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On the Trigonometric Functions in Maximum Metric

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ABSTRACT

The trigonometric functions are defined on the unit circle and the shape of the unit circle changes according to the metric. So, the values of the trigonometric functions depend on the specific metric. This paper presents definitions, rules, and identities of trigonometric functions with respect to maximum metric. Also, geometrical interpretations by using the identities of these functions are given for further work.

Key words: trigonometric functions, non- Euclidean metric, maximum metric

MSC 2000: 51K05, 51K99

O trigonometrijskim funkcijama u maksimalnoj metrici

SAŽETAK

Trigonometrijske funkcije su definirane na jediničnoj kružnici, a njezin se oblik mijenja s obzirom na metriku. Dakle, vrijednosti trigonometrijskih funkcija ovise zasebno o svakoj metrici. Ovaj članak prikazuje definicije pravila i identitete trigonometrijskih funkcija s obzirom na maksimalnu metriku. Također, za daljnje istraživanje dane su geometrijske interpretacije koje koriste identitete ovih funkcija.

Ključne riječi: trigonometrijske funkcije, neeuklidska metrika, maksimalna metrika

1 Introduction

In the plane geometry, trigonometric functions are defined as $x = \cos\theta$, $y = \sin\theta$ for all points (x,y) on the unit circle, where θ is the angle with initial side the positive $x-$axis and terminal side the radial line passing through point (x,y). The unit circle is defined as the set of all points whose distance from the origin is one and this is a different point set, a different shape in different metrics. So, the values of trigonometric functions change according to the metric which we use. The trigonometric functions on the unit circle of taxicab and Chinese Checker metrics have been defined and developed in [1,...,6]. In the present paper, the trigonometric functions are defined with angle θ in the maximum metric which is a non Euclidean metric defined in $\mathbb{R}^2$ for $X = (x_1, y_1)$, $Y = (x_2, y_2)$ as

$$\begin{aligned} d_m(X,Y) &= d_m((x_1,y_1),(x_2,y_2)) \\ &= \max\{|x_1 - x_2|, |y_1 - y_2|\} \end{aligned} \qquad (1)$$

in section 2 and several trigonometric identities of these functions are given in section 3. Also, the definitions of trigonometric functions are developed by using the reference angle α and the change of maximum length of a line segment after rotations are given in section 4.

2 $m-$trigonometric functions

We wish to maintain the standard definitions of the trigonometric functions on $m-$unit circle in the same way one determines their Euclidean analogues. $m-$ unit circle in $\mathbb{R}^2$ is the set of points (x,y) which satisfies the equation $\max\{|x|, |y|\} = 1$.The graph of unit circle is in Figure 1.

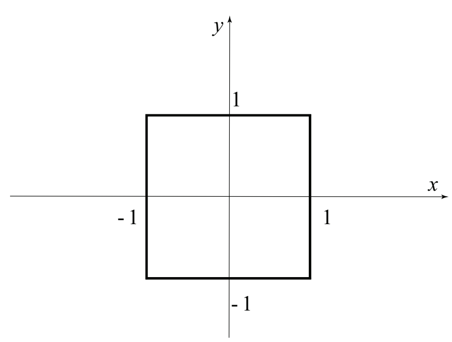

Figure 1: m – unit circle

Since the slope of radial line passing through point (x,y) does not change, tangent function does not depend on the metric. So, we can define the trigonometric functions according to the maximum metric in terms of the standard Euclidean tangent function. Consequently, the slope of the radial line goes through the point $(\cos_m\theta, \sin_m\theta)$ on $m-$ unit circle is

$$\tan\theta = \frac{\sin_m\theta}{\cos_m\theta} = \tan_m\theta. \qquad (2)$$

So, for the definitions of sine and cosine it is necessary to find only the point that will imply $(\cos_m\theta, \sin_m\theta)$ is on the line that makes an angle θ with the positive $x-$axis and on m- unit circle. The equation of the line joining (x,y) and $(0,0)$ is $y = (\tan\theta)x$. Solving the system

$$\begin{cases} y = (\tan\theta)x \\ \max\{|x|, |y|\} = 1 \end{cases}$$

we have the following chain of results

$$\begin{aligned} &\max\{|x|, |y|\} = 1 \\ &|x|\max\{1, |\tan\theta|\} = 1 \\ &|x| = \tfrac{1}{\max\{1,|\tan\theta|\}}. \end{aligned}$$

If it is made appropriate choice of sign for the absolute value based on the quadrant, we have m-cosine function,

$$\cos_m\theta = \begin{cases} 1, & -\frac{\pi}{4} \leqslant \theta < \frac{\pi}{4} \\ \cot\theta, & \frac{\pi}{4} \leqslant \theta < \frac{3\pi}{4} \\ -1 & \frac{3\pi}{4} \leqslant \theta < \frac{5\pi}{4} \\ -\cot\theta & \frac{5\pi}{4} \leqslant \theta < \frac{7\pi}{4}. \end{cases} \qquad (3)$$

Also, one can obtain easily $m-$sine function by using the identity (3) as

$$\sin_m\theta = \begin{cases} \tan\theta, & -\frac{\pi}{4} \leqslant \theta < \frac{\pi}{4} \\ 1, & \frac{\pi}{4} \leqslant \theta < \frac{3\pi}{4} \\ -\tan\theta, & \frac{3\pi}{4} \leqslant \theta < \frac{5\pi}{4} \\ -1 & \frac{5\pi}{4} \leqslant \theta < \frac{7\pi}{4}. \end{cases} \qquad (4)$$

The graphs of $m-$cosine and $m-$sine are presented in Figure 2 and Figure3.

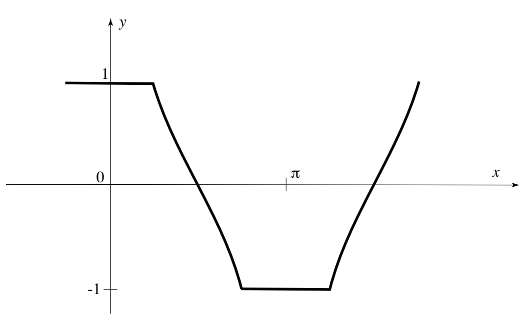

Figure 2: The graph of $y = \cos_m x$

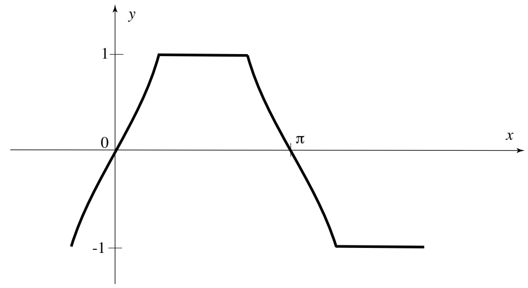

Figure 3: The graph of $y = \sin_m x$

3 Identities of the maximum trigonometric functions

The trigonometric identities for m-trigonometric functions will differ from their Euclidean counterparts in most cases. If we define the secant, cosecant and cotangent functions in maximum metric as we do in Euclidean metric, then $\csc_m\theta = \frac{1}{\sin_m\theta}$, $\sec_m\theta = \frac{1}{\cos_m\theta}$. The cotangent function does not depend on the metric as the tangent function does.

The cofunction identities of these functions are like their Euclidean counterparts, i.e. $\cos_m(\frac{\pi}{2}-\theta) = \sin_m\theta$ and $\sin_m(\frac{\pi}{2}-\theta) = \cos_m\theta$.

Using the identities $\tan(-x) = -\tan x$ and $\cot(-x) = -\cot x$ in (3) and (4),one can verify the identities $\cos_m(-\theta) = \cos_m\theta$ and $\sin_m(-\theta) = -\sin_m\theta$.

The Pythagorean identities are the fundamental identities in Euclidean trigonometry. In maximum metric case, the Pythagorean identities are different from $\cos^2\theta + \sin^2\theta = 1$ in Euclidean metric. Clearly, The Pythagorean identity follows simply from equation of the $m-$ unit circle $\max\{|x|, |y|\} = 1$, giving us

$$\max\{|\cos_m x|, |\sin_m x|\} = 1,$$

and applying the tangent identity (2), we have

$$\max\{1, |\tan_m x|\} = |\sec_m x|.$$

Since the tangent function does not change in the maximum metric, the Euclidean identity

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

holds in the maximum metric. Applying (2) to both sides, this equation can be rewritten and simplified as

$$\frac{\sin_m(u\pm v)}{\cos_m(u\pm v)} = \frac{\frac{\sin_m u}{\cos_m u} \pm \frac{\sin_m v}{\cos_m v}}{1 \mp \frac{\sin_m u \sin_m v}{\cos_m u \cos_m v}} = \frac{\sin_m u \cos_m u \pm \sin_m v \cos_m v}{\cos_m u \cos_m v \mp \sin_m u \sin_m v}$$

and since the fractions on both sides are in the lowest terms, we can equate numerator and denominator. So, the sum and difference formulas in the maximum metric is obtained:

$$\begin{aligned}\sin_m(u\pm v)&=\sin_m u\cos_m u\pm\sin_m v\cos_m v\\ \cos_m(u\pm v)&=\cos_m u\cos_m v\mp\sin_m u\sin_m v.\end{aligned}\tag{5}$$

Also, the double angles formulas are obtained by applying $u=v$ in (5).

4 m- trigonometric functions with reference angle

In Euclidean metric, an angle size and an arc length are equivalent on the unit circle. But, there is a non-uniform change in the arc length increasing the angle θ by a fixed amount in maximum metric. So, it is necessary to develop the trigonometric functions defined on the maximum unit circle by the reference angle for θ not in standard position [4]. If θ is not in standard position, it can be defined two angles in standard position. So, we can define the general trigonometric functions for the angle θ with the reference angle α on $m-$ unit circle. When θ is the angle with the reference angle α which is the angle between θ and the positive direction of the x- axis on $m-$ unit circle, we can define the general cosine and sine functions θ as follows:

$$\begin{aligned}\mathrm{mcos}\theta&=\cos_m(\alpha+\theta).\cos_m\alpha-\sin_m(\alpha+\theta).\sin_m\alpha\\ \mathrm{msin}\theta&=\sin_m(\alpha+\theta).\cos_m\alpha+\cos_m(\alpha+\theta).\sin_m\alpha\,.\end{aligned}\tag{6}$$

In these definitions, since $\alpha+\theta$ and α are in standard position, the value of $\mathrm{mcos}(\alpha+\theta)$,$\mathrm{mcos}\alpha$,$\mathrm{msin}(\alpha+\theta)$ and $\mathrm{msin}\alpha$ are calculated by using (3) and (4). If $\alpha=0$, then $\mathrm{mcos}\theta=\cos_m\theta$ and $\mathrm{msin}\,\theta=\sin_m\theta$ since θ is in standard position. The general definitions for other trigonometric functions for the angles which are not in standard position can be given similarly.

Consequently, the general definitions of trigonometric functions can be given by defining angles with the reference angle in maximum metric.

It is well known that all rotations and translations preserve the Euclidean distance. But , if a line segment is rotated , the length of it changes in non-Euclidean metrics. Thus, the change of a line segment length after rotation is given by the following theorem:

Theorem 1 *Let OA be a line segment, not on the x-axis with reference angle* α *and* $d_m(O,A)=k$. *If* OA' *is the image of OA under the rotation through the angle* θ, *then*

$$\begin{aligned}d_m(O,A')&=k\sqrt{\frac{\cos_m^2\alpha+\sin_m^2\alpha}{\cos_m^2(\alpha+\theta)+\sin_m^2(\alpha+\theta)}}=\\ &=k.\frac{\max\{|\cos(\alpha+\theta)|,|\sin(\alpha+\theta)|\}}{\max\{|\cos\alpha)|,|\sin\alpha|\}}.\end{aligned}$$

PROOF: Since the lengths under the translations are preserved in maximum metric, it is enough to consider a line segment passing through the origin. Let $d_m(O,A)$ be k. By the rotation of OA through an angle θ, we get the line segment OA'. If α is the reference angle of θ, then $A=(k.\cos_m\alpha,k.\sin_m\alpha)$. Let $d_m(O,A')$ be k'. Then $A'=(k'.\cos_m(\alpha+\theta),\,k'.\sin_m(\alpha+\theta))$. Because of the equality of Euclidean lengths of the line segments OA and OA' we get

$$d_m(O,A)=d_m(O,A')$$

and therefore

$$(k.\cos_m\alpha)^2+(k.\sin_m\alpha)^2=(k'.\cos_m(\alpha+\theta))^2+(\sin_m(\alpha+\theta))^2$$

$$k'=k\sqrt{\frac{\cos_m^2\alpha+\sin_m^2\alpha}{\cos_m^2(\alpha+\theta)+\sin_m^2(\alpha+\theta)}}.$$

and by using the identity

$$\cos_m^2\alpha+\sin_m^2\alpha=1+\tan^2\alpha=\sec^2\alpha,$$

$$k'=k.\frac{\max\{|\cos(\alpha+\theta)|,|\sin(\alpha+\theta)|\}}{\max\{|\cos\alpha)|,|\sin\alpha|\}}$$

is obtained in terms of the Euclidean sine and cosine functions.

The following corollary shows how one can find the maximum length, after a rotation of a line segment through an angle θ in standard form.

Corollary: Let OA be a line segment on the x-axis. If OA' is the image of OA under the rotation through an angle θ then

$$d_m(O,A')=\frac{k}{\sqrt{\cos^2\theta+\sin^2\theta}}=k\max\{|\cos\theta|,|\sin\theta|\}.$$

PROOF: Using the value $\alpha=0$ in the theorem, the corollary is obtained.

Consequently, this paper on the trigonometric functions in maximum metric provides good facilities for further works on the subjects as norm, inner product, cosine theorem and area of triangles in maximum metric.

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