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EFZG WORKING PAPER SERIES
EFZG SERIJA ČLANAKA U NASTAJANJU
ISSN 1849-6857
UDC 33:65

No. 2026-03

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# European and US capital markets: Which econometric approach is the best fit?

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**Abstract**

This paper examines the long-run cointegration between the German DAX and the US S&P 500 from January 2021 to December 2025, using daily closing prices expressed as natural logarithms. The central argument is that methodological choices prevalent in the existing literature systematically fail to detect genuine long-run equilibrium relationships due to the econometric costs of over-differencing and first-step OLS bias amplification. Drawing on the theoretical contributions of Granger and Newbold (1974), Granger (1981), Granger and Joyeux (1980), Engle and Granger (1987), and Phillips (1988), the paper reconstructs the conditions under which differencing destroys low-frequency spectral dynamics and renders standard cointegration tests unreliable. With this conclusion in mind, the paper tests two models that encompass long-term cointegration: ARDL and ECM models. The empirical analysis confirms that the ARDL model outperforms its competitor, making it the best-fit methodology for modelling cross-Atlantic equity market integration and carrying direct implications for portfolio diversification, financial stability monitoring, and applied econometric practice.

**Key words**
cointegration, ARDL, error correction model, DAX, S&P 500, spurious regression, fractional integration, capital market integration

**JEL classification**
C22, C51, G17

## 1. Research gaps and goals of the article

The relationship between the European and US capital markets has attracted considerable attention in empirical finance and financial econometrics. As globalisation has increased cross-border capital flows, understanding whether the German DAX and the US S&P 500 share a common long-run equilibrium has become a central question for portfolio managers, policymakers, and academic researchers. The existing literature shows highly heterogeneous approaches to modelling such cointegrations. Cerny and Koblas (2004) investigate the speed of information transmission among major stock indices, using intraday data at frequencies ranging from five minutes to daily. Applying Granger causality tests across data frequencies, they find that the S&P 500 Granger-causes the DAX 30 at five-minute and thirty-minute intervals, even at the one-percent significance level, while the reverse causality is detectable only at certain daily frequencies. Plíhal (2016) applies the Toda-Yamamoto (1995) modification of the Granger causality test to the DAX index and a range of German macroeconomic variables over the period 1999–2015. The study finds that the DAX Granger-causes industrial production and interest rates, and that bidirectional causality exists between the index and the money supply. At the aggregate market level, Li (2019) used a structural VAR with Diebold-Yilmaz forecast-error variance decompositions to study spillovers between US and European short-term rates, bond yields, and stock returns, finding that transmission within asset classes is strongest and that US assets influence European assets more than the reverse. Bakkar et al. (2020) decomposed US uncertainty into real and financial components using an SVAR, finding that US financial uncertainty shocks reduce euro area industrial production by about 0.8%, roughly double the impact of real uncertainty shocks (−0.4%), due to greater financial friction in the transmission. On intraday dynamics, Thomaidis et al. (2008) applied VAR to 5-minute returns of the S&P 500, FTSE 100, CAC 40, and DAX, finding strong causalities among European markets both before and after the NYSE opening, as well as significant return transmissions from the S&P to continental indices. A more advanced methodological approach, such as GARCH modelling, has also been applied: Kaltenhäuser (2002, 2003) produced two key studies using a two-step GARCH framework on 10 sectors across the euro area, US, and UK. He found that sectors became increasingly heterogeneous over time in their response to aggregate shocks, with IT and telecoms becoming the most globally integrated sectors, most affected by both European and US shocks, while basic industries, utilities, and resources became less affected. European industries showed this increased heterogeneity coinciding with the launch of EMU. Balli, Balli, and Louis (2013) extended this by examining return, volatility, and trend spillovers across Euro- and US-wide sector indices, finding that volatility spillovers (not return spillovers) are what matter for explaining sector equity returns, and that sectors cluster into four groups that respond similarly to local and global shocks: production/industry; consumer goods/services; financial; and technology/media/telecom. For the financial sector specifically, Elyasiani (2016) examined banking and insurance industries across the US, UK, EU, and Japan within a VAR-BEKK framework, finding strong bidirectional return and volatility contagion between US and EU banks and insurers, with the US as the leading provider of volatility information in banking.

Across both strands of the literature reviewed above, a common gap emerges. None of the authors explicitly conclude that the approach of differencing to induce stationarity weakens the dynamics of the modelled series, or that the estimated relationship may exhibit genuine long-run cointegration despite showing no evidence of short-run cointegration. Likewise, even if the results show short-term cointegration after differencing, long-term cointegration can still exist, yet none of the authors test for it. Suppose the existing VAR models, which concluded that there is cointegration between the US and EU (Li, 2019; Thomaidis et al., 2008; Elyasiani, 2016), did not test for their long-term cointegration nor whether the variables that do not show short-term cointegration still exhibit it in the long run. This crucial oversight is also present in GARCH methodology. Traditional GARCH models suffer from the problem of overestimating long-term effects of variance (persistence issue) despite their ability to incorporate heteroskedasticity. An econometric model that addresses both such issues is the ECM and ARDL model. The main goal of this article is to determine which of these two models provides a better fit when modelling the cointegration between US and EU markets (**O1**).

The remainder of the article is structured as follows. Chapter 2 traces the theoretical foundations of the problem in three steps: it first documents the spurious regression phenomenon established by Granger and Newbold (1973, 1974), then formalises the spectral definition of cointegration and the consequences of fractional integration through Granger (1981) and Granger and Joyeux (1980), showing that imposing integer differencing on a fractionally integrated process destroys the low-frequency dynamics necessary for long-run modelling. Chapter 3 introduces the ECM and ARDL frameworks, derives their formal equivalence, and explains why the Engle-Granger two-step procedure accumulates and amplifies OLS bias in a way that the ARDL Bounds Testing approach avoids. Chapter 4 presents the empirical results for the DAX and S&P 500, demonstrating that the long-run cointegrating relationship is missed by the classical ECM procedure but successfully recovered by the ARDL specification. Chapter 5 concludes.

## 2. The issues of spurious integration and differentiating into stationarity

### 2.1 Spurious regression

The paper by Granger and P. Newbold (1973) warned that econometric methods contain autocorrelation of residuals, yet this is nonetheless ignored by relevant research in the field. More explicitly, they observe that the $R^2$ and DW statistics gathered from data containing autocorrelated residuals are used in the spurious inference of empirical significance in the time series $X_t$ and $Y_t$. The authors sought to prove their assumption by constructing a regression equation of the form:

$$Y_t = \beta_0 + \beta_1 \cdot X_t + \epsilon_t \tag{1}$$

Where

$$Y_t = \phi \cdot Y_{t-1} + a_t \text{ , } X_t = \phi^* \cdot X_{t-1} + \alpha_t \tag{2}$$

are mutually independent AR(1) processes. They proceed to show:

$$Var(R) = \frac{1 + \phi \cdot \phi^*}{T \cdot (1 - \phi \cdot \phi^*)} \tag{3}$$

where $R^2$ is the square of the ordinary sample correlation between $X_t$ and $Y_t$ and use the fact that since $R \in (-1,1)$, the condition $Var(R) > \frac{1}{3}$ implies that the distribution cannot have a single mode/peak at zero. The intuition behind this is that the distribution of R will have multiple peaks and will therefore imply that $E[R^2]$ will not be concentrated only at zero, but also at larger values. It will also follow that greater significance will erroneously appear even when there is none. The second part of the paper is concerned with a more comprehensive regression simulation of:

$$\begin{gathered} Y_t = Y_{t-1} + a_t \ \wedge \ Y_t' = Y_{t-1} + a_t + b_t \ \wedge Y_0 = 100 \\ X_{j,t} = X_{j,t-1} + a_{j,t} \ \wedge \ X_{j,t}' = X_{j,t-1} + a_{j,t} + b_{j,t} \ \wedge \ X_{j,0} = 100 \end{gathered} \tag{4}$$

where $a_{j,t}, b_{j,t}, a_t, b_t$ are independent white N(0,1) noises. The set $[Y_t, X_{j,t}]$ represents variables for random walks, while $[Y_t', X_{j,t}']$ represents the ARIMA(0,1,1) series. Then the regression equation is given by:

$$Y_t = \beta_0 + \sum_{i=1}^{m} \beta_i \cdot X_{i,t} + \epsilon_t \tag{5}$$

The taxonomy of their chart is as follows: Levels of the regression refer to the raw $Y_t, X_t$ without differencing, while changes refer to the application of differencing to the raw data (detrending). They iterate m=1,..,5 for the number of regressors present in the simulation. Their findings confirm that using only levels of the data, the average $R^2$ steadily grows with the number of regressors m and so does the DW statistic d, spuriously indicating significance in both random walks and ARIMA. In contrast, the changed data for both series correctly shows significance relationships, but stressing the original hypothesis that removing autocorrelations or including the omitted variables corrects the tests.

**2.2 Fractional integration and spectral formalization**

Despite the findings from 1973, the first definition of cointegration was given by Granger (1981). Specifically, the paper analyzed the series:

$$x_t = a(B) \cdot \epsilon_t \,, B^k \epsilon_t = \epsilon_{t-k} \quad (6)$$

where the zero-mean $x_t$ is generated by a white noise series $\epsilon_t$ subject to the linear filter (polynomial in a lag operator) a(B). It is well known(Box & Jenkins, 1970) that if there exists a factorization of the form:

$$a(B) = (1 - B)^{-d} \cdot a'(B) \quad (7)$$

($x_t$ integrated of order d), then we can reduce $x_t$ to form a new ARMA series $x_t'$ of integrating order 0. The hitherto assumption (the Box-Jenkins approach) was that $d \in Z$, but it was already observed in (Granger & Joyeux, 1980a) that relaxing the condition such that $d \in S \subset R$(usually $S = [-1,1]$), might give better modeling properties, such as long-memory. In their paper (Granger & Joyeux, 1980a), they showed an important relationship between the integration order d and the variance of $x_t = a(B)\epsilon_t \sim I(d)$ , ($\epsilon_t$ zero-mean white noise) defined by its spectrum:

$$f(\omega) = \alpha \cdot (1 - cos(\omega))^{-d}, \alpha > 0 \quad (8)$$

They showed that the autocovariances of such series are given by:

$$\mu_t = \alpha \cdot 2^{1+d} \cdot sin(\pi \cdot d) \cdot \Gamma(1 - 2d) \cdot \frac{\Gamma(\tau + d)}{\Gamma(\tau + 1 - d)} \,, for\ d \in < -1, \frac{1}{2} > \quad (9)$$

and that the variance $\mu_0$ diverges to $\infty$ for $d \geq \frac{1}{2}$ They also showed an asymptotic approximation of the autocorrelations for such series:

a stationary ARMA model is

$$\rho_\tau \approx A(d) \cdot \tau^{2d-1}, d \in < -1, \frac{1}{2} >, A(d)\ constant \quad (10)$$

whereas the same quantity for

$$\rho_t \approx A \cdot \theta^\tau, |\theta| < 1 \quad (11)$$

so it can be observed that the ARMA model quantity tends exponentially to zero, faster than the $x_t$ one. But for d=0, which is the most common to differencing operations in practice, we get that for a series

$$x_t = \sum_{j=0}^{\infty} b_j \cdot \epsilon_{t-j} \quad (12)$$

the autocorrelation $\rho_j$ and $b_j$ decrease exponentially in magnitude as $j \to \infty$. In subsection 3 of (Granger, 1981), they analyze the algebra of integrated series and how certain combinations of orders can turn an inconsistent combination into a consistent one. For example suppose that you arrive at an inconsistent model defined by:

$$Y_t = \alpha + \beta \cdot X_t + f(B)\epsilon_t \tag{13}$$

$Y_t \sim I(0)$ = Change in employment, $X_t \sim I(1)$ = level of production
We can see that the integration orders of LHS and RHS differ, but if we set $\beta \to \beta \cdot (1 - B)$, then $I(d_{x_t} + d_{\epsilon_t}) = I(1 - 1) = I(0)$ on the RHS, obtaining a consistent model. The next logical question one might ask; Is it possible for a model

$$b(B) \cdot y_t = c \cdot x_t + g \cdot z_t + h(B) \cdot \epsilon_t \tag{14}$$

$$d_y > 0,\ h(B) \cdot \epsilon_t \sim I(d_y),\ var(\epsilon_t) = 1\ \forall t \tag{15}$$

to have $d_y < max(d_x, d_z)$. The answer is indeed positive and can be proved using spectral analysis by imposing three conditions:

1. $g = -c \cdot \alpha.$
2. $f_x(\omega) = \alpha^2 \cdot f_z(\omega),\ \omega \approx 0 \Longrightarrow d_x = d_z$
3. cross-spectrum between $x_t$ $and$ $y_t$ $cr(\omega) = \alpha \cdot f_z(\omega), \omega \approx 0 \Longrightarrow$ coherence $C(\omega) = 1$ , phase $\phi(\omega) = 0$

and analyzing the spectral representation of the series :

$$|b(z)|^2 \cdot f_y(\omega) = \{c^2 \cdot f_x(\omega) + g^2 \cdot f_z(\omega) + g \cdot c \cdot [cr(\omega) + \underline{cr}(\omega)]\} + \frac{|h(z)|^2}{2\pi} \tag{16}$$

Granger (1981) then defines two series $x_t$ and $y_t$ to be cointegrated if the conditions 2 and 3 are satisfied. If they are cointegrated, then the spectral representation becomes:

$$|b(z)|^2 \cdot f_y(\omega) = \frac{|h(z)|^2}{2\pi} \Leftrightarrow f_y(\omega) = |\frac{h(z)}{b(z)}|^2 \cdot \frac{1}{2\pi}, z = e^{-2\pi i\omega} \tag{17}$$

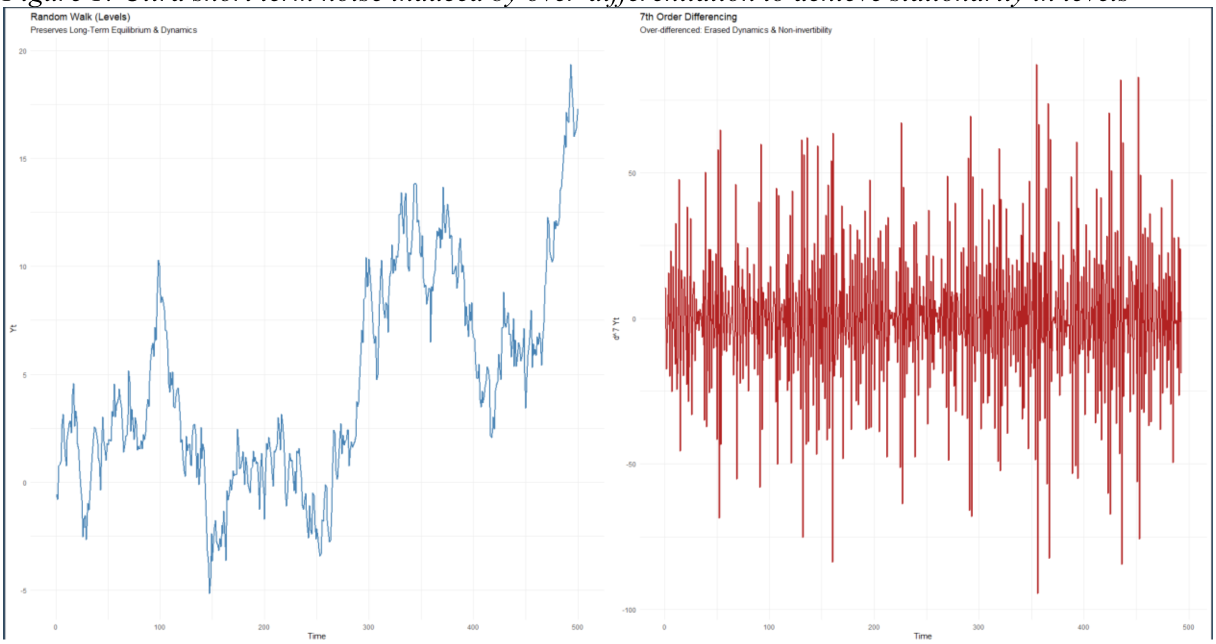
because the terms in the big bracket cancel out. Unpacking the two spectral conditions indeed conforms with our modern intuition of stationarity and cointegration as defined by differencing. As a matter of fact, intuition behind condition 2 is that the mutual spectral behavior at low frequencies(long-term/run) of $x_t$ and $z_t$ is the same up to a constant, i.e. they share long-run components. The third condition expands upon the 2., but also captures the idea that the long-run movements are perfectly linearly related ($C(\omega) = 1$), i.e. if one goes up, the second also goes up by a fixed proportion. It also captures the idea that long-run movements of the two series do not systematically lag each other ($\phi(\omega) = 0$), i.e it is not possible for them to drift infinitely far apart. What is meant when the literature refers to the cointegrated systems sharing a stochastic trend is the following: apply the IDFT and transform the equation back into the time domain at low frequencies:

$$\begin{aligned} y_t &= \frac{h(B)}{b(B)}\varepsilon_t + (stationary\ noise) \\ x_t &= a \cdot \frac{h(B)}{b(B)}\varepsilon_t + (stationary\ noise) \end{aligned} \tag{18}$$

$$z_t = a_z \cdot \frac{h(B)}{b(B)} \varepsilon_t + (stationary\ noise)$$

Therefore, all are generated by a common stochastic process at low(long-run) frequencies: $\tau = \frac{h(B)}{b(B)} \varepsilon_t$. To also see why unit roots of b(B) create problems, suppose that $b\left(e^{-2\pi i \cdot 0}\right) = b(1) = 0$ and h(1) finite, then $f_y(0) \rightarrow +\infty$ explodes at zero frequencies close to 0. The big final conclusion of the article is that there exist cases where the difference between two co-integrated series results in a I(0) series, or more generally a $I\left(d_q\right), d_q < d_x$ series. This is why VAR methodologies and modelling cointegration requires stationarity of the variables being studied. However, applying to many differences to achieve stationarity has a drawback. In their seminal work, Granger and Joyeux (1980) identify the optimal differencing parameter lies within the range $\frac{1}{2}$< d < 1. In this specific interval, the time series exhibits infinite variance, however, in empirical practice, researchers often erroneously apply differencing using a value from the set of integers, specifically d=1, rather than a real number $d \in R$ that matches the true fractional integration of the series. Granger and Joyeux demonstrate that such an integrer to a process results in 'zapping out' the vital low-frequency dynamics. Mathematically, if a series is fractionally integrated, the power spectrum $f(\omega)$ behaves as $\omega^{-2d}$ as $\omega \rightarrow 0$. By forcing a higher order of integration than required (i.e., moving from d to 1), the researcher effectively drives the spectral density at zero frequency to zero (f(0) = 0). This 'cleansing' of low frequencies is catastrophic for long-run modeling. As the authors note, while the resulting series may appear stationary, it loses the persistent autocorrelations that link the distant past to the future. In an Error Correction Model (ECM) context, over-differencing eliminates the very signal required for the system to recognize its long-run equilibrium. Consequently, the long-term forecast from such a model will invariably fail; it will either converge prematurely to a constant mean or drift aimlessly, as the autoregressive components are no longer mathematically linked to the long-term trajectory of the series.
This can be graphically observed in Figure 1, which compares the random walk in levels (blue) and in the 7th level (red):

*Figure 1: Ultra short term noise induced by over-differentiation to achieve stationarity in levels*




As such, it is customary not to go beyond the first levels, since the interpretation and validity of the dynamics are flawed. Even so, the time series function may suffer two drawbacks: removing the long-term cointegration and weakening the short-term dynamics. This is why applying ARDL and ECM models provides a better fit, yet it is not disclosed which is the better choice when studying US and EU market cointegration.

## 3. Methodology: ARDL and ECM models

From the theoretical perspective, it is clear why VAR models and Granger tests yield results that are inadequate for describing the dynamics of cointegration. Therefore, only two models remain viable: ECM and ARDL models. We must then determine which of these provides a better fit for modelling the cointegration between US and EU capital markets (**O1**).

The error correction models is defined as:

$$(1-\mathrm{B})^{\mathrm{d}}\mathrm{a}_1(\mathrm{B})\mathrm{y}_\mathrm{t} = \mathrm{m}_1 + \underbrace{\beta(\mathrm{y}_{\mathrm{t}-1} - \mathrm{A}\mathrm{x}_{\mathrm{t}-1})}_{\text{Error correcting term } \mathrm{E}_\beta} + (1-\mathrm{B})^{\mathrm{d}}\,\mathrm{b}_1(\mathrm{B})\mathrm{x}_\mathrm{t} + \mathrm{c}_1(\mathrm{B})\varepsilon_{1\mathrm{t}} \quad (19)$$

$$(1-B)^d\, a_2(B)x_t = m_2 + c_2(B)\varepsilon_{2t} \quad (20)$$

$$m_1, m_2, \beta \subset R;\ deg\big(a_1(B)\big), deg\big(b_1(B)\big) < \infty \quad (21)$$

$$Where\ \varepsilon_{1t}\ \varepsilon_{2t}\ are\ independent, zero\ mean, finite\ variance\ white\ noise;$$

By some strenuous algebra it can be shown that for $z_t = y_t - A \cdot x_t$:

$$\begin{aligned} a_2(B)[(1-B)^d a_1(B) - \beta B]z_t \\ &= a_2(B)m_1 - Aa_1(B)m_2 + b_1(B)m_2 + c_2(B)\cdot[b_1(B) - Aa_1(B)]\varepsilon_{2t} \\ &+ a_2(B)c_1(B)\varepsilon_{1t}. \end{aligned} \quad (22)$$

Assuming an univariate model(constants are zero) we will arrive at the Granger result:

$$\begin{aligned} a_2(B)\cdot[(1-B)^d \cdot a_1(B) - \beta \cdot B]\cdot z_t \\ &= c_2(B)\cdot[b_1(B) - A\cdot a_1(B)]\cdot \epsilon_{2t} + c_1(B)\cdot a_2(B)\cdot \epsilon_{1t} \end{aligned} \quad (23)$$

From this it follows that $z_t \sim I(0)$ even when $x_t, y_t \sim I(1)$. This shows that series $x_t, y_t$ generated by for d=1 are necessarily cointegrated. The contrapositive also holds for, that is if they are not cointegrated then the LHS has finite variance (since $y_t \sim I(d)$) but $E_\beta \sim I(d_\beta), d_\beta > 0$ (infinite variance), so the model is inconsistent. Cointegration is also an invariant property under linear transformations and finite filtering, that is if $x_t, y_t$ cointegrated, then $x_t^{'} = a + b \cdot x_{t-s}$, $y_t^{'} = c + f \cdot y_{t-k}$ is also a pair of cointegrated series. Granger (1981) discusses difficulties of estimating whether an ECM model is viable given a set of data, that is, pre-testing for cointegration. An essential stumbling block in using regressions to estimate cointegration is the following: Suppose that $x_t$ of is given by $x_t = x_{1t} + \gamma \cdot x_{2t}$ and $x_t, y_t$ cointegrated. Then

$$E_\beta = \beta \cdot \left(y_{t-1} - A_1 x_{1,t-1} - A_2 x_{2,t-1}\right) \quad (24)$$

To belong in $E_\beta$ we suppose that $y_t, x_{1t}, x_{2t} \sim I(1)$. Using spectral techniques and imposing, it can be shown that the coherences have to satisfy:

$$1 - C_{12}^2 - C_{1y}^2 - C_{2y}^2 + 2C_{12}C_{1y}C_{2y} = 0 \quad (25)$$

Where:

$$C_{12}^2 = coherence(x_{1t}, x_{2t})\ at\ low\ frequencies, and\ C_{jy}^2 = coh\big(x_{jt}, y_t\big), j = 1,2, \omega \approx 0$$

Problems arise when we suppose that

$$C_{1,2} = 0 \Rightarrow\ C_{1y}^2 + C_{2,y}^2 = 1 \tag{26}$$

Taking for example:

$$C_{1y} = C_{2,y} = \frac{1}{\sqrt{2}} \Rightarrow\ C_{1y}^2 + C_{2,y}^2 = 1 \tag{27}$$

but $y_t$ is not cointegrated with any $C_{jy}$ by definition. The previous observation implies that we cannot test pairwise cointegration of the components, but must form one big regression equation:

$$z_t = y_t - A_t x_{1t} - A_2 x_{2t} \tag{28}$$

and see if$z_t \sim I(0)$. Omitting crucial components $A_l x_{l,t}, l > 2$ from will result in a higher omitted variable bias:

$$Bias\big(\widehat{A_l}\big) = A_l \cdot \frac{\sigma_{22}\sigma_{1l} - \sigma_{12}\sigma_{2l}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{29}$$

and will therefore result in spurious regressions, suboptimal estimators, and weak determination of cointegration. This is what popular literature implies when it states that the lack of cointegration is due to a scarcity of modelling variables that are correlated with those included. The following paper (Engle & Granger, 1987) formalises these ideas for vector-defined time series in the Granger representation theorem. The authors define a more general cointegration order by:

$$x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{bmatrix} \sim Cl(d,b) \Leftrightarrow \forall t \forall i \in 1, \dots, n\ \left(x_{i,t} \sim I(d)\right)$$

$$\land\ \exists \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} : z_t = \alpha^T \cdot x_t \sim I(d-b) \tag{30}$$

By this multi-component definition, we lose uniqueness of the cointegrating vectors, necessitating their quantification. Consequently, they introduce:

$$\alpha_j = \begin{bmatrix} \alpha_{1,j} \\ \alpha_{2,j} \\ \vdots \\ \alpha_{n,j} \end{bmatrix},\ j \in 1, \dots, r,\ r \leq N-1, \text{cointegrating vectors}$$

$$x_t\ is\ of\ cointegrating\ rank\ r \Leftrightarrow \mathrm{rank}(\alpha) = \mathrm{r}, \alpha = \begin{bmatrix} \alpha_{1,1} & \dots & \alpha_{1,r} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \dots & \alpha_{n,r} \end{bmatrix}$$

Grangers representation theorem then gives us the following results for the following setup: If for $x_t \sim Cl(1,1), x_t\ of\ co.int\ rank\ r \Rightarrow x_{i,t} \sim I(1) \Rightarrow (1-B)x_{i,t} \sim I(0)$. Applying Wold's decomposition and subtracting the deterministic $\eta_t$ part it can be obtained:

$$(1-B)x_t = \sum_{j=1}^{\infty} b_j \cdot \epsilon_{t-j} = C(B) \cdot \epsilon_t \tag{31}$$

C(B) an IIR filter, $\epsilon_t$ uncorrelated white noise. We can rewrite a general moving average filter C(B) as:

$$C(B) = C(1) + (1-B)C^*(B) \tag{32}$$

Suppose now that $z_t = \alpha_j \cdot x_t$

$$(1-B)z_t = \alpha_j\big(C(1) + (1-B)C^*(B)\big)\epsilon_t \tag{33}$$

setting B=1, we get

$$a_j \cdot C(1) = 0 \Rightarrow null\big(C(1)\big) = \{a_j\}_{j=1}^{r} \Rightarrow rank\big(C(1)\big) = n - r \tag{34}$$

Also, because $z_t \sim I(0) \Rightarrow \alpha_j \cdot C^*(1) \neq 0$. Because if not, then by FTA: $C^*(B)(1-B)C^{\#}(B) \Rightarrow (1-B)z_t = (1-B)^2 C^{\#}(B)$ which would contradict $x_t \sim Cl(1,1)$ assumptions (it would imply b=2). Now using Lemma 1 of (Engle & Granger, 1987) and a finite NxN matrix can be obtained with polynomial with G(0) =C(1):

$$det(C(B)) = (1-B)^r g(1-B)$$

$$Adj(C(B)) = (1-B)^{r-1} H(1-B)$$

Multiplying by $Adj\big(C(B)\big)$ and utilizing $C \cdot Adj\big(C(B)\big) = det\big(C(B)\big) \cdot I_N \Rightarrow$

$$H(1-B)x_t = g(1-B)\varepsilon_t \tag{35}$$

where $H$ and $g$ are obtained from Lemma 1 (Engle & Granger, 1987). Now set A(B) = H(1-B), d(B)=g(1-B)

$$A(B)x_t = d(B)\varepsilon_t \tag{36}$$

a vector ARMA model, d(B) a scalar finite lag op ,d(1) finite, rank(A(1))=r, A(0)=0. If the property of Adj(C(B) is applied as follows:

$$C(B) \cdot H(1-B) = (1-B)g(1-B)I_n \tag{37}$$

It can be evaluated at B=1:

$$C(1)H(0) = C(1)A(1) = 0 \Rightarrow Col_j\big(A(1)\big) \in Null\big(C(1)\big)$$

$$\Rightarrow A(1) \in Span(Null(C(1))) \Rightarrow A(1) = \gamma \cdot \alpha', \gamma \in M^{N \times r} \tag{38}$$

Rewriting the A(B) the same as in and using the identity: $x_t = x_{t-1} + (1 - B)x_t$, it can be obtained that:

$$[A(1) + A^*(B)](1 - B)x_t = -A(1)x_{t-1} + d(B)\varepsilon_t \tag{39}$$

Introducing $\tilde{A}(B) = A(1) + A^*(B)$, and using the fact that $\alpha'$ is cointegrating $z_t = \alpha' x_t$; the main result of Grangers representation theorem appears.

$$\tilde{A}(B)(1 - B)x_t = -\gamma z_{t-1} + d(B)\varepsilon_t \tag{40}$$

a general cointegrated relationship gave rise to an ECM model. The methodology consists of first approximating the cointegrating vectors by OLS:

$$y_t = \hat{\alpha}' \cdot x_t + u_t \tag{41}$$

and then use the residuals to approximate an ECM model term $E_{\hat{\beta}}$. Prop 1 and theorem 2 show that if $x_t, y_t \sim I(1)$ then the OLS estimator $\hat{\beta}$ is super consistent (Engle & Granger, 1987). More technically, the estimator converges to the true value at O(T), faster than stationary regressions of $O\left(T^{\frac{1}{2}}\right)$

For the purpose of methodological findings of this paper, the analysis is applied on the assumption of n=1, that is $x_t \in R$. To determine whether the variables $Y_t$ and $X_t$ are cointegrated of order CI (1,1), the stationarity of the residuals of the estimated cointegration equation is examined. Henceforth, the first step is conducting Engle–Granger procedure that estimates the static long-run equilibrium equation using the ordinary least squares (OLS) method and then tests the stationarity of the residuals of the estimated cointegration equation. If the sequence $\{\hat{\varepsilon}_t, t = 1, \dots, n\}$ is stationary, $e_t \sim i.i.d.\left(0, \sigma^2\right)$, then the variables $Y_t$ and $X_t$ are cointegrated and the deviations from the current state can be corrected as:

$$\Delta Y_t = \lambda \Delta X_t + \pi(Y_{t-1} - \alpha - \beta X_{t-1}) + \upsilon_t \tag{42}$$

or equivalently,

$$\Delta Y_t = \lambda \Delta X_t + \pi \varepsilon_{t-1} + \upsilon_t \tag{43}$$

The equilibrium errors $\varepsilon_{t-1}$ measure the deviation of the actual value of $Y_{t-1}$ from its equilibrium value $(\alpha + \beta X_{t-1})$ and are referred to as the error correction factor, expressed as:

$$\varepsilon_{t-1} = Y_{t-1} - (\alpha + \beta X_{t-1})\varepsilon t - 1 \tag{44}$$

In such case, there exists an ECM that describes the relationship between the variables and is estimated as:

$$\Delta Y_t = \lambda \Delta X_t + \pi \widehat{\varepsilon_{t-1}} + \upsilon_t \tag{45}$$

An alternative approach to defining the long-run equation and estimating the cointegration parameters α and β can be obtained using the unrestricted Auto Regressive Distributed Lag model (ARDL). The ARDL(p,q) model defined for two variables $Y_t$ and $X_t$ includes p lags of the first (dependent) variable $Y_t$ and q lags of the second (independent) variable $X_t$ , i.e.:

$$Y_t = \mu + \gamma_1 Y_{t-1} + \gamma_2 Y_{t-2} + \dots + \gamma_p Y_{t-p} + \delta_0 X_t + \delta_1 X_{t-1} + \delta_2 X_{t-2} + \dots + \delta_q X_{t-q} + \varepsilon_t \tag{46}$$

or in summation form:

$$Y_t = \mu + \sum_{i=1}^{p} \gamma_i Y_{t-i} + \sum_{i=0}^{q} \delta_i X_{t-i} + \varepsilon_t \tag{47}$$

Alternatively, if the model is defined with one lag for both variables, it can be expressed as

$$Y_t = \mu + \gamma_1 Y_{t-1} + \delta_0 X_t + \delta_1 X_{t-1} + \varepsilon_t \tag{48}$$

Same as in the ECM model, the variables themselves don't need to be stationary, but they must be integrated at the same levels. Also, it is essential that their residuals exhibit white noise dynamics. With no unit root presence, the long-term dynamics can be expressed and meaning that the ARDL(1,1) model is equivalent to the ECM:

$$\Delta Y_t = \lambda \Delta X_t + \pi \widehat{\varepsilon_{t-1}} + e_t \tag{49}$$

where $\lambda = \delta_0$, and the speed of adjustment parameter (equilibrium correction term) equals: $\pi = -(1-\gamma_1)$. The relationship between the parameters of the cointegration equation (long-run equation $Y_t = \alpha + \beta X_t + \varepsilon_t$) and the ARDL model parameters is given by the rearranging of terms in the ARDL equation as:

$$\Delta Y_t = \mu + (\gamma_1 - 1)Y_{t-1} + \delta_0 \Delta X_t + (\delta_0 + \delta_1)X_{t-1} + e_t \tag{50}$$

or equivalently,

$$\Delta Y_t = \delta_0 \Delta X_t + \mu + (\gamma_1 - 1)Y_{t-1} + (\delta_0 + \delta_1)X_{t-1} + e_t \tag{51}$$

which can be further expressed as:

$$\Delta Y_t = \delta_0 \Delta X_t + (\gamma_1 - 1)\left[Y_{t-1} - \frac{\mu}{1-\gamma_1} - \frac{(\delta_0 + \delta_1)}{1-\gamma_1} X_{t-1}\right] + e_t \tag{52}$$

Given the general form of the Error Correction Model (ECM):

$$\Delta Y_t = \lambda \Delta X_t + \pi \varepsilon_{t-1} + \upsilon_t \tag{53}$$

it follows that:

$$\lambda = \delta_0, \quad \pi = -(1-\gamma_1) \tag{54}$$

Furthermore, if the cointegration (long-run) equation is defined as:

$$Y_t = \alpha + \beta X_t + \varepsilon_t \tag{55}$$

then the parameters of the cointegration of ARDL equation are:

$$\alpha = \frac{\mu}{1-\gamma_1}, \quad \beta = \frac{\delta_0 + \delta_1}{1-\gamma_1} \tag{56}$$

The analysis demonstrates that the primary advantage of the ARDL(1,1) framework over the traditional ECM approach to cointegration is its ability to estimate long-run parameters through a single-equation system. In contrast, the ECM approach may rely on biased OLS estimations of long-term dynamics, which increases the risk of spurious results in unit root testing. These spurious outcomes arise because the initial estimation bias is significantly exacerbated during the OLS processing of the unit root, leading to unreliable inferences regarding the level of integration between variables. This dynamics will be further discussed in the following chapter.

## 4. Empirical testing

To validate the assumptions outlined in the previous chapters, this paper applies the ARDL (1,1) and ECM models to two financial time series: the US equity market (represented by the S&P 500) and the EU equity market (represented by the DAX 30). The time series studied covers the period from 1 January 2021 to 4 December 2025, using daily closing prices that are differenced and expressed as natural logarithms, denoted in econometric as ln. Data for both stock tickers is retrieved from Yahoo Finance. The paper begins by testing for stationarity of the data in levels and their order of integration (I). To this end, we conducted the Augmented Dickey–Fuller (ADF) unit root test for both series in levels and first differences. The results indicate that both variables are non-stationary in levels, as the null hypothesis of a unit root cannot be rejected at conventional significance levels. These results are presented in Table 1:

*Table 1: Unit root test for both time series in levels*

| Series | Dickey–Fuller statistic | Lag order | p-value | Decision (5% level) | Interpretation |
|---|---|---|---|---|---|
| **DAX** | -1.8189 | 10 | 0.6550 | Fail to reject ($H_0$) | Non-stationary in levels |
| **S&P 500** | -1.5417 | 10 | 0.7723 | Fail to reject ($H_0$) | Non-stationary in levels |

Subsequently, we transformed both series into first differences and re-applied the Augmented Dickey–Fuller (ADF) test. The second-round unit-root testing indicates both time series are integrated in the first level (I), as seen in Table 2:

*Table 2: Unit root test for both time series in first levels (I)*

| Series | Dickey–Fuller statistic | Lag order | Reported p-value | Decision (5% level) | Conclusion |
|---|---|---|---|---|---|
| $(\Delta \boldsymbol{DAX})$ | -9.9175 | 10 | 0.0100* | Reject ($H_0$) | Stationary |
| $(\Delta \boldsymbol{S\&P500})$ | -10.7260 | 10 | 0.0100* | Reject ($H_0$)) | Stationary |

Regarding the earlier conclusion, integration of the same level opens a possibility that the two financial time series are cointegrated in the long run, constructing the long term cointegration function from the squared residuals as:

$$ln\ (DAX)_{\ t} = 1.090 + 1.021\ ln\ ln\ (SP500)_{\ t} + e_t$$

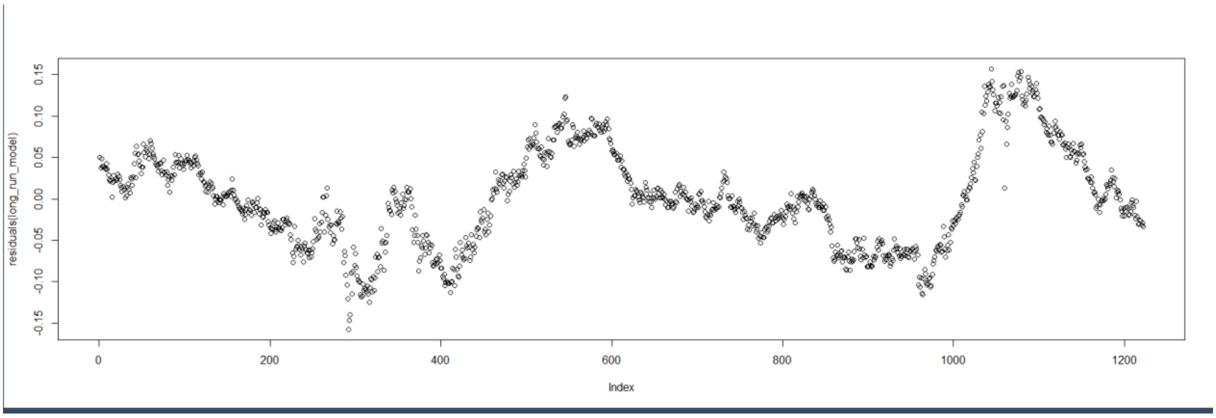
Continuing on the set theory, the stationarity of the residuals of the long term cointegration function is tested with ADF and is shown in Table 3:

*Table 3: ADF test of the residuals of the long term cointegration function*

| Variable | Coefficient | p-value |
|---|---|---|
| $\boldsymbol{\omega/\alpha_0}$ (intercept) | *1.090* | - |
| $\boldsymbol{ln\ ln}\ (\boldsymbol{sp500})$ | *1.021* | - |
| ADF test of the residuals | *−2.2977* | *0.4523* |

The results indicate that the residuals have a unit root, meaning that the two time series are not cointegrated and instead exhibit spurious regression due to the presence of a unit root in the residuals (Phillips and Ouliaris, 1990). This can be observed graphically in their plot shown in Figure 1:

*Figure 1: The plot of the residuals from the long term cointegration function*




The non-stationarity of the residuals indicates that the classical two-step ECM procedure is not applicable to these data, as the prerequisite of a stationary cointegrating residual is not met. Two options remain: differencing both series to achieve stationarity (which has been shown to weaken the dynamics), or applying the ARDL Bounds Testing procedure (Pesaran et al., 2001), which tests for a level relationship directly within a single-equation framework without requiring pre-tested stationary residuals. The results of the latter are shown in Table 4:

*Table 4: ARDL Bound testing procedure results*

| Test | Null Hypothesis ($\boldsymbol{H_0}$) | F-statistic | p-value | Result |
|---|---|---|---|---|
| Bounds F-test | No level relationship exists ($\delta_1 = \delta_2 = 0$) | 3.3498 | < 0.05 | Reject $H_0$ |

The calculated F-statistic surpasses the upper critical bound, thereby confirming a stable long-run relationship. This suggests that the classical ECM approach was unable to identify these dynamics, due to the described limitations.

Using the same data and applying the ARDL model yields successfully estimates the dynamic structure in a single step, giving the statistically significant parameters from Table 5:

*Table 5: ARDL estimation of the cointegration*

| **Variable** | **Coefficient** | **Standard error** | **t-statistic** | **p-value** |
|---|---|---|---|---|
| $\boldsymbol{\mu}$ | 0.011052 | 0.014287 | 0.774 | 0.439 |
| $\boldsymbol{\gamma_1}$ | 0.987873 | 0.004696 | 210.372 | < 2e-16 *** |
| $\boldsymbol{\delta_0}$ | 0.451094 | 0.025306 | 17.826 | < 2e-16 *** |
| $\boldsymbol{\delta_1}$ | -0.438433 | 0.025460 | -17.221 | < 2e-16 *** |
| **Model statistics** | | | | |
| $\boldsymbol{R^2}$ | 0.9974 | | | |
| **Adjusted $\boldsymbol{R^2}$** | 0.9974 | | | |
| **F- statistic** | 155.000* | (df = 3; 1217) | | p < 2.2e-16 |

From Table 5, the ARDL models is defined as follows:

$$Y_t = 0.011\mu + 0.988\gamma_1 + 0.451\delta_0 - 0.438\delta_1$$

With all the estimated parameters showing statistical significance except for the mean estimate, the contemporaneous effect of the S&P 500 on the DAX index is positive. Specifically, a 1% increase in the S&P 500 index results in an immediate appreciation of the DAX by approximately 0.45%, ceteris paribus. If the long run of the ARDL is defined as:

$$\beta = \frac{\delta_0 + \delta_1}{1 - \gamma_1} \tag{57}$$

And the short run $\alpha$ as:

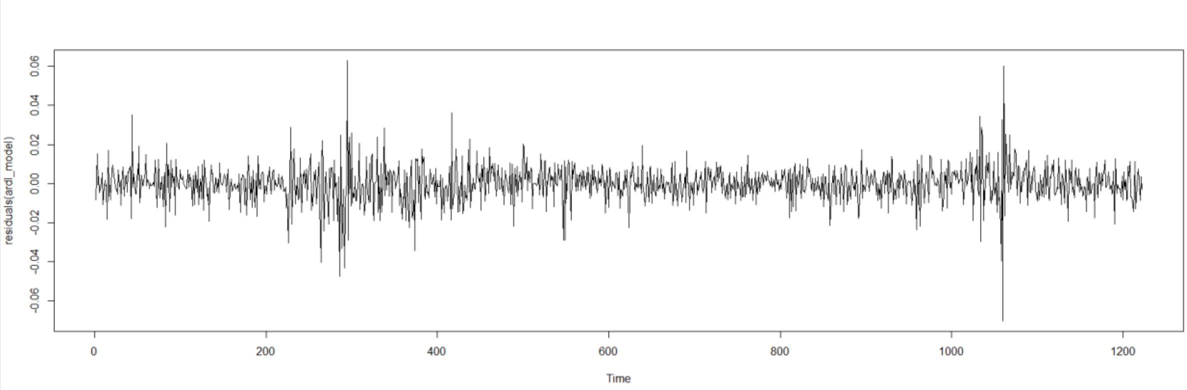
$$\alpha = \frac{\mu}{1 - \gamma_1} \tag{58}$$

Substituting the estimated parameters, we obtain long term multiplier as:

$$\beta = \frac{0.451 - 0.438}{1 - 0.988} = \frac{0.013}{0.012} \approx 1.08 \tag{59}$$

The resulting long-run multiplier of 1.08 indicates that in the long run, the DAX index follows the S&P 500 in a nearly one-to-one ratio. This validates such results that contradict the ECM model approach, the residuals of the ARDL dynamics are given in Figure 2:

*Figure 2: Residuals from the ARDL estimate*




From the given figure, it can be observed that the tested residuals follow pure random walk meaning that the parameter estimates are free of bias, making them consistent and robust for the interpretation of the long-run market dynamics and their cointegration.

## 5. Discussion and conclusion

This article examined the long-run cointegration between the German DAX and the US S&P 500 from January 2021 to December 2025, with the explicit aim of determining which econometric framework – the classical ECM or the ARDL – constitutes the more reliable methodology for detecting and estimating that relationship. The answer provided by the theoretical analysis and empirical results is unambiguously in favour of ARDL.

The theoretical argument proceeds in three stages. The first, grounded in Granger and Newbold (1973, 1974), establishes that regressing non-stationary series in levels produces spuriously significant results, motivating differencing as a corrective. The second, drawing on Granger (1981) and Granger and Joyeux (1980), demonstrates that this corrective carries a hidden cost: if the true integration order is fractional rather than integer, forcing d = 1 drives the spectral density at zero frequency to zero, eliminating precisely the low-frequency co-movement that constitutes the long-run relationship. This step shows that conducting VAR and Granger causality tests on over-differenced data weakens the modelled dynamics, meaning that either ECM or ARDL must be applied to model the cointegration correctly. The third stage shows that the Engle-Granger two-step ECM procedure inherits this problem structurally, because any bias in the first-step OLS estimation of the cointegrating vector is amplified in the second-step ADF test on residuals, making the overall procedure unreliable even when a genuine long-run relationship exists. The ARDL model, however, bypasses such issues because it can estimate the parameters directly without requiring pre-testing of residuals for stationarity, as the Bounds Testing procedure inherently accounts for the integration properties of the variables and identifies a stable long-run relationship even when individual series are non-stationary.

The empirical results directly confirm this theoretical prediction. Applied to the same dataset, the Engle-Granger procedure fails to reject the null hypothesis of no cointegration (p = 0.4523), while the ARDL Bounds F-test rejects the null of no level relationship and recovers a long-run multiplier of approximately 1.08, indicating near one-to-one co-movement between the two markets in the long run. The ARDL residuals follow a pure random walk, confirming that the parameter estimates are unbiased and robust. The divergence of these two results from identical data is itself the central finding of the paper: it provides direct empirical evidence that the inferential failure of the ECM approach is not merely theoretical but operational.

The practical implications are clear. For portfolio managers, the confirmed long-run multiplier indicates that the DAX and S&P 500 share a common stochastic trajectory and cannot be treated as independent assets over longer horizons. This means that diversification strategies assuming independence will systematically underestimate co-movement risk. For financial stability analysts, the result strengthens the case for coordinated monitoring of cross-Atlantic equity dynamics. For applied econometricians, the paper offers a replicable methodological template: when studying cointegration between integrated financial series, the ARDL Bounds Testing procedure should be the default specification, with the Engle-Granger two-step approach reserved only for contexts where its assumptions can be explicitly verified.

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