NASH EQUILIBRIUM AND NASH BARGAINING SOLUTION: THE 1994 NOBEL PRIZE FOR ECONOMICS

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Game theory, in its non-cooperative as well as in its cooperative form, has acquired a central role in modern economics. In the analysis of strategic interactions it proved superior to the traditional mathematical methods used in economic theory. This paper deals with the two major concepts in game theory introduced by John Nash, namely, Nash equilibrium, and Nash bargaining solution, the elaboration of which brought John Nash the Nobel Prize in Economics in 1994. In spite of its relative mathematical sophistication, game theory has found numerous applicati-ons in real life economic situations where strategic interactions are involved.

INTRODUCTION

Game theory, as a mathematical method for analyzing strategic interactions, has acquired a central role in economic theory. The Nobel Jury has recognized that influence by awarding the 1994 economics prize to three of the leading contributors to equilibrium analysis in non-cooperative game theory: John Harsanyi of the University of California at Berkeley, John Nash of Princeton University, and Reinhard Selten of the University of Bonn.

John Nash, whose name is attached to the concepts presented in this paper, was born in 1928. He entered the Princeton doctoral program in 1948, and got his Ph.D. in 1950 for his remarkable doctoral dissertation entitled *Non-cooperative Games.* This article attempts to give an account of the two major concepts in game theory introduced by John Nash, namely Nash equilibrium, and Nash bargaining solution. The elaboration of these concepts follows a brief history of game theory, and a review of some of its fundamental concepts.

A BRIEF HISTORY OF GAME THEORY

The history of game theory may claim to go back to 1713, when the mathematician James Waldegrave, in a letter to P. de Montmort, introduced the notion of a mixed strategy, and the minimax principle as he analyzed, and gave a solution to the card game called 'le Her' (Baumol-Goldfeld, 1968).

Emile Borel, one of the main creators of mathematics in the 20[th] century, had proved particular instances of the minimax theorem in the restrictive case of symmetric games (von Neumann, 1953). He also conjectured that it did not generally hold. However, in his pathbreaking article of 1928, which marked the beginning of the contemporary period for game theory, John von Neumann gave the first proof of the minimax theorem for an arbitrary finite payoff matrix in a zero-sum two-person game showing that the theorem does generally hold (von Neumann, 1928).

As it usually happens in mathematics, von Neumann's initially complex proof was later replaced by simpler arguments. Two decades later it evolved into a demonstration whose main ideas could be conveyed to a bright high school student (Gale, 1951).

Little happened in game theory from 1928 to 1944 when the economics profession was hit by the 625 page book *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern. Even after that, at first, significant events rarely occurred in game theory. The book by von Neumann and Morgenstern focused on the concept of a solution, now called a stable set, a concept that did not meet with success. Their volume did not emphasize the notions, many of them not yet discovered, that came to dominate game theory: Nash equilibrium, Shapley value, the core, etc. It took several decades after 1944 for game theory to start acquiring the influence on economic theory it now has.

After the publication of the book of 1944 that, unlike the article of 1928, could not be ignored, the earliest major contribution was made in 1950 by John Nash (Nash, 1950)(i). In a one-page paper that was remarkable by its scientific contribution and by its conciseness, Nash introduced, in the new context of game theory, a concept of equilibrium to which his name is attached, and proved its existence for all finite games. In doing so, he used a mathematical result (the Kakutani fixed point theorem) which, for many years, became the basic analytical tool for establishing that a social system has an equilibrium, whatever precise definition of the concept of an equilibrium is adopted.

THE CONCEPT OF NASH EQUILIBRIUM

The situation considered by Nash can be presented as follows: If we denote by n the number of players in a game, and if we characterize each one of them by an index i running from 1 to n, we observe a particular player, say the i-th one. That player has to decide on which action to take, what decision to make. In game theory terminology, he has to choose a strategy $\hat{s}_i$ in a given set S_i of strategies that are available to him. Once every player has made his choice, namely $\hat{s}_1$, an element of S_1, for the first one; $\hat{s}_2$, an element of S_2, for the second one; ...; $\hat{s}_n$, an element of S_n, for the n-th one, the resulting global choice for the game is the n-list $\hat{s} = (\hat{s}_1, \hat{s}_2, ..., \hat{s}_n)$. The problem becomes to find an $\hat{s}_i$ which forms an equilibrium, i.e., a combination of strategies in which every one of the n players considers himself to be in equilibrium, and has, therefore, no incentive to select a different strategy.

We can consider any one of those players, the i-th one as before. That player is in equilibrium, according to Nash, if taking the strategies $(\hat{s}_1, ..., \hat{s}_{i-1}, \hat{s}_{i+1}, ..., \hat{s}_n)$ of the n-1 other players as given, he reacts optimally to such a combination. It means that by choosing $\hat{s}_i$ in the given set S_i of available strategies, he maximizes his payoff, i.e., his utility. It is represented by a number that depends on the choices made by the n players: by himself, and by the others. In symbols $\hat{s}_i$ must maximize $u_i(\hat{s}_1, ..., \hat{s}_{i-1}, \hat{s}_i, \hat{s}_{i+1}, ..., \hat{s}_n)$ where $\hat{s}_1, ..., \hat{s}_{i-1}, \hat{s}_{i+1}, ..., \hat{s}_n$ are given by the global choice s_i being considered, but where s_i can be chosen freely in the set S_i by the i-th player, $u_i(\hat{s}_1, ..., \hat{s}_{i-1}, \hat{s}_i, \hat{s}_{i+1}, ..., \hat{s}_n) = Max\ u_i(\hat{s}_1, ..., \hat{s}_{i-1}, \hat{s}_i, \hat{s}_{i+1}, ..., \hat{s}_n)$

$$s_i \in S_i$$

If we denote by N the set of the n players, and by $N\backslash i$ the set of the n-1 players including every one of them except the i-th, then the (n-1)-list $(\hat{s}_1, ..., \hat{s}_{i-1}, \hat{s}_{i+1}, ..., \hat{s}_n)$ can be denoted by $\hat{s}_{N\backslash i}$, and the last equality can be written in a more compact way as

$$u_i(\hat{s}_{N\backslash i}, \hat{s}_i) = Max\ u_i(\hat{s}_{N\backslash i}, s_i)$$

$$s_i \in S_i$$

In this expression the (n-1)-list $\hat{s}_{N\backslash i}$ is given. The variable is the strategy s_i constrained only to be an element of S_i. The i-th player is in equilibrium if $\hat{s}_i$ is a maximizer of $u_i(\hat{s}_{N\backslash i}, s_i)$ in the set S_i. That best reaction strategy, however, may not be unique. The set of optimal reactions is, therefore, $M_i(\hat{s}_{N\backslash i})$, a set which depends on the strategies chosen by the n-1 players in $N\backslash i$. The i-th player is in equilibrium if his chosen strategy $\hat{s}_i$ is an element of the set $M_i(\hat{s}_{N\backslash i})$. In symbols, if

$$\hat{s}_i \in M_i(\hat{s}_{N\backslash i})$$

The global choice $\hat{s}$ is a Nash equilibrium if every one of the n players is in equilibrium in this sense, i.e., if

$$\text{for every } i = 1, ..., n,\ \hat{s}_i \in M_i(\hat{s}_{N\backslash i}).$$

In non-technical terms, the decisions made by the players in the game form a Nash equilibrium if every one of them, considering the decisions made by the others as given, cannot increase his utility by changing his own decision.

Augustin Cournot had given a similar definition in 1838 (Cournot, 1838), and the equilibrium concept of 1838-1950 is sometimes called a Cournot-Nash equilibrium. But Nash introduced the concept in the context of game theory where the decision to be made by each one of the n players is the choice under uncertainty of a mixed strategy, i.e., a probability distribution over his set of pure strategies. Nash then proved with great generality and elegance that every game with a finite number of players, every one of them having a finite number of pure strategies, has an equilibrium. By proving the existence of an equilibrium, his result contributed powerfully to its wide use.

Nash's proof also had mathematical depth, which, unlike von Neumann's proof of the minimax theorem in 1928, was necessary for his result. It introduced into the social sciences a theorem (Kakutani's fixed point theorem) which became a standard means of establishing the existence of a General Equilibrium in economic theory.

A Nash equilibrium was proved by him to be a fixed point of a suitable transformation M. Let S be the cartesian product

$$S = S_1 \times S_2 \times \ldots \times S_n$$

of the set of strategies of the n players, S_1 for the first, S_2 for the second, ..., S_n for the n-th. An n-list $s = (s_1, \ldots, s_n)$ of the strategies chosen by those n players is an element of S,

$$s \in S.$$

The set of optimal reactions of the first player to the strategies $(s_2, \ldots, s_n) = s_{N\backslash 1}$ of the others has been defined as $M_1(s_{N\backslash 1})$. Similarly, one has for the second player $M_2(s_{N\backslash 2})$, ..., and for the n-th, $M_n(s_{N\backslash n})$. The transformation M is defined by

$$M(s) = M_1(s_{N\backslash 1}) \times M_2(s_{N\backslash 2}) \times \ldots \times M_n(s_{N\backslash n}).$$

It carries an element s of S into the cartesian product $M(s)$ of $M_1(s_{N\backslash 1})$, a subset of S_1; $M_2(s_{N\backslash 2})$, a subset of S_2; ...; $M_n(s_{N\backslash n})$, a subset of S_n. The set $M(s)$ is a subset of S:

$$M(s) \subset S.$$

Now s is a Nash equilibrium if and only if for every $i = 1, \ldots, n$, $s_i \in M_i(s_{N\backslash i})$, i.e., if and only if

$$s \in M(s)$$

Thus s is a Nash equilibrium if and only if it belongs to its image $M(s)$ by the transformation M, i.e., if and only if it is a fixed point of the transformation M of s into itself. Kakutani's theorem gives conditions under which such a transformation has a fixed point. Therefore, Nash had to check that in the game situation that he considered, those conditions were satisfied.

THE CONCEPT OF NASH BARGAINING SOLUTION

Also in 1950, Nash proposed another concept, that of a bargaining solution, which became associated with his name as well (Nash,1950)(ii). Nash now considers two interacting individuals. The utility levels that they reach are denoted by u_1 for the first and by u_2 for the second. The situation that they face can be represented in a two dimensional utility space whose coordinates are u_1 as the abscissa and u_2 as the ordinate. Some points $u = (u_1, u_2)$ are attainable by the two participants. Others are out of their reach.

Let A be the set of attainable utility points. If the two individuals agree on a point of A, they each obtain the utility levels that are the coordinates of that point. If they do not agree, the disagreement point d is the result. Clearly d, too, is an element of A. Nash asks what point of A should two rational individuals agree upon. Since their utility functions are von Neumann-Morgenstern utility functions, the set A is convex. To see this, let u and v be two points in A, $u = (u_1, u_2)$ and $v = (v_1, v_2)$. One has to show that, given any number a such that $0 \leq a \leq 1$, the point $au + (1-a)v$ is also in A. Clearly, if the first participant can attain u_1 and v_1, he can also attain $au_1 + (1-a)v_1$ by doing with probability a what yielded u_1, and with probability 1-a what yielded v_1. Similarly, the second participant can attain $au_2 + (1-a)v_2$.

The two utility functions of the two participants, also because they are von Neumann-Morgenstern utilities, are determined by the two underlying preferences only up to two increasing linear transformations. $U_1(.)$ can be replaced by $k_1U_1(.) + l_1$, and $u_2(.)$ can be replaced by $k_2U_2(.) + l_2$, where k_1 and k_2 are positive, and these replacements are immaterial for any concept that is determined by the preferences of the two participants. The solution that Nash proposes satisfies those conditions.

Two rational individuals should not agree on a point of A that is weakly dominated by another point of A for which $v_1 \geq u_1$ and $v_2 \geq u_2$, at least one of the two inequalities being strict. Nash's solution, therefore, is Pareto optimal and is restricted to the North-East boundary of A.

Nash also postulates that if the two participants have agreed on the solution $s = (s_1, s_2)$, and if the set of attainable utility points A is now restricted to a subset A' of A in such a way that s is still an element of A', then s is a solution for the new situation described by A'. Indeed, if an agreement was reached on the point s, and s is still attainable after the restriction from A to A', then the two participants should keep agreeing on s.

Apparently equally innocuous is the symmetry axiom according to which a set A having the 45-degree line as a symmetry axis treats the two participants symmetrically. The solution on which they agree must be a point on the 45-degree line. Bypassing points that are inessential for the present purpose, one can show that the Nash solution is obtained by maximizing the product

$$(u_1-d_1)(u_2-d_2)$$

where $u = (u_1, u_2)$ belongs to the given convex set A, and $d = (d_1, d_2)$ is the given disagreement point.[1]

CONCLUSION

John Nash has made a lasting impact on the theory of games. His contributions (which the preceding brief and partial account only attempted to illustrate) to the non-cooperative game theory where binding agreements are not feasible and to cooperative game theory where such agreements are feasible, are remarkable by their scientific impact, but also by the fact that they were made more than 40 years ago, in a span of three years.

In spite of its relative mathematical sophistication, non-cooperative game theory has found numerous applications in real life economic situations where strategic interactions are involved, from the analysis of competitive behavior of oligopolistic firms to labor negotiations, economic policy issues, government regulations, international trade, etc. The concept of equilibrium introduced by Nash which is used to make predictions about the outcome of such interactions was later refined by his other two colleagues with whom Nash is sharing the 1994 Nobel prize for economics.

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NASHOV EKVILIBRIJ I NASHOVO POGODBENO RJEŠENJE: NOBELOVA NAGRADA ZA EKONOMSKE ZNANOSTI 1994.

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Teoriji igre, njezinom nekooperativnom kao i kooperativnom obliku, dana je središnja uloga u suvremenoj ekonomskoj znanosti. U analizi strateških međudjelovanja pokazala se nadmoćnom u odnosu na tradicionalne matematičke metode koje se koriste u ekonomskoj teoriji. Ovaj tekst bavi se dvjema glavnim koncepcijama u teoriji igre koje je uveo John Nash, a to su Nashov ekvilibrij i Nashovo pogodbeno rješenje, čija je razrada Nashu donijela Nobelovu nagradu za ekonomiju 1994. godine. Unatoč razmjernoj matematičkoj složenosti, teorija igre našla je svoju primjenu u brojnim svakodnevnim ekonomskim situacijama strateškog međudjelovanja.

NASHS ÄQUILIBRIUM UND NASHS KOMPROMISSLÖSUNG: NOBELPREIS FÜR ÖKONOMISCHE WISSENSCHAFTEN 1994

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Die zeitgenössische ökonomische Wissenschaft gesteht der Spieltheorie sowohl in ihrer nicht-kooperativen als auch kooperativen Form eine zentrale Rolle zu. Die Analyse strategischer Interaktionen hat ihre Überlegenheit in Bezug auf traditionelle mathematische Methoden, die in der ökonomischen Theorie angewandt werden, erwiesen. Vorliegender Aufsatz beschäftigt sich mit zwei Hauptkonzeptionen innerhalb der Spieltheorie, die von John Nash eingeführt wurden: Nashs Äquilibrium und Nashs Kompromisslösung, die dem Autor 1994 den Nobelpreis für ökonomische Wissenschaften eingebracht haben. Trotz ihrer relativ großen mathematischen Komplexität kommt die Spieltheorie in zahlreichen strategischen Interaktionen des ökonomischen Alltags zur Anwendung.