

NOTE ON CONCENTRATION VIA THE CONJUGATE-LINEAR HODGE STAR OPERATOR

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ABSTRACT. We construct conjugate-linear perturbations of twisted spin^c Dirac operators on compact almost Hermitian manifolds of dimension congruent to 2 or 6 modulo 8, employing the conjugate-linear Hodge star operator rescaled by unit complex numbers depending on degree. These perturbations satisfy the concentration principle.

1. INTRODUCTION

One of the most fruitful ideas in geometry is *localization*, which underlies the celebrated Witten deformation. Based on this idea, consider a first-order elliptic operator

$$D : \Gamma(E) \rightarrow \Gamma(F)$$

on a compact Riemannian manifold X , and a deformation of the form

$$D_s = D + sA,$$

where $s \in \mathbb{R}$ and $A : E \rightarrow F$ is a bundle map.

As shown in [4], if the perturbation A satisfies the following algebraic condition:

$$(1.1) \quad \sigma_{D^*}(\gamma) \circ A + A^* \circ \sigma_D(\gamma) = 0 \quad \forall \gamma \in T^*X,$$

where σ_D denotes the principal symbol of D , then as $s \rightarrow \infty$, the kernel of D_s (as well as low eigenspaces of $D_s^*D_s$) becomes concentrated near the *singular set* Z_A of A , defined by

$$Z_A := \{x \in X : \ker(A_x) \neq 0\}.$$

2020 *Mathematics Subject Classification.* 53C27, 53C55.

Key words and phrases. Twisted spin^c Dirac operator, Conjugate-linear perturbation, Concentration principle.

The most well-known example of this *concentration principle* is the Witten deformation [12], where D is the Hodge–de Rham operator. Taubes’ localization proof of the Riemann–Roch theorem is another well-known and ingenious example [11].

When D is a Dirac operator, Prokhorenkov and Richardson [8] formulated the concentration principle for complex-linear perturbations A and classified such perturbations. Maridakis [4] later extended the principle to include real elliptic operators as described above, yielding many interesting examples.

The concentration principle has appeared in various contexts; see, for example, [9, 3, 1, 7, 5, 2]. In particular, Nagy [5] constructed conjugate-linear perturbations A satisfying the concentration principle in a general setting, using nondegenerate invariant bilinear forms on spinor modules.

The present work aims to generalize the conjugate-linear perturbations introduced in [3] (see Remark 3.4) by constructing conjugate-linear perturbations A_φ of the twisted spin^c Dirac operator

$$(1.2) \quad D_E : \Gamma(\mathbb{S}^+ \otimes E) \rightarrow \Gamma(\mathbb{S}^- \otimes E)$$

on a compact almost Hermitian manifold X with $\dim X \equiv 2, 6 \pmod{8}$. These perturbations are defined via the conjugate-linear Hodge star operator rescaled by unit complex numbers depending on degree, together with a section φ of $K_X \otimes (E^*)^2$, and satisfy the condition (1.1) (see Theorem 3.2). The dimension restriction is due to X being an almost complex manifold and the fact that the perturbations A_φ interchange $\mathbb{S}^+ \otimes E$ and $\mathbb{S}^- \otimes E$ when $\dim X \equiv 2, 6 \pmod{8}$.

The rest of the paper is organized as follows. In Section 2, we briefly review basic well-known facts about twisted spin^c Dirac operators D_E , and present observations central to our construction. In Section 3, we construct conjugate-linear perturbations A_φ of the operator D_E satisfying (1.1), and provide examples where the singular set Z_{A_φ} is empty; in such cases, the index of the operator D_E is zero by the concentration principle (see Corollary 3.6).

2. PRELIMINARIES

Throughout this paper, X denotes a compact almost Hermitian manifold of dimension $2n$, equipped with an almost complex structure J and a Riemannian metric g compatible with the almost complex structure J , while E denotes a Hermitian vector bundle over X .

This section briefly reviews basic well-known facts relevant to our discussion, fixes notation, and presents key observations to our construction.

2.1. Twisted Dirac operator. The almost complex structure J , extended by complex linearity to $TX \otimes \mathbb{C}$ and $T^*X \otimes \mathbb{C}$, induces the decompositions:

$$TX \otimes \mathbb{C} = (TX)^{1,0} \oplus (TX)^{0,1} \quad \text{and} \quad T^*X \otimes \mathbb{C} = (T^*X)^{1,0} \oplus (T^*X)^{0,1},$$

where $(TX)^{1,0}$ and $(TX)^{0,1}$ are the $\pm i$ -eigenbundles of J , and similarly for $T^*X \otimes \mathbb{C}$.

Let $g_{\mathbb{C}}$ denote the complex bilinear extension of the metric g to $\Lambda_{\mathbb{C}}^*X = \Lambda^*X \otimes \mathbb{C}$. The associated Hermitian metric on $\Lambda_{\mathbb{C}}^*X$ is defined by

$$\langle \alpha, \beta \rangle := g_{\mathbb{C}}(\alpha, \bar{\beta}).$$

The almost Hermitian manifold X carries a canonical spin^c structure, which comes with a spinor bundle

$$(2.1) \quad \mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- \quad \text{where} \quad \mathbb{S}^{\pm} = \Lambda^{0,\text{even/odd}}X,$$

and a Clifford multiplication $c : T^*X \rightarrow \text{End}_{\mathbb{C}}(\mathbb{S}^{\pm}, \mathbb{S}^{\mp})$, given by

$$(2.2) \quad c(\gamma)\alpha := \sqrt{2}(\gamma^{0,1} \wedge \alpha - \iota(\gamma^{1,0})\alpha),$$

where $\iota(\gamma^{1,0})$ denotes contraction by $\gamma_{0,1}^{\sharp}$, the vector dual to $\gamma^{1,0}$ with respect to $g_{\mathbb{C}}$. Note that

$$\langle \gamma^{0,1} \wedge \alpha, \beta \rangle = \langle \alpha, \iota(\gamma^{1,0})\beta \rangle \quad \text{and} \quad \langle c(\gamma)\alpha, \beta \rangle = -\langle \alpha, c(\gamma)\beta \rangle.$$

A spin^c connection ∇ on the spinor bundle \mathbb{S} , induced from the Levi-Civita connection on X and a Hermitian connection on K_X^{-1} , preserves the chiral grading in (2.1). The induced connection ∇^+ on \mathbb{S}^+ , together with a Hermitian connection ∇^E on the vector bundle E and the Clifford multiplication, defines the twisted Dirac operator (1.2) via the composition

$$(2.3) \quad D_E : \Gamma(\mathbb{S}^+ \otimes E) \xrightarrow{\nabla^+ \otimes \text{Id}_E + \text{Id}_{\mathbb{S}^+} \otimes \nabla^E} \Gamma(T^*X \otimes \mathbb{S}^+ \otimes E) \xrightarrow{c \otimes \text{Id}_E} \Gamma(\mathbb{S}^- \otimes E).$$

Consequently, the principal symbols of D_E and its adjoint D_E^* are given by

$$\sigma_{D_E}(\gamma) = c(\gamma) \otimes \text{Id}_E \quad \text{and} \quad \sigma_{D_E^*}(\gamma) = -\sigma_{D_E}(\gamma)^* = c(\gamma) \otimes \text{Id}_E$$

for all $\gamma \in T^*X$. (See, e.g., [6, 10] for more on Dirac operators.)

2.2. Conjugate-linear Hodge star operator. The conjugate-linear operator $\bar{*}$, which maps $\Lambda^{p,q}X$ to $\Lambda^{n-p,n-q}X$, is defined by

$$\alpha \wedge \bar{*}(\beta) = \langle \alpha, \beta \rangle dv_g$$

for $\alpha, \beta \in \Omega^{p,q}(X)$, where dv_g denotes the volume form on X . The following lemma is a key observation in our construction.

LEMMA 2.1. *For any $\beta \in \Omega^{0,p}(X)$, $\gamma \in \Omega^1(X)$, and $\eta \in \Omega^{n,0}(X)$, we have:*

- (a) $|\eta|^2 \bar{*}(\gamma^{0,1} \wedge \beta) = (-1)^{n(p+1)+p} \eta \wedge \iota(\gamma^{1,0}) \bar{*}(\eta \wedge \beta)$.
- (b) $|\eta|^2 \bar{*}(\iota(\gamma^{1,0})\beta) = (-1)^{(n+1)(p-1)} \eta \wedge \gamma^{0,1} \wedge \bar{*}(\eta \wedge \beta)$.

PROOF. For any $\alpha \in \Omega^{0,p+1}X$, we have:

$$\begin{aligned} \alpha \wedge |\eta|^2 \bar{*}(\gamma^{0,1} \wedge \beta) &= |\eta|^2 \langle \alpha, \gamma^{0,1} \wedge \beta \rangle dv_g = |\eta|^2 \langle \iota(\gamma^{1,0})\alpha, \beta \rangle dv_g \\ &= \langle \eta \wedge \iota(\gamma^{1,0})\alpha, \eta \wedge \beta \rangle dv_g = \eta \wedge \iota(\gamma^{1,0})\alpha \wedge \bar{*}(\eta \wedge \beta) \\ &= (-1)^p \eta \wedge \alpha \wedge \iota(\gamma^{1,0})\bar{*}(\eta \wedge \beta) \\ &= \alpha \wedge \left((-1)^{n(p+1)+p} \eta \wedge \iota(\gamma^{1,0})\bar{*}(\eta \wedge \beta) \right), \end{aligned}$$

where the third equality follows from the definition of the Hermitian metric on $\Lambda_{\mathbb{C}}^*X$, and the sign factors follow since contraction and wedge product are antiderivations. This implies (a). The proof of (b) is similar. \square

2.2.1. *Rescaling.* For our purposes, we rescale the operator $\bar{*}$ as follows:

$$(2.4) \quad \tau := \epsilon_k \bar{*} \quad \text{on} \quad \Lambda_{\mathbb{C}}^k X, \quad \text{where} \quad \epsilon_k = i^{k(k-1)+n}.$$

The unit complex number ϵ_k , which also appears in the context of the signature operator, plays a useful role in our construction. Observe that

$$(2.5) \quad \bar{*}^2 = (-1)^k \text{Id} \quad \text{on} \quad \Lambda_{\mathbb{C}}^k X \quad \text{and} \quad \tau^2 = (-1)^n \text{Id}.$$

The following identity is central to our construction.

PROPOSITION 2.2. *For any $\beta \in \Omega^{0,p}(X)$, $\gamma \in \Omega^1(X)$, and $\eta \in \Omega^{n,0}(X)$, we have*

$$|\eta|^2 \tau \circ c(\gamma)(\beta) = (-1)^{\frac{n(n+1)}{2}+1} \eta \wedge c(\gamma) \circ \tau(\eta \wedge \beta).$$

PROOF. The proof follows from Lemma 2.1 and the identity:

$$\epsilon_{p+1} \epsilon_{n+p}^{-1} (-1)^{n(p+1)+p} = (-1)^{\frac{n(n+1)}{2}} = \epsilon_{p-1} \epsilon_{n+p}^{-1} (-1)^{(n+1)(p-1)}.$$

\square

2.2.2. *Adjoint.* Since τ is conjugate-linear, its adjoint τ^* is defined with respect to the real part of the Hermitian metric. By (2.5), we obtain:

$$\text{LEMMA 2.3.} \quad \tau^* = (-1)^n \tau.$$

PROOF. For any $\alpha \in \Omega_{\mathbb{C}}^k(X)$ and $\beta \in \Omega_{\mathbb{C}}^{2n-k}(X)$, we have

$$\begin{aligned} \langle \beta, \tau \alpha \rangle dv_g &= \bar{\epsilon}_k (-1)^k \beta \wedge \alpha = \bar{\epsilon}_k \alpha \wedge \beta = \bar{\epsilon}_k \epsilon_{2n-k} (-1)^{2n-k} \alpha \wedge \bar{*}(\tau \beta) \\ &= (-1)^n \langle \alpha, \tau \beta \rangle dv_g. \end{aligned}$$

This shows that $\text{Re}\langle \tau \alpha, \beta \rangle = \text{Re}\langle \alpha, (-1)^n \tau \beta \rangle$, and hence proves the lemma. \square

2.2.3. *Operator with values in E .* The Hermitian metric on E , still denoted by $\langle \cdot, \cdot \rangle$, defines a conjugate-linear isomorphism $E \rightarrow E^*$ by $\xi \mapsto \xi^* := \langle \cdot, \xi \rangle$. This induces the map

$$\bar{*}_E : \Lambda^{p,q} X \otimes E^* \rightarrow \Lambda^{n-p, n-q} X \otimes E$$

defined by $\bar{*}_E(\alpha \otimes \xi^*) = \bar{*}(\alpha) \otimes \xi$. Now we set

$$\tau_E := \epsilon_k \bar{*}_E \quad \text{on} \quad \Lambda_{\mathbb{C}}^k X \otimes E.$$

Note that $\tau_E(\alpha \otimes \xi^*) = \tau(\alpha) \otimes \xi$ and $\tau_E^*(\alpha \otimes \xi) = (-1)^n \tau(\alpha) \otimes \xi^*$.

3. CONJUGATE-LINEAR PERTURBATIONS

3.1. *Concentrating condition.* This subsection constructs conjugate-linear perturbations of the twisted Dirac operator D in (2.3) that satisfy the concentrating condition (1.1).

DEFINITION 3.1. *Given a section $\varphi \in \Gamma(K_X \otimes (E^*)^2)$, viewed as a complex-linear bundle map $\varphi : E \rightarrow K_X \otimes E^*$, we define a conjugate-linear bundle map*

$$A_\varphi : \Lambda^{0,p} X \otimes E \rightarrow \Lambda^{0, n-p} X \otimes E$$

by the composition

$$A_\varphi : \Lambda^{0,p} \otimes E \xrightarrow{\text{Id} \otimes \varphi} \Lambda^{0,p} \otimes K_X \otimes E^* \cong \Lambda^{n,p} \otimes E^* \xrightarrow{\tau_E} \Lambda^{0, n-p} \otimes E.$$

Noting $A_\varphi : \mathbb{S}^\pm \otimes E \rightarrow \mathbb{S}^\mp \otimes E$ when $\dim X \equiv 2, 6 \pmod{8}$, and using the decomposition

$$E^* \otimes E^* = \text{Sym}^2 E^* \oplus \Lambda^2 E^*,$$

we obtain concentrating pairs $(\sigma_{D_E}, A_\varphi)$ in the sense of [4].

THEOREM 3.2. *Let D_E be the twisted Dirac operator in (2.3) and A_φ be as in Definition 3.1. Suppose $\varphi \in \Gamma(K_X \otimes \text{Sym}^2 E^*)$ if $\dim X \equiv 2 \pmod{8}$ and $\varphi \in \Gamma(K_X \otimes \Lambda^2 E^*)$ if $\dim X \equiv 6 \pmod{8}$. Then A_φ satisfies the concentrating condition*

$$(3.1) \quad \sigma_{D_E^*}(\gamma) \circ A_\varphi + A_\varphi^* \circ \sigma_{D_E}(\gamma) = 0 \quad \forall \gamma \in T^* X.$$

PROOF. In unitary frames $\{\phi_i\}$ for E (with dual coframe $\{\phi^i\}$) and η for K_X (with dual coframe η^*), the complex bundle map $\varphi : E \rightarrow K_X \otimes E^*$ is given by

$$(3.2) \quad \varphi = \varphi_{ij} \eta \otimes \phi^i \otimes \phi^j : \xi = \xi_k \phi_k \mapsto \varphi_{ij} \xi_j \eta \otimes \phi^i.$$

It follows that for any $\beta \otimes \xi \in \Gamma(\mathbb{S}^+ \otimes E)$ and any $\gamma \in \Omega^1(X)$,

$$(3.3) \quad \begin{aligned} \sigma_{D^*}(\gamma) \circ A_\varphi(\beta \otimes \xi) &= (\text{c}(\gamma) \otimes \text{Id}_E) \circ \tau_E(\eta \wedge \beta \otimes \varphi_{ij} \xi_j \phi^i) \\ &= \text{c}(\gamma) \tau(\eta \wedge \beta) \otimes \bar{\varphi}_{ij} \bar{\xi}_j \phi_i. \end{aligned}$$

On the other hand, since $|\eta|^2 = 1$, by Lemma 2.3 and Proposition 2.2, we have

$$(3.4) \quad \begin{aligned} \tau_E^* \circ \sigma_D(\gamma)(\beta \otimes \xi) &= (-1)^n \tau(c(\gamma)\beta) \otimes \bar{\xi}_k \phi^k \\ &= (-1)^{\frac{n(n+1)}{2} + n+1} \eta \wedge c(\gamma) \circ \tau(\eta \wedge \beta) \otimes \bar{\xi}_k \phi^k. \end{aligned}$$

Observe that the adjoint $\varphi^* : K_X \otimes E^* \rightarrow E$ of the complex bundle map φ is given by

$$\varphi^* = \bar{\varphi}_{ji} \phi_i \otimes \eta^* \otimes \phi_j : \nu = \nu_k \eta \otimes \phi^k \mapsto \bar{\varphi}_{ji} \nu_j \phi_i.$$

With the canonical isomorphism $\Lambda^{n,p} X \cong \Lambda^{0,p} \otimes K_X$, from (3.4) we obtain

$$(3.5) \quad \begin{aligned} &A_\varphi^* \circ \sigma_D(\gamma)(\beta \otimes \xi) \\ &= (-1)^{\frac{n(n+1)}{2} + n+1} \text{Id} \otimes \varphi^*(c(\gamma) \circ \tau(\eta \wedge \beta) \otimes \bar{\xi}_k \eta \otimes \phi^k) \\ &= (-1)^{\frac{n(n+1)}{2} + n+1} c(\gamma) \circ \tau(\eta \wedge \beta) \otimes \bar{\varphi}_{ji} \bar{\xi}_j \phi_i. \end{aligned}$$

By the assumption on φ , (3.3), and (3.5), the proof now follows from the identity

$$(-1)^{\frac{n(n+1)}{2} + n+1} = \begin{cases} -1 & \text{if } n \equiv 1 \pmod{4}, \\ 1 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

□

The next two remarks are related to the motivation for our approach.

REMARK 3.3. For the twisted Dirac operator $D_E : \Gamma(\mathbb{S}^+ \otimes E) \rightarrow \Gamma(\mathbb{S}^- \otimes E)$, there is no nonzero complex-linear bundle map $A : \mathbb{S}^+ \otimes E \rightarrow \mathbb{S}^- \otimes E$ that satisfies the concentrating condition (3.1). This explains why the Dolbeault and signature operators do not admit such complex perturbations. (See Section 2 of [8].)

REMARK 3.4. When $\dim X = 2$, the manifold X is a compact complex curve, and the operator $D_E = \sqrt{2} \bar{\partial}_E : \Omega^0(E) \rightarrow \Omega^{0,1}(E)$ is the usual Cauchy–Riemann operator on E . In [3], when E is a holomorphic line bundle and φ is a holomorphic section of $K_X \otimes (E^*)^2$, a conjugate-linear bundle map $R : E \rightarrow \Lambda^{0,1} X \otimes E$ is defined as the composition $R = \bar{*}_E \circ \varphi$. In this case, $A_\varphi = iR$ and $A_\varphi^* = iR^*$, so our construction generalizes the conjugate-linear bundle map R .

Indeed, $A_\varphi = iR$ satisfies the following equation, which is stronger than the concentrating condition:

$$D_E^* \circ A_\varphi + A_\varphi^* \circ D_E = 0$$

(see Lemma 2.1 of [3]). When E is a theta characteristic (i.e., $E^2 \cong K_X$), this implies that A_φ restricts to a conjugate-linear isomorphism $\ker D_E \rightarrow \ker D_E^*$, which leads to a proof of the Atiyah–Mumford theorem: $h^0(E) (= \dim \ker D_E) \pmod{2}$ is *deformation invariant* (see Section 3 of [3]).

3.2. *Concentration principle.* Let $D_E, A_\varphi : \Gamma(\mathbb{S}^+ \otimes E) \rightarrow \Gamma(\mathbb{S}^- \otimes E)$ be as in Theorem 3.2 and consider the deformation of the twisted Dirac operator D_E given by

$$D_s = D_E + sA_\varphi.$$

The following concentration principle shows that as $s \rightarrow \infty$, the kernel of D_s (as well as low eigenspaces of $D_s^*D_s$) becomes concentrated near the singular set

$$Z_\varphi := Z_{A_\varphi} = \{x \in X : \det(\varphi_x) = 0\}.$$

THEOREM 3.5. *Let $D_s = D_E + sA_\varphi$ be as above. For each $\delta > 0$ and $C \geq 0$, there exists a constant $C' = C'(\delta, A_\varphi, C) > 0$ such that if $\zeta \in \Gamma(\mathbb{S}^+ \otimes E)$ satisfies $\|\zeta\|_2 = 1$ and $\|D_s\zeta\|_2^2 \leq C|s|$, where $\|\cdot\|_2$ denotes the L^2 -norm, then we have*

$$\int_{X \setminus Z_\varphi(\delta)} |\zeta|^2 dv_g < \frac{C'}{|s|},$$

where $Z_\varphi(\delta)$ is the δ -neighborhood of the singular set Z_φ .

PROOF. Since A_φ satisfies the concentrating condition (3.1), the theorem follows directly from Proposition 2.4 of [4]. \square

In applying this principle, the singular set plays a crucial role. If $Z_\varphi = X$, no concentration occurs. For example, this happens when

$$\dim X \equiv 6 \pmod{8} \quad \text{and} \quad \text{rank}(E) \text{ is odd,}$$

since $\varphi_{ij} = -\varphi_{ji}$ in (3.2), and thus $\det(\varphi_{ij}) = 0$. On the other hand, when $Z_\varphi = \emptyset$, we have:

COROLLARY 3.6. *Let D_E and A_φ be as in Theorem 3.2. If $Z_\varphi = \emptyset$, then $\text{ind } D_E = 0$.*

PROOF. By Theorem 3.5, the singular set $Z_\varphi = \emptyset$ implies that for sufficiently large s , the operator D_s has trivial kernel. It follows that

$$\text{ind } D_E = \frac{1}{2} \text{ind}_{\mathbb{R}} D_s \leq 0,$$

where the second term denotes the index of D_s as a real operator, and the equality follows since D_s is a compact perturbation of D_E . Applying the same argument to the operator $D_E^* + sA_\varphi^*$ yields $\text{ind } D_E^* = -\text{ind } D_E \leq 0$, so the claim follows. \square

3.2.1. *Compact spin almost hermitan manifolds.* Suppose $c_1(X) \equiv 0 \pmod{2}$, so that X admits a spin structure. Fix a spin structure σ on X , which is equivalent to choosing a square root N_σ of the canonical bundle K_X , i.e., $N_\sigma^2 \cong K_X$. In this case, the complex spinor bundle \mathbb{S}_σ of the spin structure σ is given by

$$\mathbb{S}_\sigma = \mathbb{S} \otimes N_\sigma.$$

Let $E = N_\sigma \otimes F$, where F is a Hermitian vector bundle F over X . A section $\psi \in \Gamma((F^*)^2)$ induces a section $\varphi \in \Gamma(K_X \otimes (E^*)^2)$ via the isomorphism $(F^*)^2 \cong K_X \otimes (E^*)^2$, which restricts to yield

$$\mathrm{Sym}^2 F^* \cong K_X \otimes \mathrm{Sym}^2 E^* \quad \text{and} \quad \Lambda^2 F^* \cong K_X \otimes \Lambda^2 E^*.$$

When $\dim X \equiv 2, 6 \pmod{8}$, we set

$$A_\psi := A_\varphi : \mathbb{S}^\pm \otimes E = \mathbb{S}_\sigma^\pm \otimes F \rightarrow \mathbb{S}^\mp \otimes E = \mathbb{S}_\sigma^\mp \otimes F,$$

where A_φ is the conjugate-linear bundle map as in Theorem 3.2. The singular set of this bundle map is then given by

$$Z_\psi := Z_\varphi = \{x \in X : \det(\psi_x) = 0\}.$$

Below are examples of bundle maps A_ψ whose singular set $Z_\psi = \emptyset$.

- (a) Let $\dim X \equiv 2 \pmod{8}$. The following are typical examples of nondegenerate symmetric complex bilinear forms $\psi \in \Gamma(\mathrm{Sym}^2 F^*)$ on F :
- $F = TX \otimes \mathbb{C}$ with $\psi = g_\mathbb{C}$, the complex bilinear extension of the metric g to $TX \otimes \mathbb{C}$, or
 - $F = \mathrm{End}_\mathbb{C}(W)$, where W is a Hermitian vector bundle over X , and ψ is the natural trace pairing given by $\psi(A, B) = \mathrm{tr}(A \circ B)$.
- (b) Let $\dim X \equiv 6 \pmod{8}$. A complex vector bundle F is called a *complex symplectic vector bundle* if it is equipped with a nondegenerate skew-symmetric complex bilinear form $\psi \in \Gamma(\Lambda^2 F^*)$. Standard examples include:
- $F = TX \otimes \mathbb{C}$ with $\psi = \omega_\mathbb{C}$, where $\omega_\mathbb{C}(u, v) = g_\mathbb{C}(u, Jv)$, or
 - $F = W \oplus W^*$ with ψ defined by $\psi((w_1, f_1), (w_2, f_2)) = f_2(w_1) - f_1(w_2)$.

In all of the above examples (F, ψ) , the bundle map $A_\psi (= A_\varphi)$ satisfies the concentrating condition, and the singular set $Z_\psi = \emptyset$. Hence, by Corollary 3.6, the twisted Dirac operator

$$D_{N_\sigma \otimes F} : \Gamma(\mathbb{S}_\sigma^+ \otimes F) \rightarrow \Gamma(\mathbb{S}_\sigma^- \otimes F)$$

has index $\mathrm{ind} D_{N_\sigma \otimes F} = 0$.

REMARK 3.7. The vanishing of the index also follows easily from the Atiyah–Singer index theorem:

$$\mathrm{ind} D_{N_\sigma \otimes F} = \int_X \mathrm{ch}(F) \hat{A}(X),$$

where $\hat{A}(X)$ is a polynomial in the Pontryagin classes $p_k(X) = (-1)^k c_{2k}(X)$ and $\mathrm{ch}(F)$ is a polynomial in the even Chern classes $c_{2\ell}(F)$ (since $F \cong F^*$ as a complex vector bundle). As $\frac{1}{2} \dim X$ is odd, there is no top-degree term in the integrand, and therefore the integral vanishes.

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Received: 17.8.2025.

Revised: 28.10.2025.

BILJEŠKA O KONCENTRACIJI PUTEM KONJUGIRANO-LINEARNOG HODGEOVA ZVJEZDASTOG OPERATORA

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SAŽETAK. Konstruiramo konjugirano-linearne perturbacije usukanih spin^c Diracovih operatora na kompaktnim gotovo Hermiteovim mnogostrukostima dimenzije kongruentne 2 ili 6 modulo 8, koristeći konjugirano-linearni Hodgeov zvjezdasti operator skaliran jediničnim kompleksnim brojevima koji ovise o stupnju. Ove perturbacije zadovoljavaju princip koncentracije.