}^2k$, ${}^3k$ located in the planes perpendicular to the axes of revolution ${}^1o$, ${}^2o$, ${}^3o$. With respect to the relative position of axes of revolutions these circles do not necessarily lie in one plane. Form of the spatial cycloidal curve is dependent on the relative position of the axes of revolutions, on the orientations of the single revolutions and on their angular velocities, and also on the position of the revolving point P with respect to the axes of revolutions. Some forms of these curves are studied in [1], [2].

In the next section there is described the creation of one type of the cycloidal cyclical surface for particular relative position of the axes of revolutions (Fig. 1).

Let axis ${}^1o$ be fixed and ${}^1o = z$ in the Cartesian coordinate system (O,x,y,z). Axis ${}^2o$ skew to ${}^1o$, ${}^2o/{}^1o$ creates a 1-sheet hyperboloid of revolutional by its revolutionary movement about axis ${}^1o$ with angular velocity $w_1 = v$ and with orientation determined by parameter q_1 (Fig. 2). Axis ${}^3o$ that is intersect to ${}^2o$, ${}^3o \times {}^2o$, creates a conical surface of revolution by revolution about axis ${}^2o$ with angular velocity $w_2 = m_1 w_1 = m_1 v$ and with orientation determined by parameter q_2 (Fig. 3). Axis ${}^3o$ parallel to ${}^1o$, ${}^3o \parallel {}^1o$, creates a cylindrical surface of revolutional by revolving about axis ${}^1o$ (Fig. 4). In Fig. 5 there are displayed all three surfaces of revolution together. Axis ${}^3o$, which revolves about two axis simultaneously, about axis ${}^2o$ and axis ${}^1o$, creates a two-axial suface of revolution of Euler type - composite ruled, as decribed in [3], (Fig. 6). This surface has six identical branches, because axis ${}^3o$ revolves about axis ${}^2o$ with angular velocity, which is 6-multiple of angular velocity of revolution of the axis ${}^2o$ about axis ${}^1o$.

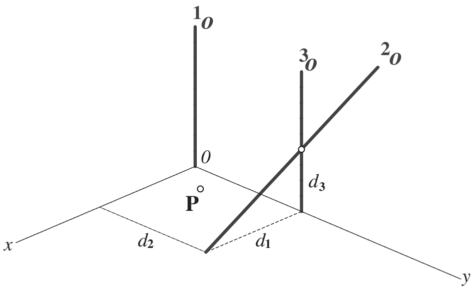
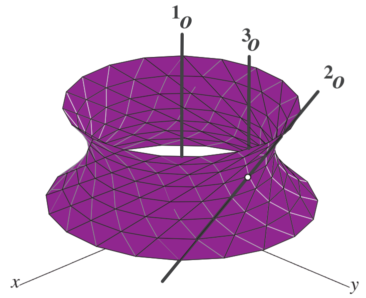


Figure 1



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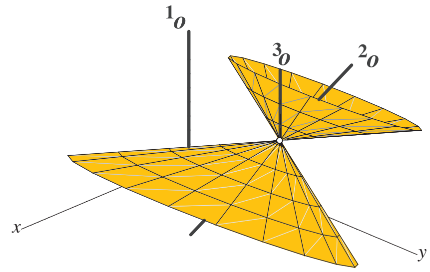

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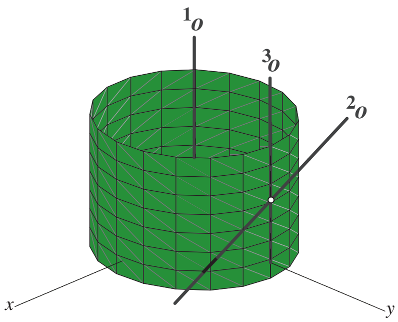

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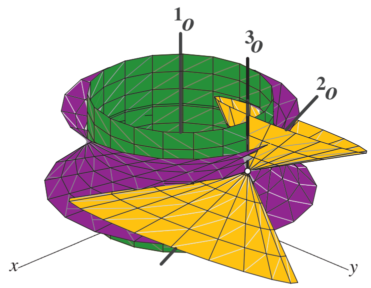

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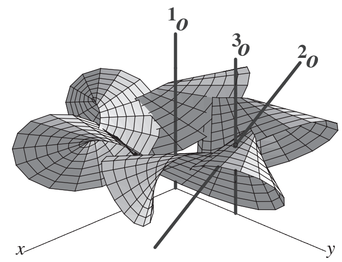


Figure 6

The point P revolves about axis ^3o with angular velocity $w_3 = m_2 w_2 = m_2 m_1 v$ with orientation determined by parameter q_3, where perameters $q_1, q_2, q_3 = \pm 1$ (if $q_i = +1$, $i = 1, 2, 3$ then revolution is right-handed, if $q_i = -1$ then revolution is left-handed). Trajectory of the point P movement created by its revolution about axis ^1o is circle ^1k (Fig. 7), the circle ^2k is the trajectory of the point P movement about axis ^2o (Fig. 8) and the circle ^3k is the trajectory of the point P movement about axis ^3o (Fig. 9).

The curve k as trajectory of the point P composite revolutionary movement is created by rolling of the circle ^3k on the circle ^2k, which rolls on the circle ^1k simultaneously (Fig. 10). Form of this spatial cycloidal curve is dependent on the relative position of the axes of revolutions ^1o, ^2o, ^3o, on the orientations of the single revolutions and on their angular velocities, and also on the position of the revolving point P with respect to three axes of revolutions.

The cycloidal cyclical surface can be created by moving a circle alongside the curve k, while the circle lies always in the normal plane of the curve k and its centre is on the curve (Fig. 11, Fig. 12-view from above).

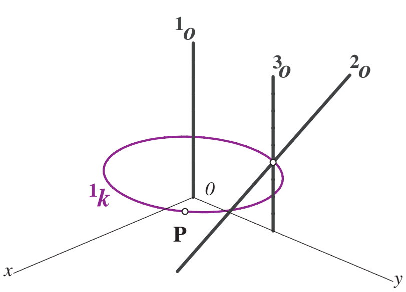
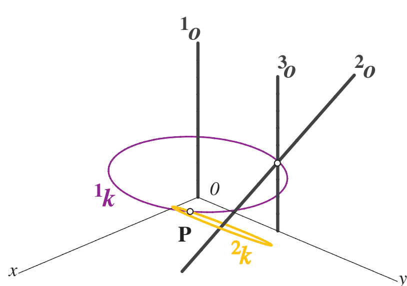


Figure 7



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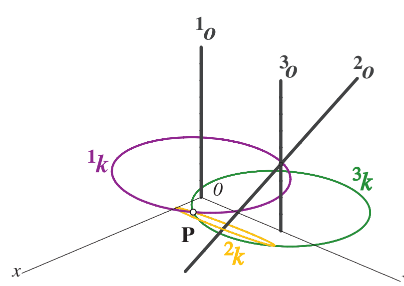


Figure 9

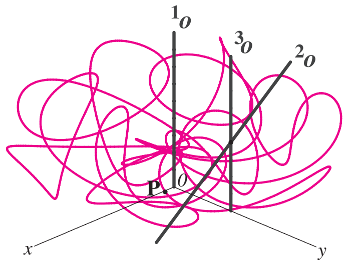


Figure 10

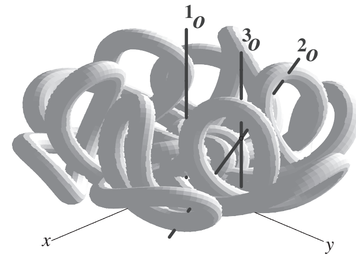


Figure 11

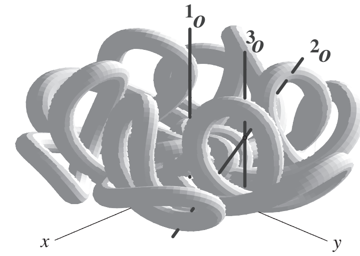


Figure 12

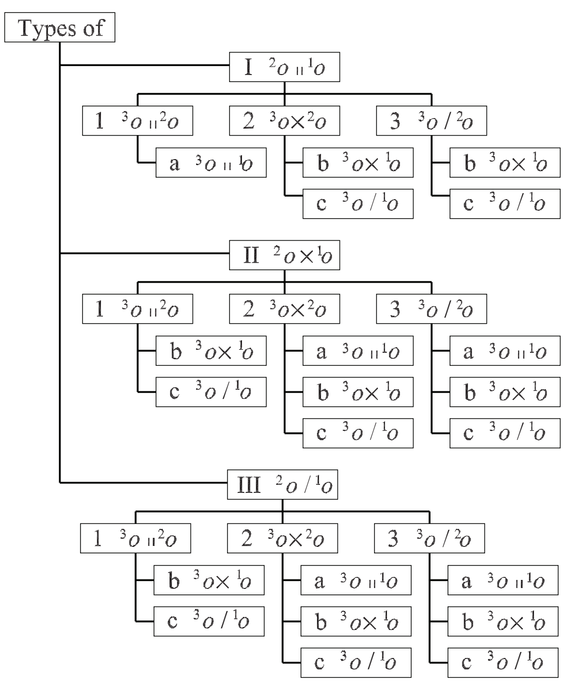
2 Classification of Family of Cycloidal Cyclical Surfaces

The classification of the family of cycloidal cyclical surfaces can be done according to the relative position of axes of revolutions ${}^3o$, ${}^2o$ and ${}^1o$, which may be parallel, intersect or skew. The distribution of surfaces within the family is illustrated in the next Graph 1.

Cycloidal cyclical surfaces are distributed in the first level into the three types I, II, III with respect to the relative position of the axes ${}^2o$ and ${}^1o$.

Surfaces in all three subclasses I, II, III are distributed in the second level into the three types 1, 2, 3 with respect to the relative position of the axes ${}^3o$ and ${}^2o$.

Finally, in the third level, each subgroup of types 1, 2, 3 can be further classified with respect to the relative position of the axes ${}^3o$ and ${}^1o$ into types a, b or c.



Graph 1

3 Analytical Representation of Cycloidal Cyclical Surface

Let us derive the vector function of the cycloidal cyclical surface for one particular position of the axes of revolutions and for one special position of the point P with respect of these axes, particularly for the surface of type III 2 a. Derivation of the vector function of all other types of surfaces is analogous.

Let the axes of revolution be in the following relative positions: ${}^1o = z, {}^2o / {}^1o$ (skew), ${}^3o \times {}^2o$ (intersect), ${}^3o \parallel {}^1o$ (parallel). The position of axis ${}^2o$ in the plane parallel to the coordinate plane (xz), ${}^2o \subset v'$, $v' \parallel v$, is determined by parameters d_1, d_2, d_3, which determine the position of the intersection points of axis ${}^2o$ with the coordinate planes (xy) and (yz) in the Cartesian coordinate system (O,x,y,z). Then $\alpha = \arctan \dfrac{d_3}{d_1}$ is the angle formed by axis ${}^2o$ with the coordinate plane (xy) and the position of axis ${}^3o$ is determined by parameter d_2, which is the distance between axes ${}^3o$ and ${}^1o$, (Fig. 1).

The revolution about axis ${}^1o$ with angular velocity $w_1 = v$, in the direction determined by parameter $q_1 = \pm 1$, is represented by matrix

$$\mathbf{T}_1\left(w_1(v), q_1\right) = \mathbf{T}_z\left(w_1, q_1\right), \tag{1}$$

where the matrix $\mathbf{T}_z\left(w_1, q_1\right)$ represents revolution about axis z by angle w_1 in the direction determined by parameter q_1 and for $i = 1$ it can be derived from (2)

$$\mathbf{T}_z\left(w_i, q_i\right) = \begin{pmatrix} \cos w_i & q_i \sin w_i & 0 & 0 \\ -q_i \sin w_i & \cos w_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The revolution about axis ${}^2o$ with angular velocity $w_2 = m_1 w_1$, in the direction determined by parameter $q_2 = \pm 1$, is represented by matrix

$$\begin{aligned} \mathbf{T}_2\left(w_2(v), q_2\right) \quad = \quad & \mathbf{T}\left(-d_1, -d_2, 0\right) \cdot \mathbf{T}_y\left(\alpha, +1\right) \cdot \mathbf{T}_x\left(w_2, q_2\right) \cdot \\ & \cdot \mathbf{T}_y\left(\alpha, -1\right) \cdot \mathbf{T}\left(d_1, d_2, 0\right), \end{aligned} \tag{3}$$

where the matrix $\mathbf{T}_y(\alpha,\pm 1)$ expressed in (4) represents the revolution about axis y by angle α in positive or negative direction, matrix $\mathbf{T}_x(w_2,q_2)$ represents revolution about axis x by angle $w_2 = m_1 v$ in the direction determined by parameter q_2 (5), matrix $\mathbf{T}(\pm d_1,\pm d_2,0)$ represents translation with translation vector $(\pm d_1,\pm d_2,0)$ in (6).

$$\mathbf{T}_y(\alpha,\pm 1) = \begin{pmatrix} \cos\alpha & 0 & \pm\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \mp\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (4)$$

$$\mathbf{T}_x(w_2,q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos w_2 & q_2\sin w_2 & 0 \\ 0 & -q_2\sin w_2 & \cos w_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (5)$$

$$\mathbf{T}(\pm d_i,\pm d_j,\pm d_k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm d_i & \pm d_j & \pm d_k & 1 \end{pmatrix}. \qquad (6)$$

The revolutionary movement of the point $P = (x_0,y_0,z_0,1)$ about axis ${}^3o$ with angular velocity $w_3 = m_2 w_2 = m_2 m_1 v$ and in the direction determined by parameter $q_3 = \pm 1$ is represented by matrix

$$\mathbf{T}_3(w_3(v),q_3) = \mathbf{T}(0,-d_2,0)\cdot\mathbf{T}_z(w_3,q_3)\cdot\mathbf{T}(0,d_2,0), \quad (7)$$

where matrix $\mathbf{T}(0,\pm d_2,0)$ in (6) represents translation with translation vector $(0,\pm d_2,0)$, and matrix $\mathbf{T}_z(w_3,q_3)$ is for $i=3$ represented by (2).

A vector function of the cycloidal curve k created by simultaneous revolution of the point $P = (x_0,y_0,z_0,1)$ about axes ${}^3o$, ${}^2o$ and ${}^1o$ is

$$\mathbf{r}(v) = \mathbf{R}\cdot\mathbf{T}_3(w_3(v),q_3)\cdot\mathbf{T}_2(w_2(v),q_2)\cdot\mathbf{T}_1(w_1(v),q_1), \quad v\in\langle 0,2\pi\rangle, \quad (8)$$

where $\mathbf{T}_3(w_3(v),q_3)$, $\mathbf{T}_2(w_2(v),q_2)$, $\mathbf{T}_1(w_1(v),q_1)$ are matrices of particular revolutions expressed in (6), (3), (1) and $\mathbf{R}(x_0,y_0,z_0,1)$ is the positioning vector of the point P.

Let the new coordinate system be defined at the arbitrary regular point $P\in k$, identical to the trihedron (P,t,n,b) determined by tangent t, basic normal n and binormal b to the curve k with unit vectors expressed in (9).

$$\mathbf{t}(v) = (t_1,t_2,t_3) = \frac{\mathbf{r}'(v)}{|\mathbf{r}'(v)|},$$
$$\mathbf{n}(v) = (n_1,n_2,n_3) = \frac{\mathbf{r}''(v)}{|\mathbf{r}''(v)|},$$
$$\mathbf{b}(v) = (b_1,b_2,b_3) = \mathbf{t}(v)\times\mathbf{n}(v). \qquad (9)$$

The cycloidal cyclical surface can be created by movement of the circle $c' = (P,r)$ with centre P and radius r alongside the curve k so that the circle is located in the normal plane of the curve in the point $P\in k$, which is determined by basic normal n and binormal b to this curve. Vector function of this surface is

$$\mathbf{P}(u,v) = \mathbf{r}(v) + \mathbf{c}(u)\cdot\mathbf{M}(v), \quad u\in\langle 0,2\pi\rangle, v\in\langle 0,2\pi\rangle, \quad (10)$$

where $\mathbf{r}(v)$ is vector function of the cycloidal curve k expressed in (8), $\mathbf{c}(u) = (0, r\cos u, r\sin u)$ is vector function of the circle c with centre in the origin of the coordinate system (O,x,y,z) and radius r, located in the coordinate plane (yz). Matrix

$$\mathbf{M}(v) = \begin{pmatrix} t_1 & t_2 & t_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (11)$$

transforms the circle c on the circle c' with the centre in the origin of the coordinate system (P,t,n,b) and radius r located in the normal plane (nb) of the cycloidal curve k in the point P. Entries of the first row of this matrix are coordinates of the unit vector of the tangent t, entries of the second row are coordinates of the unit vector of the basic normal n and the entries of the third row are coordinates of the unit vector of the binormal b expressed in equations (9), (Fig. 13).

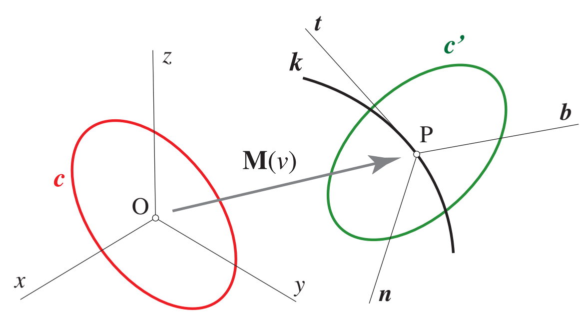


Figure 13

The form of the cycloidal curve k and created cycloidal cyclical surface changes in dependence on the relative position of the axes of revolutions that are determined by parameters $d_i, i = 1,2,3$. Surface has m_1 identical external branches, where every branch has m_2 identical internal branches. Point P revolves about axis ${}^3o$ with angular velocity w_3, which is m_2-multiple of the angular velocity w_2 of the revolution about axis ${}^2o$ and w_2 is m_1-multiple of the angular velocity w_1 of the revolution about axis ${}^1o$. Many different forms of cycloidal cyclical surfaces can be created by change of their determining parameters.

Variations of the surface form are shown by change of some parameters of the surface of type III 2 a displayed in Fig. 11. Presented surface is determined by parameters $m_1 = 6, m_2 = 3, q_1 = q_2 = q_3 = +1$, then it has 6 external and 3 internal branches, and all three revolutions are right-handed. Surface in Fig. 14 is determined by parameter $m_2 = 6$, in Fig. 15 by $m_1 = 4, m_2 = 2$, and in Fig. 16 by $m_1 = 3, m_2 = 4$, and there are changes in the number of external and internal branches.

In Fig. 17 depicted surface is determined by parameters $m_1 = 3, m_2 = 3, q_2 = -1, q_3 = +1$, in Fig. 18 by $q_2 = +1$ and $q_3 = -1$ and in Fig. 19 by $q_2 = -1$ and $q_3 = -1$, then there are changes in the orientations of particular revolutions.

In Fig. 20, there is presented surface with parameters identical to parameters of surface in Fig. 19, but the position of the point $P(x_0, y_0, z_0, 1)$ was changed from $(\frac{d_1}{2}, \frac{d_2}{2}, \frac{d_3}{3}, 1)$ to $(\frac{d_1}{2}, 3\frac{d_2}{2}, \frac{d_3}{3}, 1)$ and in Fig. 21 to $(d_1, d_2, d_3, 1)$. Surface with parameters $m_1 = 6, m_2 = 4, q_2 = +1, q_3 = +1$ is illustrated in Fig. 22, but relative position of the axes has been changed to position ${}^2o/{}^1o, {}^2o \perp {}^1 o$ and ${}^2o \perp {}^3 o$. Surfaces in Figures 14 - 21 are displayed by view from above, because in these views the changes of parameters are more illustrative.

As the conclusion it can be summarised that the presented family of cycloidal cyclical surfaces serves as an endlessly rich source of inspiration for artistic and design purposes. Their unusually complex forms obtained in a relatively simple way of composite spatial transformation. Special skew symmetry and harmonical periodicity reflect their simplistic generating priciple based on the naturally basic movement of our universe, revolution about an axis in the space. Several surface types from the presented classification frame are displayed in the Fig. 23 without commentary, as the most persuasive evidence.

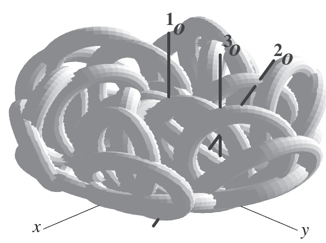
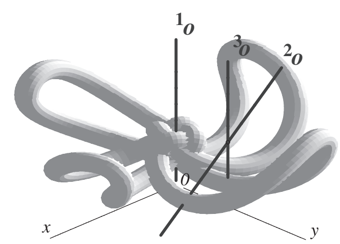


Figure 14



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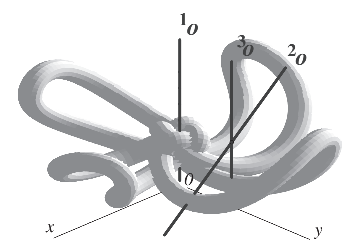

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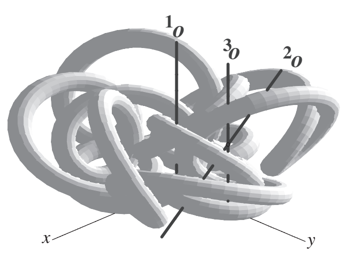
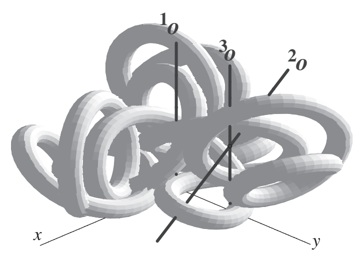


Figure 17



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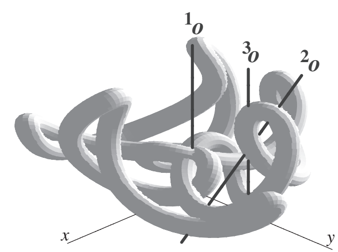

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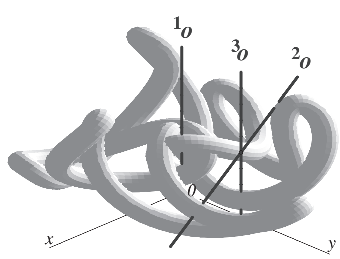

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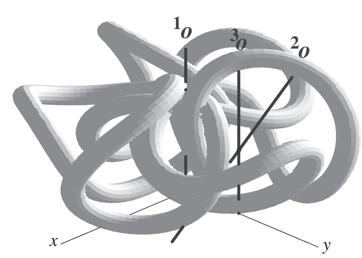

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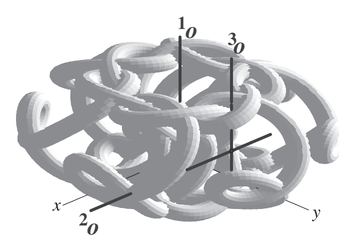

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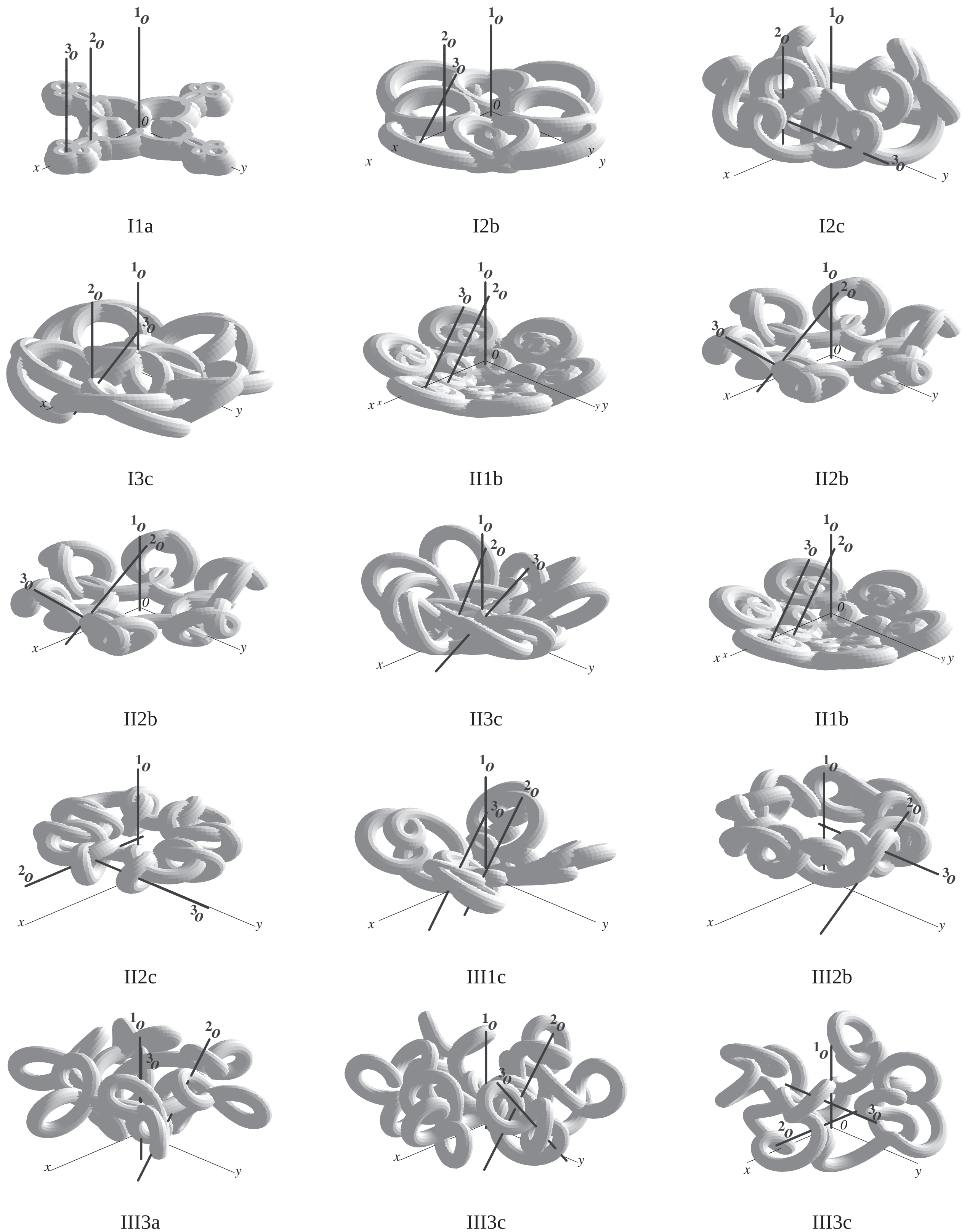
I1a
I2b
I2c
I3c
II1b
II2b
II2b
II3c
II1b
II2c
III1c
III2b
III3a
III3c
III3c

Figure 23

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