GLASNIK MATEMATIČKI
Vol. 61(81)(2026), 161 – 174

FINITE-TIME BLOW-UP OF A CLASSICAL SOLUTION TO THE TWO-FLUID MODEL WITH DENSITY-DEPENDENT VISCOSITY

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Abstract. This paper concerns the initial-boundary value problem for a compressible two-fluid model with density-dependent viscosities (possibly degenerating in vacuum), subject to Dirichlet boundary conditions. We prove that the two-fluid system with non-monotone pressure will blow up in finite time under the assumption that the initial densities include an isolated mass group.

1. Introduction

The utilization of the two-phase flow model has been extensively employed in various industries such as the petroleum industry, low temperature industrial biomedical microtechnology and other related fields. For more background, we refer the reader to [1, 10, 11, 28, 33]. In this paper, we study the compressible two-fluid model in the following form:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t n + \operatorname{div}(n\mathbf{u}) = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ \partial_t((\rho+n)\mathbf{u}) + \operatorname{div}((\rho+n)\mathbf{u}\otimes\mathbf{u}) + \nabla P = \operatorname{div}[2h(\rho,n)\mathbb{D}(\mathbf{u})] \\ \qquad + \nabla(g(\rho,n)\operatorname{div}\mathbf{u}). \end{cases} \tag{1.1}$$

The densities of two different fluids are represented by $\rho(x,t)$ and $n(x,t)$ respectively, $\mathbf{u}(x,t)$ stands for the mixed velocity. The shear viscosity coefficient $h(\rho,n)$ and the bulk viscosity coefficient $g(\rho,n)$ satisfy the following physical

2020 *Mathematics Subject Classification.* 76T10, 35Q30, 35M33.
Key words and phrases. Two-fluid model, blow-up, non-monotone pressure.
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restrictions:

$$h(\rho,n)\geq 0,\ 2h(\rho,n)+dg(\rho,n)\geq 0. \tag{1.2}$$

We consider the non-monotone pressure $P(\rho,n)\in C^2([0,\ \infty)^2)$, which satisfies $P(0,0)=0$ and the bounds

$$c_1(\rho^\gamma+n^\alpha-1)\leq P(\rho,n)\leq c_2(\rho^\gamma+n^\alpha+1), \tag{1.3}$$

where the constants c_1, c_2, γ, α satisfy

$$c_1>0,\ c_2>0,\ \gamma\geq 1,\ \alpha\geq 1,\ \max\{\gamma,\alpha\}\geq 2. \tag{1.4}$$

The strain tensor is given by $\mathbb{D}(\mathbf{u}):=\frac{\nabla\mathbf{u}+(\nabla\mathbf{u})^t}{2}$. We consider the system (1.1) in a bounded smooth domain $\Omega\subset\mathbb{R}^d(d\geq 2)$, and pose the usual Dirichlet boundary condition

$$\mathbf{u}|_{\partial\Omega}=0,\ \text{for } t\geq 0. \tag{1.5}$$

Moreover, we complement (1.1) with the initial conditions

$$\rho(x,0)=\rho_0(x),\ n(x,0)=n_0(x),\ \mathbf{u}(x,0)=\mathbf{u}_0(x). \tag{1.6}$$

It is well known that the compressible Navier-Stokes (CNS) model holds a pivotal role in the field of mechanics. Consequently, we first recall some classical advancements in the CNS model. For monotonic pressure, Kazhikhov and Shelukhin [16] established an existence result for the model in one-dimensional space with initial density bounded away from zero. Then Matsumura and Nishida [23] proved the global existence with small initial data in multidimensional space. For non-monotonic pressure, Feireisl [7] proved the global existence of weak solutions on a compact set. Recently Bresch and Jabin [2] focused on more general pressure laws that are not thermodynamically stable and obtained global weak solutions to the system. For more results and physical background, see [8, 21].

The subject of two-phase flow has become increasingly important in a wide variety of engineering systems and biological systems. From the mathematical point of view, researchers have noted the similar mathematical structure between the CNS model and the two-phase model. Therefore, in light of the results observed in the CNS model, there has been a significant surge in research focused on investigating the well-posedness of the two-phase model in recent years. For the system (1.1) with standard polytropic pressure $P(\rho,n)=\rho^\gamma+n^\alpha$ and constant viscosity, the existence of global weak solutions in the three-dimensional space has been established in [3, 27]. Due to the observed phenomena, the two-fluid model with non-monotonic pressure is closer to the real industrial applications, such as, the model of gas-kick flow scenarios in oil wells is related to a compressible gas-liquid model with non-monotonic pressure $P(n,m)=C(\frac{n}{\rho_l-m})^\gamma$ (see [6]). The mathematical analysis of global solutions to the two-phase system with non-monotone pressure has been established in [4, 6, 14, 15, 17, 20, 24, 29, 32]. More precisely, Novotný and Pokorný [24] proved the existence of weak solutions for large initial data on an arbitrarily large time interval where more general pressure functions are considered. In the one-dimensional space, the global existence and uniqueness of the classical solution for the two-fluid model with density-dependent viscosity was proved by Chen et al. [4]. Li et al. [20] established the global existence of weak solutions to the initial value problem (IVP) for the drift-flux system in the periodic domain $T^d := R^d/Z^d (d \geq 2)$. In addition, there has been important progress made recently about the outflow/inflow problem for viscous multi-phase flow [9, 12, 25].

Currently, extensive research has been dedicated to the blow-up phenomenon of compressible fluid. In the context of the CNS model, Xin [30] established the blow-up phenomenon when the initial densities possess compact support. Then Rozanova [26] generalized [30] to the case with initial data that rapidly decays at far fields. Furthermore, Xin and Yan [31] showed that any classical solutions of viscous compressible fluids lacking heat conduction will blow up in a finite time, provided that the initial data exhibits an isolated mass group. Inspired by the work [31], researchers extended the corresponding result to other models, such as [5, 19]. Jiu, Wang and Xin [13] proved the blow-up in the smooth solutions to the Cauchy problem for both the full CNS equations and isentropic CNS equations, considering constant and degenerate viscosities in arbitrary dimensions, subject to certain limitations on the initial data. In the case where the initial density ρ_0 possesses compact support, Li et al. [18] established the non-existence of any non-trivial classical solutions with finite energy to the Cauchy problem of the full CNS and isentropic CNS equations in the standard inhomogeneous Sobolev space for any short time.

Due to the structural similarity between the CNS model and the two-fluid model, the purpose of this article is to investigate the blow-up phenomenon of the classical solution for the compressible two-fluid model with non-monotonic pressure, under the condition that the initial density allows for the presence of an isolated mass group. To obtain our main results, it is essential to introduce various physical quantities, including mass, momentum, moment of inertia, internal energy, potential energy, total energy. The crucial step in the proof is to provide an estimate for physical measurement $G(t) := \frac{1}{2}\int_{U(t)} |x|^2\big(\rho(x,t) + n(x,t)\big)dx$. Thus we need to control the term $|\int_{U(t)} \big(2h(\rho,n) + dg(\rho,n)\big)dx|$. However, compared with [5, 19], the energy inequality cannot be employed for a direct estimate of $|\int_{U(t)} \big(2h(\rho,n) + dg(\rho,n)\big)dx|$ due to the disparity in the pressure term. In order to overcome the difficulty, we utilize the Helmholtz free energy function introduced by [4].

The paper is organized as follows. Section 2 is devoted to presenting our main results. In Section 3, we will give some basic properties of the physical quantities and prove the main results.

2. Main Results

Before stating our main results, we give the following definitions.

Definition 2.1. *For $T > 0$, a triple $(\rho(x,t), n(x,t), \mathbf{u}(x,t))$ is called a classical solution to the Dirichlet problem for System* (1.1)–(1.6) *on $\partial\Omega$ if the nonnegative function $\rho, n \in C^1(\Omega \times [0,T))$, and the vector field $\mathbf{u} \in C^1([0,T); C^2(\Omega))$ satisfies the system* (1.1) *point-wisely on $\Omega \times [0,T)$, take on the initial condition* (1.6) *continuously, and satisfy the boundary condition* (1.5) *continuously.*

In light of [31], we introduce the following definition of the isolated mass group.

Definition 2.2. *Let U and V be two bounded open subsets of Ω and $V \subset U$. The pair $(U,\ V)$ is called an isolated mass group of initial density $\rho_0(x),\ n_0(x)$ if it holds that*

$$\begin{cases} V \subset \overline{V} \subset U \subset \overline{U} \subset \Omega,\ U \text{ is connected}, \\ \rho_0(x) = 0,\ n_0(x) = 0,\ x \in U \backslash V, \\ \int_V \rho_0(x)dx > 0,\ \int_V n_0(x)dx > 0. \end{cases}$$

Let $(U,\ V)$ be an isolated mass group of $\rho_0(x),\ n_0(x)$ in Ω. and $H(\rho_0, n_0)$ be the Helmholtz free energy function. Then we set

- Mass
$$M_0 := \int_V \rho_0(x)dx + \int_V n_0(x)dx := M_{\rho_0} + M_{n_0},$$
- Momentum
$$\mathbb{P}_0 := \int_V \rho_0(x)\mathbf{u}_0(x)dx + \int_V n_0(x)\mathbf{u}_0(x)dx := \mathbb{P}_{\rho_0} + \mathbb{P}_{n_0},$$
- Momentum weight
$$F_0 := \int_V \rho_0(x)\mathbf{u}_0(x)\cdot xdx + \int_V n_0(x)\mathbf{u}_0(x)\cdot xdx := F_{\rho_0} + F_{n_0},$$
- Momentum of inertia
$$G_0 := \frac{1}{2}\int_V \rho_0(x)|x|^2dx + \frac{1}{2}\int_V n_0(x)|x|^2dx := G_{\rho_0}(t) + G_{n_0}(t),$$
- Total energy
$$E_0 := \int_V \frac{1}{2}(\rho_0(x) + n_0(x))|\mathbf{u}_0(x)|^2 + H(\rho_0(x), n_0(x))dx.$$

Moreover, we assume that

$$\frac{\mathbb{P}_0^2}{M_0} + dc_1(|B_R(0)|^{1-\gamma}M_{\rho_0}^{\gamma} + |B_R(0)|^{1-\alpha}M_{n_0}^{\alpha} - |B_R(0)|) > 0. \tag{2.1}$$

It is well known that the Helmholtz free energy function $H(\rho, n)$ corresponding to pressure P is a solution of the following partial differential equation of the first order in $(0,\ \infty)^2$:

$$P(\rho, n) = \rho \frac{\partial H(\rho, n)}{\partial \rho} + n \frac{\partial H(\rho, n)}{\partial n} - H(\rho, n). \tag{2.2}$$

To derive the pressure estimate, we adopt the approach presented in [4], employing the Helmholtz free energy function $H(\rho, n)$ that corresponds to $P(\rho, n)$:

$$H(\rho, n) := (\rho + n) \int_1^{\rho+n} \frac{P(s\frac{\rho}{\rho+n}, s\frac{n}{\rho+n})}{s^2} ds, \text{ if } \rho + n > 0,\ H(0,\ 0) = 0. \tag{2.3}$$

We assume that the viscosity coefficients $h(\rho, n)$ and $g(\rho, n)$ satisfy the following form

$$h(\rho, n) = A\rho^\lambda + Bn^\beta,\ g(\rho, n) = C\rho^\tau + Dn^q, \tag{2.4}$$

where $A > 0,\ B > 0,\ 2A + dC \geq 0,\ 2B + dD \geq 0,$ and one of the following conditions:

(i) $\lambda = \tau,\ \beta = q,\ 2A + dC = 0,\ 2B + dD = 0;$
(ii) $\lambda,\ \tau \in (0,\ \gamma],\ \beta,\ q \in (0,\ \alpha].$

Our main results are stated as follows.

THEOREM 2.3. *Let* $(\rho(x, t),\ n(x, t),\ \mathbf{u}(x, t))$ *be a classical solution to the Dirichlet problem of the compressible two-phase model* (1.1) *with initial data* (1.6). *Suppose that the initial density* $\rho_0(x),\ n_0(x)$ *admit an isolated mass group* $(U,\ V)$ *and* $h(\rho, n),\ g(\rho, n)$ *satisfy* (1.2) *if one of the following conditions holds:*

(1) *condition (i) holds, and* M_0 *is finite;*
(2) *condition (ii) holds, and* $M_0,\ E_0$ *are finite.*

Then the classical solution $(\rho(x, t),\ n(x, t),\ \mathbf{u}(x, t))$ *will blow up in finite time.*

REMARK 2.4. If $M_0,\ E_0$ are finite and $h(\rho, n)$ and $g(\rho, n)$ satisfy: $\lambda = \tau,\ 2A + dC = 0,\ \beta,\ q \in (0,\ \alpha]$ or $\lambda,\ \tau \in (0,\ \gamma],\ \beta = q,\ 2B + dD = 0,$ Theorem 2.3 still holds.

3. PROOF OF THEOREM 2.3

Let $(\rho(x, t),\ n(x, t),\ \mathbf{u}(x, t))$ be a classical solution to the compressible two-fluid model (1.1) on $\Omega \times (0,\ T)$, where T is the maximal time of existence. We denote by $X(a,\ t)$ the particle path starting from a when $t = 0,\ i.e.$

$$\begin{cases} \frac{d}{dt} X(a,\ t) & = \mathbf{u}(X(a,\ t),\ t), \\ X(a,\ 0) & = a, \end{cases} \tag{3.1}$$

and set

$$U(t) = \{X(a,\ t) | a \in U\},\ V(t) = \{X(a,\ t) | a \in V\}. \tag{3.2}$$

The pair $(U(t),\ V(t))$ is an isolated mass group of the density $\rho(x,t),\ n(x,t)$ in Ω at time t, and it will not disappear for any $t < T$. In fact, $\mathbf{u} \in C^1([0,\ T); C^2(\Omega))$ ensures that $X(a,\ t)$ is well-defined (existence and uniqueness) by the classical theory of ordinary differential equations. Furthermore, $V \subset \overline{V} \subset U \subset \overline{U} \subset \Omega$ implies that

$$V(t) \subset \overline{V(t)} \subset U(t) \subset \overline{U(t)} \subset \Omega.$$

Since $\rho_0(x) = 0,\ n_0(x) = 0$ in $U \setminus V$, we can immediately obtain the following lemma from mass equations $(1.1)_1$ and $(1.1)_2$.

LEMMA 3.1. *Suppose* $(\rho(x,t),\ n(x,t),\ \mathbf{u}(x,t))$ *is a classical solution to system* (1.1)- (1.6) *with a suitable boundary condition on* $\partial\Omega$*, then it holds that*

$$\rho(x,t) = 0,\ n(x,t) = 0 \text{ in } U(t) \setminus V(t). \tag{3.3}$$

Following [19], we assume that the density ρ, n equal to 0 on $\partial U(t)$, otherwise we could choose $R(t)$ instead of $U(t)$ satisfying $V(t) \subset \overline{V(t)} \subset R(t) \subset \overline{R(t)} \subseteq U(t)$, then ρ, n equal to 0 on $\partial R(t)$.

The following lemma is the famous transport formula. It can be found in [22].

LEMMA 3.2. *Let* $U(t)$ *be defined as* (3.2)*, for any* $f(x,t) \in C^1(\mathbb{R}^d \times \mathbb{R}^+)$*, we have*

$$\frac{d}{dt}\int_{U(t)} f(x,t)dx = \int_{U(t)} f_t(x,t)dx + \int_{\partial U(t)} f(x,t)(\mathbf{u}\cdot\nu)dx, \tag{3.4}$$

where ν *is the unit outward normal to* $\partial U(t)$*.*

Before showing Lemma 3.3, we set $G(t) := \frac{1}{2}\int_{U(t)}(\rho+n)|x|^2dx, F(t) := \int_{U(t)}(\rho+n)\mathbf{u}\cdot xdx, E(t) := \int_{U(t)}\left[\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho,n)\right]dx := E_k(t) + I(t)$.

LEMMA 3.3. *Suppose* $(\rho(x,t),\ n(x,t),\ \mathbf{u}(x,t))$ *is a classical solution to system* (1.1)- (1.6) *in* $\Omega \times (0,\ T)$ *with a suitable boundary condition on* $\partial\Omega$*, then for each* $0 < t < T$*, we have*

$$\int_{U(t)} \rho dx = M_{\rho_0},\ \int_{U(t)} ndx = M_{n_0}, \tag{3.5}$$

$$\int_{U(t)} (\rho+n)\mathbf{u}dx = \mathbb{P}_0, \tag{3.6}$$

$$\frac{d}{dt}G(t) = F(t), \tag{3.7}$$

$$\frac{d}{dt}F(t) = 2E_k(t) + d\int_{U(t)} Pdx - \int_{U(t)}\left[\big(2h(\rho,n) + dg(\rho,n)\big)\operatorname{div}\mathbf{u}\right]dx, \tag{3.8}$$

$$
\begin{aligned}
&\int_{U(t)} \Big[\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho,n)\Big]dx \\
&\quad + \int_0^t \int_{U(s)} \Big[h(\rho,n)\Big(\frac{\nabla\mathbf{u}+(\nabla\mathbf{u})^t}{2}\Big)\cdot\nabla\mathbf{u} + g(\rho,n)|\mathrm{div}\,\mathbf{u}|^2\Big]dxds \\
&\quad = E_0.
\end{aligned}
\tag{3.9}
$$

Proof. According to (1.1), Lemma 3.2 and the assumption: $\rho, n|_{\partial U(t)} = 0, P(0,0) = 0$, we obtain the conservation of mass and momentum:

$$
\begin{aligned}
\frac{d}{dt}\int_{U(t)} \rho(x,t)dx &= \int_{U(t)} \big(\rho_t + \mathrm{div}\,(\rho\mathbf{u})\big)dx = 0,\\
\frac{d}{dt}\int_{U(t)} n(x,t)dx &= \int_{U(t)} \big(n_t + \mathrm{div}\,(n\mathbf{u})\big)dx = 0,\\
\frac{d}{dt}\int_{U(t)} (\rho+n)\mathbf{u}dx &= \int_{U(t)} \Big((\rho+n)_t + \mathrm{div}\,\big((\rho+n)\mathbf{u}\otimes\mathbf{u}\big)\Big)dx\\
&= \int_{U(t)} \Big(\mathrm{div}\,[2h(\rho,n)\mathbb{D}(u)]\\
&\quad + \nabla(g(\rho,n)\mathrm{div}\,u) - \nabla P\Big)dx\\
&= 0,
\end{aligned}
$$

which imply (3.5) and (3.6). Meanwhile, we can obtain (3.7) and (3.8) by using integration by parts as:

$$
\begin{aligned}
\frac{d}{dt}G(t) &= \frac{d}{dt}\int_{U(t)} \frac{1}{2}(\rho+n)|x|^2dx\\
&= \frac{1}{2}\int_{U(t)} \big((\rho+n)|x|^2\big)_t + \mathrm{div}\,\big((\rho+n)|x|^2\mathbf{u}\big)dx\\
&= \frac{1}{2}\int_{U(t)} \Big((\rho+n)_t + \mathrm{div}\,\big((\rho+n)\mathbf{u}\big)\Big)|x|^2 + 2(\rho+n)\mathbf{u}\cdot xdx\\
&= \int_{U(t)} (\rho+n)\mathbf{u}\cdot xdx = F(t).
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
\frac{d}{dt}F(t) &= \frac{d}{dt}\int_{U(t)} (\rho+n)\mathbf{u}\cdot xdx\\
&= \int_{U(t)} \Big(\big((\rho+n)\mathbf{u}\big)_t + \mathrm{div}\,\big((\rho+n)\mathbf{u}\otimes\mathbf{u}\big)\Big)\cdot x + (\rho+n)|\mathbf{u}|^2dx\\
&= \int_{U(t)} (\rho+n)|\mathbf{u}|^2 + \Big(\mathrm{div}\,[2h(\rho,n)\mathbb{D}(\mathbf{u})] + \nabla(g(\rho,n)\mathrm{div}\,\mathbf{u}) - \nabla P\Big)\cdot xdx\\
&= 2E_k(t) + d\int_{U(t)} Pdx - \int_{U(t)} \Big(\big(2h(\rho,n) + dg(\rho,n)\big)\mathrm{div}\,\mathbf{u}\Big)dx.
\end{aligned}
$$

Taking the time derivative of $E(t)$, and combining with (2.2), we get

$$\begin{aligned}\frac{d}{dt}E(t) = \frac{d}{dt}\int_{U(t)} \big[\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho,n)\big]dx\\
&= \int_{U(t)} \Big[\Big(\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho,n)\Big)_t\\
&\quad + \operatorname{div}\Big(\big(\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho,n)\big)\mathbf{u}\Big)\Big]dx\\
&= \int_{U(t)} \Big(\big((\rho+n)\mathbf{u}\big)_t + \operatorname{div}\big((\rho+n)\mathbf{u}\otimes\mathbf{u}\big)\Big)\cdot\mathbf{u}\\
&\quad - \frac{1}{2}\Big((\rho+n)_t + \operatorname{div}\big((\rho+n)\mathbf{u}\big)\Big)|\mathbf{u}|^2\\
&\quad + H_\rho\rho_t + H_n n_t + (H_\rho\nabla\rho + H_n\nabla n)\cdot\mathbf{u} + H\operatorname{div}\mathbf{u}dx\\
&= \int_{U(t)} \big[(\operatorname{div}(2h\mathbb{D}(\mathbf{u})) + \nabla(g\operatorname{div}\mathbf{u}) - \nabla P)\cdot\mathbf{u} - P\operatorname{div}\mathbf{u}\big]dx\\
&= -\int_{U(t)} \big[2h(\rho,n)\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{u}) + g(\rho,n)|\operatorname{div}\mathbf{u}|^2\big]dx.\end{aligned}$$

Integrating the above equation, we obtain that

$$\begin{aligned}&\int_{U(t)} \Big[\frac{1}{2}(\rho+n)|\mathbf{u}|^2 + H(\rho,n)\Big]dx - E(0)\\
&\quad = -\int_0^t\int_{U(s)} \Big[h(\rho,n)\Big(\frac{\nabla\mathbf{u}+(\nabla\mathbf{u})^t}{2}\Big)\cdot\nabla\mathbf{u} + g(\rho,n)|\operatorname{div}\mathbf{u}|^2\Big]dxds,\end{aligned}$$

which implies (3.9). □

Inspired by [5], we will estimate $|\int_0^t\int_{U(s)}(2h(\rho,n)+dg(\rho,n))\operatorname{div}\mathbf{u}\,dxds|$.

LEMMA 3.4. *Suppose* $(\rho(x,t),\ n(x,t),\ \mathbf{u}(x,t))$ *is a classical solution to system* (1.1)- (1.6) *in* $\Omega\times(0,\ T)$ *with a suitable boundary condition on* $\partial\Omega$. *If* $h(\rho,n)$ *and* $g(\rho,n)$ *satisfy the conditions in Theorem 2.3, then for each* $0<t<T$, *it holds that*

$$\left|\int_0^t\int_{U(s)} \big(2h(\rho,n)+dg(\rho,n)\big)\operatorname{div}\mathbf{u}\ dxds\right| \le C_3 t^{\frac{1}{2}}, \tag{3.11}$$

where C_3 *is a positive constant independent of* t.

PROOF. By virtue of Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_0^t \int_{U(s)} \big(2h(\rho,n)+dg(\rho,n)\big)\operatorname{div}\mathbf{u}\; dxds\right| \\
&\quad\le \int_0^t \int_{U(s)} \big(2h(\rho,n)+dg(\rho,n)\big)|\operatorname{div}\mathbf{u}|dxds \\
&\quad\le \left(\int_0^t \int_{U(s)} \big(2h(\rho,n)+dg(\rho,n)\big)|\operatorname{div}\mathbf{u}|^2 dxds\right)^{\frac{1}{2}} \\
&\qquad \left(\int_0^t \int_{U(s)} \big[2h(\rho,n)+dg(\rho,n)\big] dxds\right)^{\frac{1}{2}} \\
&\quad\le E_0^{\frac{1}{2}}\Big(\int_0^t \int_{U(s)} \big[2h(\rho,n)+dg(\rho,n)\big] dxds\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.12}$$

When $h(\rho,n),\ g(\rho,n)$ satisfy Condition (i), the result is obviously true. When $h(\rho,n),\ g(\rho,n)$ satisfy Condition (ii), according to (1.3) and (2.3) , we have

$$\begin{aligned}
&c_1(\rho+n)\int_1^{\rho+n} \frac{\big(\frac{s\rho}{\rho+n}\big)^{\gamma}+\big(\frac{sn}{\rho+n}\big)^{\alpha}-1}{s^2}ds \\
&\quad\le H(\rho,n) \\
&\quad\le c_2(\rho+n)\int_1^{\rho+n} \frac{\big(\frac{s\rho}{\rho+n}\big)^{\gamma}+\big(\frac{sn}{\rho+n}\big)^{\alpha}+1}{s^2}ds.
\end{aligned} \tag{3.13}$$

By virtue of the fact that $\big(\frac{\rho}{\rho+n}\big)^{\gamma-1}<1, \big(\frac{n}{\rho+n}\big)^{\alpha-1}<1$, we have

$$\rho+n+\frac{\rho^{\gamma}}{(\rho+n)^{\gamma-1}(\gamma-1)}+\frac{n^{\alpha}}{(\rho+n)^{\alpha-1}(\alpha-1)}\le\frac{\gamma\rho}{\gamma-1}+\frac{\alpha n}{\alpha-1}.$$

Combining with (3.13), we get the upper and lower bounds of the Helmholtz function $H(\rho,n)$

$$c_1\big(\frac{\rho^{\gamma}}{\gamma-1}+\frac{n^{\alpha}}{\alpha-1}+1-\frac{\gamma\rho}{\gamma-1}-\frac{\alpha n}{\alpha-1}\big)\le H(\rho,n)\le c_2\big(\frac{\rho^{\gamma}}{\gamma-1}+\frac{n^{\alpha}}{\alpha-1}-1+\rho+n\big).$$

Integrating over the domain, we have

$$\begin{aligned}
&c_1\int_{U(t)}\Big(\frac{\rho^{\gamma}}{\gamma-1}+\frac{n^{\alpha}}{\alpha-1}+1-\frac{\gamma\rho}{\gamma-1}-\frac{\alpha n}{\alpha-1}\Big)dx \\
&\quad\le \int_{U(t)} H(\rho,n)dx \\
&\quad\le c_2\int_{U(t)}\Big(\frac{\rho^{\gamma}}{\gamma-1}+\frac{n^{\alpha}}{\alpha-1}-1+\rho+n\Big)dx.
\end{aligned} \tag{3.14}$$

Then according to (1.4), there exist two positive numbers C_1 and C_2 such that

(3.15)
$$\begin{aligned}
\int_{U(t)}&\big[2h(\rho,n)+dg(\rho,n)\big]dx\\
&\le C_1\int_{U(t)}(\rho^\gamma+n^\alpha+1)dx\\
&\le C_1\max\{\gamma-1,\ \alpha-1\}\int_{U(t)}\Big(\frac{\rho^\gamma}{\gamma-1}+\frac{n^\alpha}{\alpha-1}+1-\frac{\gamma\rho}{\gamma-1}-\frac{\alpha n}{\alpha-1}\Big)dx\\
&\quad+C_1\max\{\gamma-1,\ \alpha-1\}\int_{U(t)}\Big(\frac{\gamma\rho}{\gamma-1}+\frac{\alpha n}{\alpha-1}\Big)dx\\
&\le C_2(E_0+1).
\end{aligned}$$

Integrating (3.15) on time, we get

$$\int_0^t\int_{U(s)}\big[2h(\rho,n)+dg(\rho,n)\big]dxds\le C_2(E_0+1)t.$$

This implies

$$\left|\int_0^t\int_{U(s)}\big(2h(\rho,n)+dg(\rho,n)\big)\mathrm{div}\,u\ dxds\right|\le C_3t^{\frac12},\tag{3.16}$$

$C_3:=\big(C_2(E_0+1)E_0\big)^{\frac12}$. □

In light of [13], we derive the estimate of $G(t)$.

LEMMA 3.5. *Suppose* $(\rho(x,t),\ n(x,t),\ \mathbf{u}(x,t))$ *is a classical solution to system* (1.1)- (1.6) *in* $\Omega\times(0,\ T)$. *If* $h(\rho,n)$ *and* $g(\rho,n)$ *satisfy the conditions in Theorem 2.3, then for each* $0<t<T$, *it holds that*

$$G(t)\ge G_0+F_0t+\frac12(C_4+dC_5)t^2-\frac23C_3t^{\frac32}.\tag{3.17}$$

PROOF. Owing to (3.7) and (3.8), taking a time derivative, we get

$$\frac{d^2}{dt^2}G(t)=\frac{d}{dt}F(t)=\int_{U(t)}\big[(\rho+n)|\mathbf{u}|^2+dP-(2h(\rho,n)+dg(\rho,n))\mathrm{div}\,\mathbf{u}\big]dx.$$

Given the Hölder inequality, we can get

$$\int_{U(t)}(\rho+n)|\mathbf{u}|^2dx\ge\frac{\mathbb{P}_0^2}{M_0}:=C_4.\tag{3.18}$$

By virtue of the mass conservation equation and the fact that $U(t)$ is always in a bounded domain $\Omega \subset B_R(0)$ for some $R > 0$, we obtain that

$$\begin{aligned}\int_{U(t)} P dx &\geq c_1 \int_{U(t)} \big(\rho^\gamma + n^\alpha - 1\big) dx \\ &\geq c_1 \big(|U(t)|^{1-\gamma} M_{\rho_0}^\gamma + |U(t)|^{1-\alpha} M_{n_0}^\alpha - |U(t)|\big) \\ &\geq c_1 \big(|B_R(0)|^{1-\gamma} M_{\rho_0}^\gamma + |B_R(0)|^{1-\alpha} M_{n_0}^\alpha - |B_R(0)|\big) := C_5.\end{aligned} \tag{3.19}$$

Combining (3.18), (3.19) and Lemma 3.4, we have

$$\frac{d}{dt} G(t) \geq F_0 + (C_4 + dC_5)t - C_3 t^{\frac{1}{2}},$$

which implies that

$$G(t) \geq G_0 + F_0 t + \frac{1}{2}(C_4 + dC_5)t^2 - \frac{2}{3} C_3 t^{\frac{3}{2}}. \tag{3.20}$$

□

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Combining the previous assumption (2.1), we deduce the following inequality from Lemma 3.5

$$G(t) \geq G_0 + F_0 t + \frac{1}{2}(C_4 + dC_5)t^2 - \frac{2}{3} C_3 t^{\frac{3}{2}} \to +\infty \text{ as } \ t \to +\infty.$$

On the other hand, we have

$$G(t) = \frac{1}{2} \int_{U(t)} |x|^2 (\rho(x,t) + n(x,t)) dx \leq \frac{1}{2} R^2 (M_{\rho_0} + M_{n_0}). \tag{3.21}$$

Combining (3.20) with (3.21) yields that

$$G_0 + F_0 t + \frac{1}{2}(C_4 + dC_5)t^2 - \frac{2}{3} C_3 t^{\frac{3}{2}} \leq \frac{1}{2} R^2 (M_{\rho_0} + M_{n_0}), \tag{3.22}$$

Let $t \to +\infty$, we find the left hand side of (3.22) goes to infinity, while the right hand side of (3.22) is bounded. Which means that the time span of the classical solutions $\big(\rho(x,t),\ n(x,t),\ \mathbf{u}(x,t)\big)$ is finite. Therefore, the proof of Theorem 2.3 is completed. □

Acknowledgements.

The authors are grateful to the referees and the editors whose comments and suggestions greatly improved the presentation of this paper. T. Tang is partially supported by NSFC No. 12371246, Qing Lan Project of Jiangsu Province and Yangzhou Innovation Capability Enhancement Fund under grant No. YZ2024245.

References

[1] C. E. Brennen, Fundamentals of multiphase flow, Cambridge University Press, Cambridge, 2005.

[2] D. Bresch and P. E. Jabin, *Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor*, Ann. of Math. (2) **188** (2018), 577–684.

[3] D. Bresch, P. B. Mucha and E. Zatorska, *Finite-energy solutions for compressible two-fluid Stokes system*, Arch. Ration. Mech. Anal. **232** (2019), 987–1029.

[4] S. M. Chen and C. J. Zhu, *The global classical solution to a 1D two-fluid model with density-dependent viscosity and vacuum*, Sci. China Math. **65** (2022), 2563–2582.

[5] B. Duan, Z. Luo and W. Yan, *Finite-time blow-up of classical solutions to the rotating shallow water system with degenerate viscosity*, Z. Angew. Math. Phys. **70** (2019), Paper No. 54.

[6] S. Evje, *A compressible two-phase model with pressure-dependent well-reservoir interaction*, SIAM J. Math. Anal. **45** (2013), 518–546.

[7] E. Feireisl, *Compressible Navier-Stokes equations with a non-monotone pressure law*, J. Differential Equations **184** (2002), 97–108.

[8] E. Feireisl, Dynamics of viscous compressible fluids, Oxford University Press, Oxford, 2004.

[9] F. F. Hao, H. L. Li, L. Y. Shang and S. Zhao, *Recent progress on outflow/inflow problem for viscous multi-phase flow*, Commun. Appl. Math. Comput. **5** (2023), 987–1014.

[10] M. Ishii, Thermo-fluid dynamic theory of two-fluid flow, Eyrolles, Paris, 1975.

[11] M. Ishii, *One-dimensional drift-flux model and constitutive equations for relative motion between phases in various two-fluid flow regimes*, Argonne National Laboratory Report, USA, 1977, ANL-77-47.

[12] B. J. Jin, Y. S. Kwon, Š. Nečasová and A. Novotný, *Existence and stability of dissipative turbulent solutions to a simple bi-fluid model of compressible fluids*, J. Elliptic Parabol. Equ. **7** (2021), 537–570.

[13] Q. S. Jiu, Y. X. Wang and Z. P. Xin, *Remarks on blow-up of smooth solutions to the compressible fluid with constant and degenerate viscosities*, J. Differential Equations **259** (2015), 2981–3003.

[14] M. Kalousek, S. Mitra and Š. Nečasová, *The existence of a weak solution for a compressible multicomponent fluid structure interaction problem*, J. Math. Pures Appl. (9) **184** (2024), 118–189.

[15] M. Kalousek and Š. Nečasová, *On existence of weak solutions to a Baer-Nunziato type system*, J. Differential Equations **452** (2026), Paper No. 113804.

[16] A. V. Kazhikhov and V. V. Shelukhin, *Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech. **41** (1977), 273–282.

[17] S. Kracmar, Y. S. Kwon, Š. Nečasová and A. Novotný, *Weak solutions for a bifluid model for a mixture of two compressible noninteracting fluids with general boundary data*, SIAM J. Math. Anal. **54** (2022), 818–871.

[18] H. L. Li, Y. X. Wang and Z. P. Xin, *Non-existence of classical solutions with finite energy to the Cauchy problem of the compressible Navier-Stokes equations*, Arch. Ration. Mech. Anal. **232** (2019), 557–590.

[19] M. L. Li, Z. A. Yao and R. F. Yu, *Non-existence of global classical solutions to barotropic compressible Navier-Stokes equations with degenerate viscosity and vacuum*, J. Differential Equations **306** (2022), 280–295.

[20] H. L. Li and L. Y. Shou, *Global existence of weak solutions to the drift-flux system for general pressure laws*, Sci. China Math. **66** (2023), 251–284.

[21] P. L. Lions, Mathematical topics in fluid dynamics. Vol. 2, Oxford University Press, New York, 1998.

[22] A. J. Majda and A. L. Bertozzi, Vorticity and incompressible flow, Cambridge University Press, England, 2002.

[23] A. Matsumura and T. Nishida, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ. **20** (1980), 67–104.

[24] A. Novotný and M. Pokorný, *Weak solutions for some compressible multicomponent fluid models*, Arch. Ration. Mech. Anal. **235** (2020), 355–403.

[25] M. Pokorný, A. Wróblewska-Kamińska and E. Zatorska, *Two-phase compressible/incompressible Navier-Stokes system with inflow-outflow boundary conditions*, J. Math. Fluid Mech. **24** (2022), Paper No. 87.

[26] O. Rozanova, *Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes equations*, J. Differential Equations **245** (2008), 1762–1774.

[27] A. Vasseur, H. Y. Wen and C. Yu, *Global weak solution to the viscous two-fluid model with finite energy*, J. Math. Pures Appl. **125** (2019), 247–282.

[28] G. B. Wallis, One-dimensional two-phase flow, McGraw-Hill, New York, 1969.

[29] H. Y. Wen, *On global solutions to a viscous compressible two-fluid model with unconstrained transition to single-phase flow in three dimensions*, Calc. Var. Partial Differential Equations **60** (2021), Paper No. 158.

[30] Z. P. Xin, *Blowup of smooth solutions to the compressible Navier-Stokes equations with compact density*, Comm. Pure Appl. Math. **51** (1988), 229–240.

[31] Z. P. Xin and W. Yan, *On blowup of classical solutions to the compressible Navier-Stokes equations*, Comm. Math. Phys. **321** (2013), 529–541.

[32] L. Yao, T. Zhang and C. J. Zhu, *Existence of asymptotic behavior of global weak solutions to a 2D viscous liquid-gas two-phase flow model*, SIAM J. Math. Anal. **42** (2010), 1874–1897.

[33] N. Zuber, *On the dispersed two-phase flow in the laminar flow regime*, Chem. Eng. Sci. **19** (1964), 897–917.

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Received: 26.1.2025.
Revised: 3.8.2025.

"BLOW-UP" U KONAČNOM VREMENU KLASIČNOG RJEŠENJA ZA DVOFLUIDNI MODEL S VISKOZNOŠĆU OVISNOM O GUSTOĆI

K. Wang i T. Tang

Sažetak. Ovaj se rad bavi početno-rubnom zadaćom za kompresibilni dvofluidni model s viskoznostima ovisnima o gustoći, koje mogu degenerirati u vakuumu, uz Dirichletove rubne uvjete. Dokazujemo da će za dvofluidni sustav s nemonotonim tlakom doći do "blow-upa" u konačnom vremenu pod pretpostavkom da početne gustoće sadržavaju izolirano područje mase.