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On a conjecture of McNeil

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Abstract. We determine the maximum length of a walk on the grid graph $P_m \times P_n$, up to an additive error of 1. This nearly settles McNeil's conjecture for the square grid graph $P_n \times P_n$.

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1. Introduction

Let m and n be two positive integers and consider the grid graph $P_m \times P_n$. Assume that the mn vertices of $P_m \times P_n$ are labeled bijectively by $\{1, 2, \ldots, mn\}$. The labeling induces a walk on $P_m \times P_n$, beginning with the vertex labeled 1, proceeding to the vertex labeled 2, and so on, until finally the vertex labeled mn is reached. This work is concerned with the following question: What is the maximum possible length of such a walk, denoted by $M(P_m \times P_n)$, if the distance between consecutive labeled vertices is the Manhattan distance? See Figure 1 for a visualization.

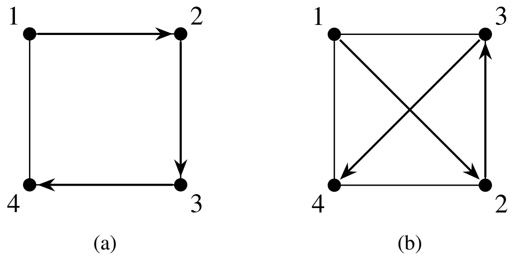

Figure 1: Two walks on the grid graph $P_2 \times P_2$. The left walk has length $3 = 1 + 1 + 1$, while the right walk has length $2 + 1 + 2 = 5$, which is maximal. Thus, $M(P_2 \times P_2) = 5$.

This question was studied in two special cases: the case $P_1 \times P_n$ reduces to standard permutations since $P_1 \times P_n$ is isomorphic to P_n. It was shown by Bulteau et al. [1] that $M(P_n) = \lfloor n^2/2 \rfloor - 1$. The second case, with $m = n$, was studied by McNeil (see sequence A179094 in [3]). Based on empirical evidence, McNeil proposed the following conjecture.

Conjecture 1. *Assume that $n \geq 2$. Then*

$$M(P_n \times P_n) = \begin{cases} n^3 - 3, & \text{if } n \text{ is even,} \\ n^3 - n - 1, & \text{if } n \text{ is odd,} \end{cases}$$

Our results are summarized in the following theorem. In particular, they nearly settle McNeil's conjecture.

Theorem 1. *We have $M(P_2 \times P_n) \in \{r, r + 1\}$, where $r = (n + 1)^2 - 4$. Furthermore, if $m, n \geq 3$, then:*

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1. *If m and n are both even, then $M(P_m \times P_n) \in \{r, r+1\}$, where*

$$r = \frac{mn(m+n)}{2} - 3.$$

2. *If m and n are both odd, then $M(P_m \times P_n) \in \{r, r+1\}$, where*

$$r = \frac{mn(m+n)}{2} - \frac{m+n}{2} - 1.$$

3. *If m is odd and n is even, then*

$$M(P_m \times P_n) = \frac{mn(m+n)}{2} - \frac{n}{2} - 1.$$

2. Main results

Throughout the paper, let $m, n \geq 2$ be two integers. We set $[n] = \{1, 2, \dots, n\}$ and identify the vertex set of $P_m \times P_n$ with $V = [m] \times [n]$. Thus, the labelings we consider are bijections $\sigma : V \longrightarrow [mn]$. The Manhattan distance between two vertices $(i, j), (i', j') \in V$ is

$$d((i, j), (i', j')) = |i - i'| + |j - j'|.$$

If c is a condition, let $\mathbf{1}_c$ denote its indicator function, which is equal to 1 if c holds and 0 otherwise. For two sequences a, b, we write $a \circ b$ for their concatenation. If $a = (a_1, \dots, a_n)$ is a sequence, we denote by $\mathrm{TV}(a)$ the total variation of a, i.e., $\mathrm{TV}(a) = \sum_{i=1}^{n-1} |a_{i+1} - a_i|$. The proof of Theorem 1 consists of an upper bound (Lemma 3) and corresponding lower bounds (Lemmas 4, 5, 6, and 7).

2.1. The upper bound

A key ingredient in the proof of the upper bound (Lemma 3) is an extension of the maximum total variation formula of Bulteau et al. [1] from permutations ($m = 1$) to permutations of a multiset consisting of $m \geq 2$ copies of $[n]$ (Lemma 2). The proof of this extension uses the following result concerning the maximum number of transitions in binary words.

Lemma 1. *The maximum number of transitions from 0 to 1 or from 1 to 0 in a binary word of length n with k ones is*

$$2\min\{k, n-k\} - \mathbf{1}_{k=\frac{n}{2}}.$$

Proof. Recall that a run in a binary word is a subword consisting of consecutive zeros or ones that is not preceded or followed by the same digit. It is well known (e.g., [4, p. 52]) that the maximum number of runs in a binary word of length n with k ones is

$$2\min\{k, n-k\} + 1 - \mathbf{1}_{k=\frac{n}{2}}.$$

Since the number of transitions is equal to the number of runs minus one, the assertion follows. □

Lemma 2. *Denote by $S(m, n)$ the set of permutations of the elements of the multiset containing m copies of $[n]$. Then*

$$\max_{\sigma \in S(m,n)} \mathrm{TV}(\sigma) = 2m \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - \mathbf{1}_{2|n}. \tag{1}$$

Proof. For $\sigma = (\sigma_1, \dots, \sigma_{mn}) \in S(m, n)$ set

$$C_k(\sigma) = \sum_{i=1}^{mn-1} \mathbf{1}_{(\sigma_i \leq k < \sigma_{i+1}) \text{ or } (\sigma_{i+1} \leq k < \sigma_i)}.$$

We claim that

$$\mathrm{TV}(\sigma) = \sum_{k=1}^{n-1} C_k(\sigma). \tag{2}$$

Indeed, for $x, y \in [n]$ and $k \in [n-1]$, we have

$$|x - y| = \sum_{k=1}^{n-1} \mathbf{1}_{(x \le k < y) \text{ or } (y \le k < x)}.$$

Thus,

$$\begin{aligned}
\mathrm{TV}(\sigma) &= \sum_{i=1}^{mn-1} |\sigma_{i+1} - \sigma_i| \\
&= \sum_{i=1}^{mn-1} \sum_{k=1}^{n-1} \mathbf{1}_{(\sigma_i \le k < \sigma_{i+1}) \text{ or } (\sigma_{i+1} \le k < \sigma_i)} \\
&= \sum_{k=1}^{n-1} C_k(\sigma).
\end{aligned}$$

Now, let $p_k \colon S(m,n) \to \{0,1\}^{mn}$ be the projection defined as follows: For $i \in [mn]$, let the ith coordinate of $p_k(\sigma)$ be $\mathbf{1}_{\sigma_i \le k}$. It follows that $C_k(\sigma)$ is equal to the number of transitions in $p_k(\sigma)$. Indeed,

$$\begin{aligned}
C_k(\sigma) &= \sum_{i=1}^{mn-1} \mathbf{1}_{(\sigma_i \le k < \sigma_{i+1}) \text{ or } (\sigma_{i+1} \le k < \sigma_i)} \\
&= \sum_{i=1}^{mn-1} \mathbf{1}_{((p_k(\sigma))_i = 1 \text{ and } (p_k(\sigma))_{i+1} = 0) \text{ or } ((p_k(\sigma))_i = 0 \text{ and } (p_k(\sigma))_{i+1} = 1)} \\
&= \text{number of transitions in } p_k(\sigma).
\end{aligned}$$

Applying Lemma 1 to the binary word $p_k(\sigma)$, which has length mn and mk ones, we conclude that

$$C_k(\sigma) \le 2m \min\{k, n-k\} - \mathbf{1}_{k=\frac{n}{2}}. \tag{3}$$

Summing (3) over $k \in [n-1]$ and using the identity

$$\sum_{k=1}^{n-1} \min\{k, n-k\} = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil,$$

which may be found, for example, in sequence A002620 in [3], together with $\sum_{k=1}^{n-1} \mathbf{1}_{k=\frac{n}{2}} = \mathbf{1}_{2|n}$, we conclude from (2) that

$$\mathrm{TV}(\sigma) \le 2m \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - \mathbf{1}_{2|n}.$$

This proves that the left-hand side of (1) is at most the right-hand side. To prove the equality, we shall construct a permutation that attains the upper bound. We distinguish between even and odd n. Assume first that n is odd, i.e., $n = 2k+1$ for some integer $k \ge 1$. For $j \in [k]$, define a sequence $B_{j,m,n}$ of length $2m$ by

$$B_{j,m,n} = (j, n+1-j,\ j, n+1-j,\ \ldots,\ j, n+1-j).$$

Now define σ as follows:

$$\sigma = (k+1) \circ B_{1,m,n} \circ \cdots \circ B_{k,m,n} \circ (\underbrace{k+1, \ldots, k+1}_{m-1 \text{ times}}).$$

To calculate $\mathrm{TV}(\sigma)$, notice that $\mathrm{TV}(B_{j,m,n}) = (2m-1)(n+1-2j)$. Furthermore, for $j \in [k-1]$, the absolute difference between the last element of $B_{j,m,n}$ and the first element of $B_{j+1,m,n}$ is $n-2j$. It follows that

$$\begin{aligned}\mathrm{TV}(\sigma) &= |k+1-1| + \sum_{j=1}^{k}(2m-1)(n+1-2j) + \sum_{j=1}^{k-1}(n-2j) + |n+1-k-(k+1)| \\ &= 2mk(n-k) \\ &= 2m\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil.\end{aligned}$$

Assume now that n is even, i.e., $n = 2k$ for some integer $k \geq 1$. For $j \in [k]$ define a sequence $B_{j,m,n}$ of length $2m$ by

$$B_{j,m,n} = (n+1-j, j,\ n+1-j, j,\ \ldots,\ n+1-j, j).$$

Now define σ in two steps: First, let

$$\sigma' = B_{1,m,n} \circ \cdots \circ B_{k,m,n}.$$

Then let σ be the sequence obtained from σ' by cutting its last element (which is k) and inserting it at the beginning of σ'. As in the odd case, $\mathrm{TV}(B_{j,m,n}) = (2m-1)(n+1-2j)$, and, for $j \in [k-1]$, the absolute difference between the last element of $B_{j,m,n}$ and the first element of $B_{j+1,m,n}$ is $n-2j$. It follows that

$$\begin{aligned}\mathrm{TV}(\sigma) &= |k-n| + \sum_{j=1}^{k}(2m-1)(n+1-2j) + \sum_{j=1}^{k-1}(n-2j) - 1 \\ &= 2mk(n-k) - 1 \\ &= 2m\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil - 1.\end{aligned}$$

Thus, in both cases there is a permutation attaining the upper bound and the proof is complete. □

We may now prove the upper bound on $M(P_m \times P_n)$.

Lemma 3. *We have*

$$M(P_m \times P_n) \leq 2n\left\lfloor\frac{m}{2}\right\rfloor\left\lceil\frac{m}{2}\right\rceil - \mathbf{1}_{2|m} + 2m\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil - \mathbf{1}_{2|n}.$$

Proof. Consider a labeling σ of $P_m \times P_n$, i.e., $\sigma\colon V \to [mn]$ is a bijection. For $t \in [mn]$, write $(i_t, j_t) = \sigma^{-1}(t)$ and notice that the multiset $\{i_t : t \in [mn]\}$ is equal to n copies of $[m]$ and the multiset $\{j_t : t \in [mn]\}$ is equal to m copies of $[n]$. Thus, by Lemma 2,

$$\begin{aligned}\sum_{t=1}^{mn-1} d(\sigma^{-1}(t), \sigma^{-1}(t+1)) &= \sum_{t=1}^{mn-1} |i_{t+1} - i_t| + \sum_{t=1}^{mn-1} |j_{t+1} - j_t| \\ &\leq 2n\left\lfloor\frac{m}{2}\right\rfloor\left\lceil\frac{m}{2}\right\rceil - \mathbf{1}_{2|m} + 2m\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil - \mathbf{1}_{2|n}.\end{aligned}$$ □

2.2. The lower bound

The proof of the lower bound consists of four constructions, which, taken together, cover all cases listed in Theorem 1.

Lemma 4. *Assume that n is odd. Then $M(P_2 \times P_n) \geq (n+1)^2 - 4$.*

Proof. Let $\sigma\colon V \to [2n]$ be the bijection given by

$$\begin{bmatrix} n-1 & \cdots & 4 & 2 & n+1 & 2n & n+3 & \cdots & 2n-2 \\ 2n-1 & \cdots & n+4 & n+2 & 1 & 3 & 5 & \cdots & n \end{bmatrix}.$$

The walk $1 \to 2 \to \cdots \to n$ has length $\sum_{i=1}^{n-1}(i+1)$. Now, $\frac{n+1}{2}$ steps lead to $n+1$ and additional 2 steps to $n+2$. Then, a walk of length $\sum_{i=3}^{n-1}(i+1)$ leads to $2n-1$. Finally, $\frac{n+3}{2}$ steps lead to $2n$ and the walk is completed with a total length of $(n+1)^2-4$. □

Lemma 5. *Assume that m and n are both even. Then*

$$M(P_m \times P_n) \geq \frac{mn(m+n)}{2} - 3.$$

Proof. Let $r = \frac{mn}{2}$ and let $\sigma\colon V \to [mn]$ be the bijection given by

$$\left[\begin{array}{cccc|cccc} r-1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & mn-1 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \cdots & \cdots & 3 & 1 & r+1 & r+3 & \cdots & \cdots \\ \hline \cdots & \cdots & \cdots & mn & r & \cdots & \cdots & \cdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ r+2 & r+4 & \cdots & \cdots & \cdots & \cdots & 4 & 2 \end{array}\right].$$

First, we describe the bijection explicitly. The array of size $m \times n$ is divided into four regions, each of size $\frac{m}{2} \times \frac{n}{2}$. The labels are distributed as follows:

1. The top-left region contains the odd labels $1, 3, \ldots, r-1$ arranged so that 1 is at index $(\frac{m}{2}, \frac{n}{2})$ and each row is strictly increasing from right to left and each column is strictly increasing from bottom to top.

2. The top-right region contains the odd labels $r+1, r+3, \ldots, mn-1$ arranged so that $r+1$ is at index $(\frac{m}{2}, \frac{n}{2}+1)$ and each row is strictly increasing from left to right and each column is strictly increasing from bottom to top.

3. The bottom-right region contains the even labels $2, 4, \ldots, r$ arranged so that 2 is at index (m, n) and each row is strictly increasing from right to left and each column is strictly increasing from bottom to top.

4. The bottom-left region contains the even labels $r+2, \ldots, mn$ arranged so that $r+2$ is at index $(m, 1)$ and each row is strictly increasing from left to right and each column is strictly increasing from bottom to top.

We split $\sum_{t=1}^{mn-1} d(\sigma^{-1}(t), \sigma^{-1}(t+1))$ into its horizontal and vertical parts.

1. Horizontal part - There are $\frac{mn}{4}$ walks from the top-left region to the bottom-right region, each has length $\frac{n}{2}$. The return walks have length $\frac{n+2}{2}$, except those that go from column $\frac{n+2}{2}$. These have length 1, with the exception of the walk from r to $r+1$ which has no horizontal part. By symmetry, the walks between the other pair of diagonal regions contribute exactly the same distance. It follows that the total horizontal contribution is

$$2\left(\frac{mn^2}{8} + \left(\frac{mn}{4} - \frac{m}{2}\right)\frac{n+2}{2} + \frac{m}{2} - 1\right) = \frac{mn^2}{2} - 2.$$

2. Vertical part - There are $\frac{mn}{4}$ walks from the top-left region to the bottom-right region, each has length $\frac{m}{2}$. The walks back are also of length $\frac{m}{2}$, except those that go from column $\frac{n+2}{2}$. These have length $\frac{m+2}{2}$, with the exception of the walk from r to $r+1$ which has length 1. By symmetry, the walks between the other pair of diagonal regions contribute exactly the same distance, except that we halt at label mn. It follows that the total vertical contribution is

$$2\left(\frac{m^2n}{8} + \left(\frac{mn}{4} - \frac{m}{2}\right)\frac{m}{2} + \frac{m-2}{2} \cdot \frac{m+2}{2} + 1\right) - 1 = \frac{m^2n}{2} - 1.$$

Combining the two parts yields a total distance of

$$\frac{mn^2}{2} - 2 + \frac{m^2n}{2} - 1 = \frac{mn(m+n)}{2} - 3,$$

as asserted. □

Lemma 6. *Assume that $m, n \geq 3$ are both odd. Then*

$$M(P_m \times P_n) \geq \frac{mn(m+n)}{2} - \frac{m+n}{2} - 1.$$

Proof. Let $r = \frac{(m+1)(n-1)}{2}$ and let $\sigma \colon V \to [mn]$ be the bijection given by

$$\left[\begin{array}{cccc|c|cccc}
r-1 & \cdots & \cdots & \cdots & r+1 & mn-2 & \cdots & \cdots & \cdots \\
\vdots & & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & & & \vdots & r+m-2 & \cdots & \cdots & r+m+2 & r+m \\ \cline{6-9}
\cdots & \cdots & 3 & 1 & mn & r & \cdots & \cdots & \cdots \\ \hline
mn-1 & \cdots & \cdots & \cdots & r+2 & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & \vdots & & & \vdots \\
\cdots & \cdots & r+m+3 & r+m+1 & r+m-1 & \cdots & \cdots & 4 & 2
\end{array}\right].$$

First, we describe the bijection explicitly. The array of size $m \times n$ is divided into seven regions and the labels are distributed as follows:

1. The top-left region contains the odd labels $1, 3, \ldots, r-1$ arranged so that 1 is at index $(\frac{m+1}{2}, \frac{n}{2})$ and each row is strictly increasing from right to left and each column is strictly increasing from bottom to top.

2. The top-middle region contains the odd labels $r+1, \ldots, r+m-2$ one above the other, arranged so that $r+1$ is in the first row and each row is strictly increasing from top to bottom.

3. The top-right region contains the odd labels $r+m, r+m+2, \ldots, mn-2$ arranged so that $r+m$ is at index $(\frac{m-1}{2}, n)$ and each row is strictly increasing from right to left and each column is strictly increasing from bottom to top.

4. The bottom-right region contains the even labels $2, 4, \ldots, r$ and is defined similarly to the top-left region.

5. The bottom-middle region contains the even labels $r+2, \ldots, r+m-1$ and is defined similarly to the top-middle region.

6. The bottom-left region contains the even labels $r+m+1, r+m+3, \ldots, mn-1$ and is defined similarly to the top-right region.

7. The central region consists only of the label mn at index $(\frac{m+1}{2}, \frac{n+1}{2})$.

We split $\sum_{t=1}^{mn-1} d(\sigma^{-1}(t), \sigma^{-1}(t+1))$ into its horizontal and vertical parts.

1. Horizontal part - There are $\frac{(m+1)(n-1)}{4}$ walks from the top-left region to the bottom-right region, each has length $\frac{n+1}{2}$. The return walks have length $\frac{n+3}{2}$, except those that go from column $\frac{n+3}{2}$. These have length 2, with the exception of the walk from r to $r+1$, which has length 1. Next, the walk $r+1 \to r+2 \to \cdots \to r+m-1$ has no horizontal part. Thus, we continue from $r+m-1$. Moving to $r+m$ gives $\frac{n-1}{2}$. Now, there are $\frac{(m-1)(n-1)}{4}$ walks from the top-right region to the bottom-left region, each has length $\frac{n+1}{2}$. The return walks have length $\frac{n-1}{2}$, except those that go from the first column.

These have length $n-1$, with the exception of the walk from $mn-1$ to mn, which has length $\frac{n-1}{2}$. It follows that the total horizontal contribution is

$$\begin{aligned}&\frac{(m+1)(n^2-1)}{8}+\left(\frac{(m+1)(n-1)}{4}-\frac{m+1}{2}\right)\frac{n+3}{2}+2\cdot\frac{m-1}{2}+1+\frac{n-1}{2}\\&+\frac{(m-1)(n^2-1)}{8}+\left(\frac{(m-1)(n-1)}{4}-\frac{m-1}{2}\right)\frac{n-1}{2}+\left(\frac{m-1}{2}-1\right)(n-1)+\frac{n-1}{2}\\&=\frac{mn^2-m}{2}-1.\end{aligned}$$

2. Vertical part - There are $\frac{(m+1)(n-1)}{4}$ walks from the top-left region to the bottom-right region, each has length $\frac{m-1}{2}$. The return walks have length $\frac{m-1}{2}$, except those that go from column $\frac{n+1}{2}+1$. These have length $\frac{m+1}{2}$, with the exception of the walk from r to $r+1$, which has length $\frac{m-1}{2}$. Now, there are $\frac{m-1}{2}$ walks from the top-middle region to the bottom-middle region, all having length $\frac{m+1}{2}$. The return walks have length $\frac{m-1}{2}$, except the last one from $r+m-1$ to $r+m$, which has length $\frac{m+1}{2}$. Now, there are $\frac{(m-1)(n-1)}{4}$ walks from the top-right region to the bottom-left region, each has length $\frac{m+1}{2}$. The return walks also have length $\frac{m+1}{2}$, except those that go from the first column. These have length $\frac{m+3}{2}$, with the exception of the walk from $mn-1$ to mn, which has length 1. It follows that the total vertical distance is

$$\begin{aligned}&\frac{(m^2-1)(n-1)}{8}+\left(\frac{(m+1)(n-1)}{4}-\frac{m+1}{2}\right)\frac{m-1}{2}+\frac{m-1}{2}\cdot\frac{m+1}{2}\\&+\frac{m-1}{2}+\frac{m-1}{2}\cdot\frac{m+1}{2}+\frac{m-3}{2}\cdot\frac{m-1}{2}+\frac{m+1}{2}+\frac{(m^2-1)(n-1)}{8}\\&+\left(\frac{(m-1)(n-1)}{4}-\frac{m-1}{2}\right)\frac{m+1}{2}+\frac{m-3}{2}\cdot\frac{m+3}{2}+1\\&=\frac{m^2n-n}{2}.\end{aligned}$$

Combining the two parts yields a total distance of

$$\frac{mn^2-m}{2}-1+\frac{m^2n-n}{2}=\frac{mn(m+n)}{2}-\frac{m+n}{2}-1,$$

as asserted. □

Lemma 7. *Assume that $m\geq 3$ is odd and $n\geq 3$ is even. Then*

$$M(P_m\times P_n)\geq\frac{mn(m+n)}{2}-\frac{n}{2}-1.$$

Proof. Set $r=\frac{(m+1)n}{2}-3$ and let $\sigma\colon V\to[mn]$ be the bijection given by

$$\left[\begin{array}{cccc|cccc}
r & \cdots & & \cdots & \cdots & \cdots & r+5 & r+3\\
\vdots & & & \vdots & \vdots & & & \vdots\\
\vdots & & & \vdots & mn-2 & \cdots & \cdots & \cdots\\
\hline
mn-1 & \cdots & 3 & 1 & mn & r+1 & \cdots & \cdots\\
mn-3 & \cdots & \cdots & \cdots & \vdots & & & \vdots\\
\vdots & & & \vdots & \vdots & & & \vdots\\
\cdots & \cdots & r+4 & r+2 & \cdots & \cdots & 4 & 2
\end{array}\right].$$

First, we describe the bijection explicitly. The array of size $m\times n$ is divided into four regions and the labels are distributed as follows:

1. The top-left region is a rectangular array of size $\frac{m+1}{2} \times \frac{n}{2}$ missing its bottom-left cell. It contains the odd labels $1, 3, \ldots, r$ arranged so that 1 is at index $(\frac{m+1}{2}, \frac{n}{2})$ and each row is strictly increasing from right to left and each column is strictly increasing from bottom to top.

2. The top-right region is a rectangular array of size $\frac{m-1}{2} \times \frac{n}{2}$ to which an additional cell is attached beneath its bottom-left cell. It contains the even labels $r+3, r+5, \ldots, mn$ arranged so that $r+3$ is at index $(1, n)$ and each row is strictly increasing from right to left and each column is strictly increasing from top to bottom.

3. The bottom-right region is a rectangular array of size $\frac{m+1}{2} \times \frac{n}{2}$ missing its top-left cell. It contains the even labels $2, 4, \ldots, r+1$ arranged so that 2 is at index (m, n) and each row is strictly increasing from right to left and each column is strictly increasing from bottom to top.

4. The bottom-left region is a rectangular array of size $\frac{m-1}{2} \times \frac{n}{2}$ to which an additional cell is attached above its top-left cell. It contains the odd labels $r+2, r+4, \ldots, mn-1$ arranged so that $r+2$ is at index $(m, \frac{n}{2})$ and each row is strictly increasing from right to left and each column is strictly increasing from bottom to top.

We split $\sum_{t=1}^{mn-1} d(\sigma^{-1}(t), \sigma^{-1}(t+1))$ into its horizontal and vertical parts.

1. Horizontal part - The first $\frac{n-2}{2}$ walks from the top-left region to the bottom-right region have length $\frac{n}{2}$. The next walk is only 1 step long. Now come $\frac{n-2}{2}$ walks having length $\frac{n+2}{2}$. This pattern of $1 + \frac{n-2}{2} \cdot \frac{n+2}{2}$ repeats $\frac{m-1}{2}$ times. Now consider the return walks. At the beginning, there are $\frac{n-4}{2}$ walks having length $\frac{n+2}{2}$. Now come two walks of length 2, followed by $\frac{n-4}{2}$ walks having length $\frac{n+4}{2}$. This pattern of $2 \cdot 2 + \frac{n-4}{2} \cdot \frac{n+4}{2}$ repeats $\frac{m-1}{2}$ times. At this point we arrive at $r+1$, and two steps bring us to $r+2$. All $\frac{m-1}{2} \cdot \frac{n}{2} + 1$ walks from the bottom-left region to the top-right region have length $\frac{n}{2}$. The walks back follow the pattern of $\frac{n-2}{2}$ walks having length $\frac{n+2}{2}$ and then one step. This pattern repeats $\frac{m-1}{2}$ times, except that the very last walk has length $\frac{n}{2}$, instead of 1. It follows that the total horizontal contribution is

$$\begin{aligned} &\frac{n-2}{2} \cdot \frac{n}{2} + \left(1 + \frac{n-2}{2} \cdot \frac{n+2}{2}\right)\frac{m-1}{2} + \frac{n-4}{2} \cdot \frac{n+2}{2} + \left(2 \cdot 2 + \frac{n-4}{2} \cdot \frac{n+4}{2}\right)\frac{m-1}{2} \\ &+ 2 + \left(\frac{m-1}{2} \cdot \frac{n}{2} + 1\right)\frac{n}{2} + \left(\frac{n-2}{2} \cdot \frac{n+2}{2} + 1\right)\frac{m-1}{2} - 1 + \frac{n}{2} \\ &= \frac{mn^2}{2} - 1. \end{aligned}$$

2. Vertical part - The first $\frac{n-2}{2}$ walks from the top-left region to the bottom-right region have length $\frac{m-1}{2}$. The next walk has length $\frac{m+1}{2}$. This pattern repeats $\frac{m+1}{2}$ times, except the very last step. Now consider the return walks. There are $\frac{n-4}{2}$ walks having length $\frac{m-1}{2}$, followed by two walks having length $\frac{m+1}{2}$. This pattern repeats $\frac{m+1}{2}$ times, except the very last two steps. At this point we arrive at $r+1$ and $\frac{m-1}{2}$ steps bring us to $r+2$. The walks from the bottom-left region to the top-right region follow this rule: For each $i \in \left[\frac{m-1}{2}\right]$, there are $\frac{n}{2}$ walks having length $m+1-2i$. Similarly, the walks back follow this rule: For each $i \in \left[\frac{m-1}{2}\right]$, there are $\frac{n-2}{2}$ walks having length $m+1-2i$ and one of length $m-2i$. This brings us to $mn-1$. The walk to mn has no vertical part. It follows that the total vertical contribution is

$$\begin{aligned} &\left(\frac{n-2}{2} \cdot \frac{m-1}{2} + \frac{m+1}{2}\right)\frac{m+1}{2} - \frac{m+1}{2} + \left(\frac{n-4}{2} \cdot \frac{m-1}{2} + 2 \cdot \frac{m+1}{2}\right)\frac{m+1}{2} \\ &- 2 \cdot \frac{m+1}{2} + \frac{m-1}{2} + \sum_{i=1}^{\frac{m-1}{2}} \frac{n}{2}(m+1-2i) + \sum_{i=1}^{\frac{m-1}{2}} \left(\frac{n-2}{2}(m+1-2i) + (m-2i)\right) \\ &= \frac{n(m^2-1)}{2}. \end{aligned}$$

Combining the two parts yields a total distance of

$$\frac{mn^2}{2} - 1 + \frac{n(m^2-1)}{2} = \frac{mn(m+n)}{2} - \frac{n}{2} - 1,$$

as asserted. □

3. Conclusion

It is natural to ask what happens if we consider graphs different from the grid graph. To make it more concrete, suppose G is a graph of order n with vertex set V. The distance between two vertices $x, y \in V$, denoted by $d(x, y)$, is the length of a shortest path along the edges of G joining x and y. Set

$$\Sigma = \{\sigma : V \to [n] \ : \ \sigma \text{ is a bijection}\}.$$

Then define

$$M(G) = \max_{\sigma\in\Sigma} \sum_{i=1}^{n-1} d(\sigma^{-1}(i), \sigma^{-1}(i+1)).$$

We propose to call $M(G)$ the *disorder number of* G, echoing the name *additive* y*-disorder* used in [1].

Example 1. Dominus [2] proved that

$$M(Q_n) = (2^{n-1} - 1)(2n - 1) + n,$$

where Q_n is the hypercube graph with 2^n vertices. See also sequence A271771 in [3].

To our knowledge, the cycle graph C_n has not been studied in this regard. We obtain the following result. See also sequence A000982 in [3].

Theorem 2. *For* $n \geq 3$*, we have*

$$M(C_n) = \left\lceil \frac{(n-1)^2}{2} \right\rceil.$$

Proof. Identify the vertices of C_n with the set $[n]$. The distance $d(x, y)$ between two vertices $x, y \in [n]$ is then $\min\{|x - y|, n - |x - y|\}$. We distinguish between two cases.

1. n is even. There are $\frac{n}{2}$ pairs of vertices $x, y \in [n]$ with $d(x, y) = \frac{n}{2}$. The remaining $\frac{n-2}{2}$ steps therefore have length at most $\frac{n-2}{2}$. Thus,

   $$M(C_n) \leq \left(\frac{n}{2}\right)^2 + \left(\frac{n-2}{2}\right)^2 = \frac{(n-1)^2 + 1}{2}.$$

   On the other hand, the bijection $[n] \to [n]$ given by

   $$\left[1, \frac{n}{2} + 1, 2, \frac{n}{2} + 2, \dots, \frac{n}{2}, n\right]$$

   obviously attains the upper bound.

2. n is odd. The maximum distance between any two vertices is $\frac{n-1}{2}$. Thus,

   $$M(C_n) \leq (n-1)\frac{n-1}{2}.$$

   On the other hand, with $k = \frac{n-1}{2}$ and interpreting the residue 0 as n, the bijection $[n] \to [n]$ given by

   $$[1, 1 + k, 1 + 2k, \dots, 1 + (n-1)k] \pmod{n}$$

   obviously attains the upper bound. □

Remark 1. The Manhattan distance used in this work is induced by the ℓ_1 norm. It is natural to ask what happens if we replace it by a distance induced by another norm $||\cdot||$ on $\mathbb{R}^2$ (e.g., the Euclidean norm $||\cdot||_2$). Let Σ be the set of bijections $\sigma\colon V \to [mn]$. Then one may consider

$$M_{||\cdot||}(P_m \times P_n) = \max_{\sigma\in\Sigma} \sum_{t=1}^{mn-1} ||\sigma^{-1}(t+1) - \sigma^{-1}(t)||.$$

This seems interesting and may require methods different from those used in this work.

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