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Global stability and Neimark–Sacker bifurcation of a rational difference equation with delay

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Abstract. This paper investigates the dynamic properties of a specific rational difference equation with delay. By identifying the invariant (and attractive) interval and applying the so-called "$m - M$" theorem, we establish conditions for the global stability of the positive equilibrium point. It is shown that, at this equilibrium, the equation admits no other codimension-one bifurcations, except for the Neimark–Sacker bifurcation, which arises when the parameter p reaches a critical value. Using the algorithm developed by K. Murakami, we determine the direction and type of the Neimark–Sacker bifurcation. Furthermore, an explicit formula is derived for approximating the invariant closed curve generated by this bifurcation. These results extend previous findings for the same equation in the special case $r = 1$. Several numerical examples and visualizations are included to validate the theoretical results and illustrate the complex behavior of the equation under consideration.

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1. Introduction

In this paper, we investigate the global dynamics of the following difference equation with delay:

$$y_{n+1} = 1 + p\frac{y_n}{y_{n-k}^r}, \tag{1}$$

where the parameter p, and the initial values $y_{-k},..., y_{-1}, y_0$ are arbitrary positive numbers and $r \in \mathbb{R}$, $r > 0$ and $k \in \{2, ...\}$ are fixed numbers.

Difference equations with time delays are essential mathematical tools for modeling discrete dynamical systems. In these models, the current value depends not only on the present state but also on past values of the variables. This approach effectively represents processes with time delays, which frequently occur in fields such as biology, economics, engineering, and computer science. These equations can capture memory effects, making them valuable for analyzing stability, bifurcations, and complex dynamical behaviors.

Equation (1) is a typical example of a nonlinear difference equation with delay. This equation arises in contexts such as population dynamics, chaos theory, and feedback mechanisms in systems with delayed responses. Its structure shows dependence on the state that occurred k steps earlier, which can lead to complex dynamics, including the emergence of a Neimark–Sacker bifurcation—the discrete analogue of the Hopf bifurcation—where a stable fixed point transforms into an invariant closed curve. Under certain parameter conditions, this process may further develop into chaotic behavior, characterized by sensitivity to initial conditions.

It is essential to note that difference equations with delay often arise from discretizing differential equations with delay, which describe continuous-time systems with delayed effects. In numerical methods such as the Euler or Runge–Kutta schemes, the time domain is divided into discrete steps, and differential equations with delay are converted into corresponding difference equations. This discretization facilitates

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their use in computer simulations and in the analysis of systems that naturally evolve in discrete time intervals. Thanks to this connection, theoretical results from differential equations analysis can be applied to discrete models, and vice versa.Thus, difference equations with delay not only expand the mathematical framework for modeling real-world phenomena but also bridge the discrete and continuous realms into a unified analytical structure.

Due to its nonlinearity and the presence of delay, Equation (1) represents a challenging yet highly interesting subject of research. Its analysis requires a combination of theoretical methods and numerical simulations to understand the long-term behavior of solutions depending on the parameter p, including the emergence of bifurcations, invariant structures, and chaotic behavior.

It is known that simpler equations, which represent special cases of Equation (1), often exhibit highly complex behavior, sometimes resulting in chaos. The simplest case occurs when $k = r = 1$, which corresponds to Equation (5.9) in [10]. By substituting $y_n = 1 + pz_n$, this equation reduces to Equation (4.18) in the same paper, which can be expressed as

$$z_{n+1} = \frac{q + z_n}{q + z_{n-1}}, \tag{2}$$

where $q = \frac{1}{p}$. Equation (2) was studied in [9], specifically in Corollary 3.4.1 (e), where it was shown that its equilibrium point is globally asymptotically stable.

However, for the particular case of Equation (1), when $k = 1$ and $r = 2$, the dynamics become much more complicated, featuring the emergence of the Neimark–Sacker bifurcation, which can lead to chaotic behavior (see [3]). Similar behavior has been observed in various equations (e.g., [4, 5, 13, 14, 22]).

In [8], the author studied the following equation:

$$y_{n+1} = \beta + \frac{y_n}{y_{n-k}}, \tag{3}$$

where $y_{-k}, ..., y_{-1}, y_0, \beta \in (0,\infty)$ and $k \in \{2, 3, ...\}$. By introducing the shift $y_n = \beta u_n$ and dividing the resulting equation by β, we can obtain equation (1) in the special case $r = 1$. The author demonstrated that equation (3) undergoes a Neimark–Sacker bifurcation when the parameter β reaches a critical value. Using normal form theory and the algorithm developed by K. Murakami in [19], we will also determine the direction of the Neimark–Sacker bifurcation and the stability of the equilibrium point in Equation (1). This analysis will lead us to an explicit approximation of the invariant closed curve that arises from the Neimark–Sacker bifurcation. However, the behavior of Equation (1) becomes intricate, complicating our analysis. Specifically, calculating the equilibrium point can be incredibly challenging, requiring various analytical approaches to determine its existence and, in particular, its application. Furthermore, for determining local stability, we cannot apply Theorem 1.1.1 in [10]; instead, we must use a more general theorem (Theorem 1.3.7 in [9]). On the other hand, research into global stability is mainly reduced to the so-called *"m-M" theorem* (Theorem 4 in [12] or Theorem 1.15 in [6]), indicating that the behavior of the equation is quite complex (for further reference, see [1]). For the same reasons, the investigation of the Neimark–Sacker bifurcation will be more complicated than in the case of Equation (3). Additionally, we will present results related to the boundedness, oscillations, and the rate of convergence of the solutions of Equation (1), which, together with the results on local and global stability and the Neimark–Sacker bifurcation, provide a more complete understanding of the rich dynamics of the equation.

Note that the complexity further increases significantly when r is not an integer. An interesting attempt to address this issue can be found in [2], where the following equation is investigated:

$$x_{n+1} = s + \frac{x_n^r}{x_{n-l}^r},$$

with $r \in (0, 1)$, $s \in [0, \infty)$, $l \in \{1, 2, \cdots\}$, and the initial values are positive real numbers.

The case of Equation (1) when $r = 2$ and k is an integer was considered in [23], where the author investigated the periodicity, boundedness, stability, and the rate of convergence of the following quadratic-rational difference equation:

$$u_{n+1} = \alpha + \beta \frac{u_n}{u_{n-l}^2},$$

with α, β and the initial conditions being positive real numbers and $l \in \{2, 3, \cdots\}$ (the specific case of the equation when $l = 1$ was considered in [3]).

Since there are very few papers devoted to the study of higher-order difference equations and bearing in mind the results obtained in the above-mentioned equations, we believe that it would be useful to consider the stability and the Neimark-Sacker bifurcation of a more general equation

$$u_{n+1} = \alpha + \beta \frac{u_n}{u_{n-l}^r}, \tag{4}$$

where the initial values are positive real numbers, $r \in \mathbb{R}$, and $l \in \{2, 3, ...\}$.

By substituting $y_n = \frac{u_n}{\alpha}$ in Equation (4), we obtain Equation (1). In this context, obtaining results regarding boundedness, local and global stability, and the rate of solution convergence becomes more challenging. The specific motivation for studying Equation (1) is to find the critical value of the parameter p that leads to the occurrence of a Neimark–Sacker bifurcation. Therefore, we will consider Equation (1).

This paper is organized as follows. We begin Section 2 by studying the boundedness of solutions to Equation (1), which allows us to identify an invariant and attracting interval. Then we apply Theorem 1.3.7 from [9] and Theorem 4 from [12] (or Theorem 1.15 from [6]) to investigate the local and global stability results for the unique equilibrium point of Equation (1). Additionally, we demonstrate the non-existence of periodic solutions with a minimal period of two. Section 3 focuses on the Neimark–Sacker bifurcation, where we use the algorithm from [19] to determine the asymptotic approximation of the invariant curve. In Section 4, we further investigate the oscillatory behavior of the solution to Equation (1). Section 5 addresses the rate of convergence of the solutions, which present additional challenges due to the potential irrationality of r. Finally, in Section 6, we present numerical examples and visualizations that confirm our theoretical results.

2. Boundedness and stability

In this section, we reveal that every solution of Equation (1) is bounded from below and above. Also, we study the existence of minimal period-two solutions and the asymptotic stability of solutions of Equation (1).

Lemma 1. *Every solution of Equation (1) is bounded from below and above.*

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of Equation (1). It is obvious that $y_n > 1$, for all $n \geq 1$. On the other hand, with this in mind, Equation (1) implies that for every $n \geq k + 2$

$$y_n = 1 + p\frac{y_{n-1}}{y_{n-k-1}^r} = 1 + \frac{p}{y_{n-k-1}^{r-1}}\left(\frac{y_{n-1}}{y_{n-2}}\frac{y_{n-2}}{y_{n-3}}\cdots\frac{y_{n-k}}{y_{n-k-1}}\right) < 1 + p\left(\frac{y_{n-1}}{y_{n-2}}\frac{y_{n-2}}{y_{n-3}}\cdots\frac{y_{n-k}}{y_{n-k-1}}\right),$$

and

$$\frac{y_{n-i}}{y_{n-i-1}} = \frac{1}{y_{n-i-1}} + \frac{p}{y_{n-i-k-1}^r} < 1 + p, \tag{5}$$

for $n \geq 2k + 2$ and $i = 1, 2, \cdots, k$. Therefore, using (5), we obtain

$$y_n < 1 + p\left(\frac{1}{y_{n-2}} + \frac{p}{y_{n-k-2}^r}\right)\left(\frac{1}{y_{n-3}} + \frac{p}{y_{n-k-3}^r}\right)\cdots\left(\frac{1}{y_{n-k-1}} + \frac{p}{y_{n-2k-1}^r}\right) < 1 + p\,(1+p)^k,$$

for $n \geq 2k + 2$. It implies that $y_n \leq \max\{y_{-k}, y_0, y_1, ..., y_{2k+1}, 1 + p\,(1+p)^k\}$ for all $n \geq 1$. □

Remark 1. Lemma 1 implies that $[1, 1 + p(1+p)^k]$ is an invariant and attracting interval for Equation (1). This significant result allows us to determine the conditions under which the unique positive equilibrium point of Equation (1) will be globally stable.

Now, let us determine the equilibrium points of Equation (1).

Lemma 2. *Equation (1) has a unique positive equilibrium point, $\bar{y}$, such that $\bar{y} > \max\left\{1, p^{\frac{1}{r}}\right\}$.*

Proof. Note that an equilibrium point $\bar{y}$ of Equation (1) is the root of the following function:

$$h(y) = \begin{cases} y - py^{1-r} - 1 & \text{for } 0 < r < 1, \\ y^r - y^{r-1} - p & \text{for } r > 1. \end{cases} \tag{6}$$

For function (6) and $y > 0$ the following holds:

$$h'(y) = \begin{cases} 1 - p(1-r)\, y^{-r} & \text{for } 0 < r < 1, \\ ry^{r-1} - (r-1)\, y^{r-2} & \text{for } r > 1, \end{cases}$$

and

$$\lim_{x\to\infty} h(y) = +\infty, \quad h(1) = -p < 0, \quad h\left(p^{\frac{1}{r}}\right) = -1 < 0.$$

Therefore, Equation (1) has a unique positive equilibrium, $\bar{y}$, such that $\bar{y} > \max\left\{1, p^{\frac{1}{r}}\right\}$. □

Lemma 3. *If $p < \frac{(r+1)^{r-1}}{r^r}$, then the unique positive equilibrium point $\bar{y}$ of Equation (1) is locally asymptotically stable.*

Proof. It is obvious that the linearized equation of Equation (1) around its unique positive equilibrium point $\bar{y}$ has the following form:

$$z_{n+1} - \frac{p}{\bar{y}^r} z_n + \frac{rp}{\bar{y}^r} z_{n-k} = 0,$$

and the corresponding characteristic equation is

$$\lambda^{k+1} - \frac{p}{\bar{y}^r}\lambda^k + \frac{rp}{\bar{y}^r} = 0. \tag{7}$$

According to Theorem 1.3.7 in [9], equilibrium point $\bar{y}$ of Equation (1) is locally asymptotically stable if

$$\left|-\frac{p}{\bar{y}^r}\right| + \left|\frac{rp}{\bar{y}^r}\right| = \frac{(r+1)\, p}{\bar{y}^r} < 1,$$

from which we obtain

$$0 < ((r+1)\, p)^{1/r} < \bar{y}.$$

By using the previously observed properties of the function h from (6), we see that the last condition is equivalent to the following:

$$h\left(((r+1)\, p)^{1/r}\right) < 0,$$

from which we obtain $p < \frac{(r+1)^{r-1}}{r^r}$. □

Remark 2. The result on local asymptotic stability obtained in Lemma 3, although crude since it is only a sufficient condition, is nevertheless of exceptional importance. Although there is a theorem on the necessary and sufficient conditions for asymptotic stability (Theorem 4 in [17]), it has no practical significance here. Namely, it follows from this result that the equilibrium point of Equation (1) is asymptotically stable if and only if the following holds:

$$\left(r^2 - 1\right)\left(\frac{p}{\bar{y}^r}\right)^2 + 2(\cos\phi)\,\frac{p}{\bar{y}^r} - 1 < 0, \tag{8}$$

and additionally, for odd k,

$$\frac{p}{\bar{y}^r}\,|r-1| < 1, \tag{9}$$

where ϕ is the solution in $\left(0, \frac{\pi}{k+1}\right)$ of the equation $\sin k\psi = \frac{\bar{y}^r}{p}\sin(k+1)\,\psi$.

(Note that the condition $\frac{p}{\bar{y}^r} < \frac{k+1}{k}$ is satisfied since $\bar{y}^r > \frac{pk}{k+1}$ implies that $h\left(\left(\frac{pk}{k+1}\right)^{\frac{1}{r}}\right) < 0$ holds, using (6).)

Unfortunately, here the angle ψ depends on the parameter p, so instead of obtaining the complete set of values of that parameter for which the equilibrium point of Equation (1) is asymptotically stable, we only have the possibility of checking conditions (8) and (9) for a fixed value of the parameter p and this can mainly be achieved by numerical calculations. At the same time, one can numerically obtain the eigenvalues of (7) for that specific value of p and directly check whether they lie inside the unit disk.

It hints at the complexity of the dynamics of Equation (1), which becomes increasingly apparent with the results that follow.

Lemma 4. *If k is odd, then Equation (1) has no minimal period-two solution.*

Proof. Suppose the opposite, there exists a minimal period-two solution of Equation (1) as follows:

$$\cdots, \alpha, \beta, \alpha, \beta, \cdots, \tag{10}$$

where α, β are positive numbers and $\alpha \neq \beta$. Without loss of generality, assume that $y_n = \alpha < \beta$. Since k is odd, then $y_{n-k} = y_{n+1} = \beta$ and

$$1 + p\frac{\beta}{\beta^r} < 1 + p\frac{\beta}{\alpha^r} = \alpha < \beta = 1 + p\frac{\alpha}{\beta^r} < 1 + p\frac{\beta}{\beta^r},$$

which is a contradiction. It means that Equation (1) has no minimal period-two solution. □

Remark 3. The statement of Lemma 4 does not hold in general when k is even. To illustrate this, consider the case of Equation (1) for $k = 6$, $p = 9$, and $r = 1 + log_2 3$. Specifically, if in (10) we take $\alpha = 4$, $\beta = 2$, we observe an exception. Indeed, since k is even, let us consider $y_n = \alpha$, so that $y_{n-k} = \alpha$, $y_{n+1} = \beta$. In this case, the minimal period-two solution of the form (10) clearly satisfies the system of equations

$$\beta = 1 + p\frac{\alpha}{\alpha^r} = 1 + \frac{p}{\alpha^{r-1}},$$
$$\alpha = 1 + p\frac{\beta}{\beta^r} = 1 + \frac{p}{\beta^{r-1}}.$$

Theorem 1. *If $r\,p\,(1+p)^k \leq 1$, then the positive equilibrium point of Equation (1) is a global attractor.*

Proof. Now, we consider the function $f : [1, 1 + p(1+p)^k]^{k+1} \to [1, 1 + p(1+p)^k]$ associated with the right-hand side of Equation (1) as follows:

$$f\,(y_n, y_{n-1}, ..., y_{n-k}) = 1 + p\frac{y_n}{y_{n-k}^r}.$$

The function $f(u_1, u_2, ..., u_{k+1})$ is nondecreasing in u_1, u_2,...,u_k and decreasing in u_{k+1}, where $u_2 = ... = u_k = 0$. We can apply Theorem 4 in [12] (or Theorem 1.15 in [6]) using that $m_2 = ... = m_k = 0$, $M_2 = ... = M_k = 0$, $m_1 = m$, and $M_1 = M$. Let (m, M) be a solution of the system

$$m = f(m, M) \text{ and } M = f(M, m).$$

i.e.,

$$m = 1 + p\frac{m}{M^r},$$
$$M = 1 + p\frac{M}{m^r},$$

which is equivalent to

$$p = \frac{M^r\,(m-1)}{m} = \frac{m^r\,(M-1)}{M}.$$

It implies that

$$\frac{M^{r+1}}{M-1} = \frac{m^{r+1}}{m-1}. \tag{11}$$

Let $\Phi : (1, +\infty) \to (0, +\infty)$ be a function such that $\Phi(x) = \frac{x^{r+1}}{x-1}$. Then, (11) can be written in the following form:

$$\Phi(M) = \Phi(m). \tag{12}$$

Since

$$\Phi'(x) = \frac{x^r(rx - r - 1)}{(x-1)^2},$$

it is obvious that Φ is monotone decreasing for $1 < x \leq \frac{r+1}{r}$. Assume that

$$\frac{r+1}{r} \geq 1 + p(1+p)^k,$$

i.e., $rp(1+p)^k \leq 1$. Then, Φ is an injection function on the invariant interval $[1, 1+p(1+p)^k]$, so (12) implies that $m = M = \overline{y}$. According to Theorem 4 in [12] (or Theorem 1.15 in [6]), every solution of Equation (1) with initial conditions in interval $[1, 1+p(1+p)^k]$ converges to $\overline{y}$. However, $[1, 1+p(1+p)^k]$ is an attracting interval, meaning every solution eventually enters that interval (after a finite number of iterations). It follows that the equilibrium point $\overline{y}$ is a global attractor. □

See also [15] for the use of the "$m - M$" theorem.

Now, we obtain the following global result as a direct consequence of Lemma 1 and Theorem 2.

Theorem 2. *If $p < \frac{(r+1)^{r-1}}{r^r}$ and $rp(1+p)^k \leq 1$, then the unique positive equilibrium point $\bar{y}$ of Equation (1) is globally asymptotically stable.*

When the equilibrium point $\bar{y}$ is locally asymptotically stable, we have not been able to prove its global stability fully. However, many numerical simulations suggest such behavior of the solution of Equation (1), thus confirming our suspicions that the equilibrium point $\bar{y}$ is globally asymptotically stable.

Conjecture 1. *If $p < \frac{(r+1)^{r-1}}{r^r}$, then the unique positive equilibrium point $\bar{y}$ of Equation (1) is globally asymptotically stable.*

3. Neimark–Sacker bifurcation

Let $r > 0$ and $k \in \{2, 3, ...\}$ be arbitrarily chosen, but fixed numbers. Now, we determine for which value of the parameter p the equilibrium $\overline{y}$ of Equation (1) undergoes a Neimark–Sacker bifurcation. Let $z_n = y_n - \overline{y}$. Then, the equilibrium point shifts to the origin, and we have the new equation of the form

$$z_{n+1} = 1 - \overline{y} + p\frac{z_n + \overline{y}}{(z_{n-k} + \overline{y})^r}, \quad n = 0, 1, 2, ... \tag{13}$$

If we use a new variable $Z_n = (z_n, z_{n-1}, ..., z_{n-k})^T$, then Equation (13) transforms into the system

$$Z_{n+1} = G(Z_n, p), \quad n = 0, 1, 2, ..., \tag{14}$$

where $G = (g_0, g_1, ..., g_k)^T$ with

$$g_l = \begin{cases} 1 - \overline{y} + p\frac{z_n + \overline{y}}{(z_{n-k} + \overline{y})^r}, & l = 0 \\ z_{n-l+1} & 1 \leq l \leq k. \end{cases}$$

It is easy to see that the unique equilibrium point of System (14) is the origin, and the linearized system of (14) about the origin has the following form:

$$Z_{n+1} = J_G(\mathbf{0}, p) Z_n, \quad n = 0, 1, 2, ...,$$

where

$$J_G(\mathbf{0}, p) = \begin{pmatrix} \frac{p}{\overline{y}^r} & 0 & \cdots & 0 & -\frac{rp}{\overline{y}^r} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The corresponding characteristic equation is

$$\kappa(\lambda) := \lambda^{k+1} - \frac{p}{\overline{y}^r}\lambda^k + \frac{rp}{\overline{y}^r} = 0. \tag{15}$$

The following result gives us conditions for the existence of a pair of roots with modulus one of Equation (15).

Lemma 5. *Assume that $r > 0$ and $k \in \{2, 3, ...\}$ are arbitrarily chosen, but fixed numbers, and $p > 0$. Then, Equation (15) has a pair of roots with modulus one when $p = p^* := \left(\frac{\sin k\varphi^*}{\sin k\varphi^* - \sin(k+1)\varphi^*}\right)^r \frac{\sin(k+1)\varphi^*}{\sin k\varphi^*}$, where $\varphi^* \in (0, \pi)$ is the root of the equation $\sin(k+1)\varphi^* = \frac{1}{r}\sin\varphi^*$.*

Proof. If $e^{i\varphi}$ $(0 < \varphi < \pi)$ would be a solution of Equation (15), then we obtain that

$$e^{i(k+1)\varphi} - \frac{p^*}{\overline{y}^r}e^{ik\varphi} + \frac{rp^*}{\overline{y}^r} = 0,$$

from which

$$e^{i\varphi} - \frac{p^*}{\overline{y}^r} + \frac{rp^*}{\overline{y}^r}e^{-ik\varphi} = 0,$$

and

$$\cos\varphi = \frac{p^*}{\overline{y}^r} - \frac{rp^*}{\overline{y}^r}\cos k\varphi, \tag{16}$$

$$\sin\varphi = \frac{rp^*}{\overline{y}^r}\sin k\varphi. \tag{17}$$

From (17) it implies that $\sin k\varphi \neq 0$ since $0 < \varphi < \pi$ and $\sin\varphi \neq 0$, which yields

$$\frac{\sin\varphi}{\sin k\varphi} = \frac{rp^*}{\overline{y}^r},$$

and, by subtituting that into (16), we have that $\varphi = \varphi^*$ is the solution of the equation

$$\sin(k+1)\varphi^* = \frac{1}{r}\sin\varphi^*, \tag{18}$$

and also

$$p^* := \overline{y}^r \frac{\sin(k+1)\varphi^*}{\sin k\varphi^*}. \tag{19}$$

Since the equilibrium point $\overline{y}$ is a function of the parameter p, that is, $y = y(p)$, this value of p^* is the solution of Equation (19). However, we can explicitly determine the function $y(p)$ in very few cases with a rational value of r, and determining the value of p^* becomes a significant problem. Fortunately, $p = y^r - y^{r-1}$ follows from Lemma 2, so by substituting in (19), we get

$$\overline{y}_* := \overline{y}(p^*) = \frac{\sin k\varphi^*}{\sin k\varphi^* - \sin(k+1)\varphi^*},$$

and

$$p^* = \left(\frac{\sin k\varphi^*}{\sin k\varphi^* - \sin(k+1)\varphi^*}\right)^r \frac{\sin(k+1)\varphi^*}{\sin k\varphi^*}. \tag{20}$$

□

Remark 4. (a) Using (6), it can be shown that the characteristic equation (15) cannot have eigenvalue 1, but can have eigenvalue -1 for even k, $r > 2$, and $p = \frac{1}{r-1}\left(\frac{r-1}{r-2}\right)^r$, where $\overline{y} = \frac{r-1}{r-2}$.

(b) Note that Equation (18) implies in Lemma 5 that admissible solutions must satisfy $\sin((k+1)\varphi^*) > 0$, since $\sin\varphi^* > 0$ for $\varphi^* \in (0,\pi)$. This yields a restriction of the admissible domain for φ^* to those subintervals of $(0,\pi)$ where $\sin((k+1)\varphi) > 0$; namely,

$$\varphi^* \in \bigcup_{i=0}^{\lfloor k/2 \rfloor} \left(\frac{2i\pi}{k+1}, \frac{(2i+1)\pi}{k+1} \right).$$

For specific values of k, additional reductions may occur due to further constraints imposed by Lemma 5 ($\varphi^* \neq \frac{\pi}{2}, \frac{2\pi}{3}$, but possibly others). For instance, when $k = 2$, condition (18) yields $\varphi^* \in (0, \pi/3)$, as $\cos\varphi^* > 0$ further restricts the admissible domain.

Lemma 6. *Let $r > 0$ and $k \in \{2, 3, ...\}$ be arbitrarily chosen, but fixed numbers, $p > 0$, $p^* = \overline{y}^r \frac{\sin(k+1)\varphi^*}{\sin\varphi^*}$, where $\varphi = \varphi^*$ is the solution of Equation (18). Then, the following is true:*

(i) $\lambda^m(p_*) \neq 1$ *for* $m = 1, 2, 3, 4$,

(ii) $d_* = d(p) = \frac{d}{dp}\left(|\lambda(p)|\right)\Big|_{p=p_*} > 0$.

Proof. (i) It is easy to see that $e^{im\varphi^*} \neq 1$ for $m \in \{1,2,3,4\}$ if $\varphi^* \neq \frac{\pi}{2}, \frac{2\pi}{3}$, which is satisfied since $0 < \varphi^* < \frac{\pi}{k+1}$.

(ii) For $p > 0$, $r > 0$ and $k \in \{2, 3, ...\}$, we have

$$\begin{aligned} d(p) &= \frac{d}{dp}\left(|\lambda(p)|\right)\Big|_{p=p^*,\varphi=\varphi^*} = \frac{d}{dp}\left(\sqrt{\lambda(p)\,\overline{\lambda(p)}}\right)\Big|_{p=p^*,\varphi=\varphi^*} \\ &= \frac{1}{2\sqrt{\lambda(p)\,\overline{\lambda(p)}}}\left(\lambda(p)\frac{d\overline{\lambda(p)}}{dp} + \overline{\lambda(p)}\frac{d\lambda(p)}{dp}\right)\Big|_{p=p^*,\varphi=\varphi^*} \\ &= \frac{1}{\sqrt{\lambda(p)\,\overline{\lambda(p)}}}\mathfrak{Re}\left(\overline{\lambda(p)}\frac{d\lambda(p)}{dp}\right)\Big|_{p=p^*,\varphi=\varphi^*}. \end{aligned}$$

By using the fact that the equilibrium $\overline{y}$ depends on parameter p, i.e., $\overline{y} = \overline{y(p)}$, and differentiating Equation (15) with respect to p, we have

$$(k+1)\lambda^k\frac{d\lambda}{dp} - \frac{\overline{y} - pr\overline{y}'}{\overline{y}^{r+1}}\lambda^k - \frac{kp}{\overline{y}^r}\lambda^{k-1}\frac{d\lambda}{dp} + r\frac{\overline{y} - pr\overline{y}'}{\overline{y}^{r+1}} = 0,$$

which yields

$$\frac{d\lambda}{dp} = \frac{\left(\overline{y} - pr\overline{y}'\right)\left(\lambda^k - r\right)}{\overline{y}^{r+1}\kappa'(\lambda)} = \frac{\left(\overline{y} - pr\overline{y}'\right)\frac{\overline{y}^r}{p}\lambda^{k+1}}{\overline{y}^{r+1}\kappa'(\lambda)} = \frac{\left(\overline{y} - pr\overline{y}'\right)\lambda^{k+1}}{p\overline{y}\kappa'(\lambda)}.$$

Now,

$$\begin{aligned} d_* = d(p)|_{p=p^*} &= \frac{1}{\sqrt{\lambda(p)\,\overline{\lambda(p)}}}\mathfrak{Re}\left(\frac{\overline{\lambda}\lambda\left(\overline{y} - pr\overline{y}'\right)\lambda^k}{p\overline{y}\kappa'(\lambda)}\right)\Big|_{p=p^*,\varphi=\varphi^*} \\ &= \frac{\overline{y}_* - p^*r\overline{y}'_*}{p^*\overline{y}_*^{r+1}\left|\kappa'(\lambda(p^*))\right|^2}\mathfrak{Re}\left\{\lambda^k(p^*)\,\overline{\lambda(p^*)}^{k-1}\left[(k+1)\,\overline{y}_*^r\overline{\lambda(p^*)} - kp^*\right]\right\} \\ &= \frac{\overline{y}_* - p^*r\overline{y}'_*}{p^*\overline{y}_*^{r+1}\left|\kappa'(\lambda_0)\right|^2}\mathfrak{Re}\left\{(k+1)\,\overline{y}_*^r - kp^*\lambda_0\right\} \\ &> 0, \end{aligned}$$

where $\lambda(p^*) = \lambda_0$.

Indeed,

$$\begin{aligned}\mathfrak{Re}\left[(k+1)\,\overline{y}_*^r - kp^*\lambda_0\right] &= (k+1)\,\overline{y}_*^r - kp^*\cos\varphi^* = p^*k\left[\frac{\overline{y}^r}{p^*k}(k+1) - \cos\varphi^*\right]\\ &= p^*k\left[\frac{\overline{y}_*^r}{p^*}\left(1+\frac{1}{k}\right) - \cos\varphi^*\right]\\ &> p^*k\left[\left(1+\frac{1}{k}\right) - \cos\varphi^*\right] > 0,\end{aligned}$$

since $\overline{y} > \max\left\{1, p^{\frac{1}{r}}\right\}$.

Also,

$$\overline{y}_* - pr\overline{y}'_* = \overline{y}_* - \frac{pr\overline{y}_*}{\overline{y}_*^r - p + pr} = \overline{y}_*\frac{\overline{y}_*^r - p}{\overline{y}_*^r - p + pr} > 0,$$

where $\overline{y}'$ is obtained by differentiating the following equation:

$$\overline{y} - p\overline{y}^{1-r} - 1 = 0,$$

with respect to p, since $\overline{y}$ depends on p (see (6)). □

Remark 5. Lemmas 5 and 6 show that System (14) satisfies the transversality condition and the non-resonance (or non-degeneracy) condition in the N-S bifurcation theorem (see [18], Theorem 4.6 (C1) and (C2)).

System (14) can be written in the following form:

$$Z_{n+1} = J_G(\mathbf{0}, p)\,Z_n + \widehat{G}(Z_n, p), \tag{21}$$

where $\widehat{G} = (\widehat{g}_0, \widehat{g}_1, ..., \widehat{g}_k)$ with

$$\widehat{g}_l = \begin{cases} 1 - \overline{y} + p\frac{z_n+\overline{y}}{(z_{n-k}+\overline{y})^r} + \frac{p}{\overline{y}^r}(rz_{n-k} - z_n), & l = 0\\ 0, & 1 \le l \le k. \end{cases}$$

Let $\mathbf{q} \in \mathbb{C}^{k+1}$ be a right eigenvector and $\mathbf{p} \in \mathbb{C}^{k+1}$ be a left eigenvector of the matrix $J_G(\mathbf{0}, p^*)$ associated with $\lambda_0 = e^{i\varphi^*}$ satisfying the conditions

$$J_G(\mathbf{0}, p^*)\,\mathbf{q} = \lambda_0\mathbf{q}, \quad \mathbf{p}J_G(\mathbf{0}, p^*) = \lambda_0\mathbf{p}, \quad \mathbf{pq} = 1.$$

If $\mathbf{q} = (q_0, q_1, ..., q_k)^T$, then

$$\begin{cases} c\frac{p^*}{\overline{y}_*^r}q_0 - \frac{rp^*}{\overline{y}_*^r}q_k = \lambda_0 q_0,\\ q_l = \lambda_0 q_{l+1},\ l \in \{0, 1, ..., k-1\}. \end{cases} \tag{22}$$

If we set $q_0 = 1$ in (22), this implies that

$$\mathbf{q} = \left(1, \overline{\lambda}_0, \overline{\lambda}_0^2, ..., \overline{\lambda}_0^k\right)^T.$$

Analogously, if $\widehat{\mathbf{p}} = (p_0, p_1, ..., p_k)$ is a left eigenvector of the matrix $J_G(\mathbf{0}, p^*)$, then we obtain that

$$\begin{cases} \frac{p^*}{\overline{y}_*^r}q_0 + p_1 = \lambda_0 p_0,\\ p_l = \lambda_0 p_{l-1},\ l \in \{2, \ldots, k\}, \end{cases}$$

from which

$$\widehat{\mathbf{p}} = \left(p_0, -\frac{rp^*}{\overline{y}_*^r}\overline{\lambda}_0^k p_0, -\frac{rp^*}{\overline{y}_*^r}\overline{\lambda}_0^{k-1} p_0, ..., -\frac{rp^*}{\overline{y}_*^r}\overline{\lambda}_0 p_0\right).$$

Setting $p_0 = 1$ and since $\widehat{\mathbf{p}}\mathbf{q} = 1 - \frac{krp^*}{\overline{y}_*^r}\overline{\lambda}_0^{k+1} = 1 - k\left(\frac{p^*}{\overline{y}_*^r} - \lambda_0\right)\overline{\lambda_0} = 1 + k - \frac{kp^*}{\overline{y}_*^r}\overline{\lambda}_0 = \alpha$, we obtain that $\mathbf{pq} = 1$ for $\mathbf{p} = \frac{1}{\alpha}\widehat{\mathbf{p}}$.

As in [19], we define $\Phi = (\mathbf{q}, \overline{\mathbf{q}})$ and $\Psi = \begin{pmatrix} \mathbf{p} \\ \overline{\mathbf{p}} \end{pmatrix}$ and change the variable

$$y = \Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix} + w$$

for p near p^*, where

$$\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = z\mathbf{q} + \overline{z}\overline{\mathbf{q}}, \quad z = \mathbf{p}y, \quad w = y - \Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}.$$

If we set $p = p^* + \nu$, $J_{G,\nu}(\mathbf{0}, p^*) = J_G(\mathbf{0}, p^* + \nu)$, $H_\nu(Z_n) = \widehat{G}(Z_n, p^* + \nu)$, $Z_n = \Phi\begin{pmatrix} z_n \\ \overline{z}_n \end{pmatrix} + w_n$, then System (21) can be decomposed into

$$\begin{cases} z_{n+1} = \lambda(\nu) z_n + \mathbf{p}H_\nu\left(\Phi\begin{pmatrix} z_n \\ \overline{z}_n \end{pmatrix} + w_n\right), \\ w_{n+1} = J_{G,\nu}(\mathbf{0}, p^*) w_n + (I - \Phi\Psi) H_\nu\left(\Phi\begin{pmatrix} z_n \\ \overline{z}_n \end{pmatrix} + w_n\right). \end{cases} \tag{23}$$

By using the center manifold theorem, for $|z|$ sufficiently small, there exists a function $w = \omega(z, \overline{z})$ such that the solution of System (23) in the neighborhood of the origin can be written approximately in the following form:

$$Z_n = \Phi\begin{pmatrix} z_n \\ \overline{z}_n \end{pmatrix} + \omega(z_n, \overline{z}_n). \tag{24}$$

Note that dynamics of (24) can be described by

$$z_{n+1} = \lambda(\nu) z_n + \mathbf{p}H_\nu\left(\Phi\begin{pmatrix} z_n \\ \overline{z}_n \end{pmatrix} + \omega(z_n, \overline{z}_n)\right). \tag{25}$$

The Taylor expansion of $H_0\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)$ near $z = 0$ has the following form:

$$H_0\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right) = \sum_{l+m=2} \frac{1}{l!m!}\mathbf{h}_{lm} z^l \overline{z}^m + O\left(|z|^3\right). \tag{26}$$

For System (21) we have

$$H_0\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right) = \begin{pmatrix} 1 - \overline{y}_* + p^* \frac{z+\overline{z}+\overline{y}}{\left(z\overline{\lambda}_0^k + \overline{z}\lambda_0^k + \overline{y}\right)^r} + \frac{p^*}{\overline{y}_*^r}\left(r\left(z\overline{\lambda}_0^k + \overline{z}\lambda_0^k\right) - z - \overline{z}\right) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now, by using (26), we have that

$$\mathbf{h}_{20} = \frac{\partial^2}{\partial z^2} H_0\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} \frac{p^* r}{\overline{y}_*^{r+1}}\left((r+1)\overline{\lambda}_0^{2k} - 2\overline{\lambda}_0^k\right) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\mathbf{h}_{11}=\frac{\partial^2}{\partial z\partial\overline{z}}H_0\left(\Phi\begin{pmatrix}z\\ \overline{z}\end{pmatrix}\right)\Big|_{z=0}=\begin{pmatrix}\frac{p^*r}{\overline{y}_*^{r+1}}\left((r+1)-\overline{\lambda}_0^k-\lambda_0^k\right)\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix},$$

and $\mathbf{h}_{02}=\overline{\mathbf{h}}_{20}$.

Since the function $\omega(z,\overline{z})$ can be approximated by

$$\omega(z,\overline{z})=\sum_{l+m=2}\frac{1}{l!m!}\omega_{lm}z^l\overline{z}^m+O\left(|z|^3\right),$$

where

$$\begin{aligned}\omega_{20}&=\left(\lambda^2I-J_G(\mathbf{0},p^*)\right)^{-1}(I-\Phi\Psi)\mathbf{h}_{20},\\ \omega_{11}&=\left(\lambda I-J_G(\mathbf{0},p^*)\right)^{-1}(I-\Phi\Psi)\mathbf{h}_{21},\\ \omega_{02}&=\left(\overline{\lambda}^2I-J_G(\mathbf{0},p^*)\right)^{-1}(I-\Phi\Psi)\mathbf{h}_{02},\end{aligned}$$

by substituting $\omega(z,\overline{z})$ into (25), we obtain the two-dimensional equation restricted on the center manifold. Then, Equation (25) can be transformed into its normal form

$$\xi_{n+1}=\lambda(\nu)\xi_n+c(\nu)\xi_n^2\overline{\xi}_n+O\left(\left|\xi_n^4\right|\right). \tag{27}$$

Now, using (27) we can approximate (25) as follows:

$$z_n=\Phi\begin{pmatrix}\xi_n\\ \overline{\xi}_n\end{pmatrix}+\frac{1}{2}\mathbf{k}_{20}^2\xi_n+\mathbf{k}_{11}\xi_n\overline{\xi}_n+\frac{1}{2}\mathbf{k}_{02}\overline{\xi}_n^2+O\left(\left|\xi_n^3\right|\right).$$

The expressions $\mathbf{k}_{20}$, $\mathbf{k}_{11}$ and $\mathbf{k}_{02}$ are calculated as follows:

$$\begin{aligned}\mathbf{k}_{20}&=\left(\lambda_0^2I_{k+1}-J_G(\mathbf{0},p^*)\right)^{-1}\mathbf{h}_{20}\\ &=\begin{pmatrix}\lambda_0^2-\frac{p^*}{\overline{y}_*^r}&0&0&\cdots&0&\frac{p^*r}{\overline{y}_*^r}\\ -1&\lambda_0^2&0&\cdots&0&0\\ 0&-1&\lambda_0^2&\cdots&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&0&\cdots&\lambda_0^2&0\\ 0&0&0&\cdots&-1&\lambda_0^2\end{pmatrix}^{-1}\begin{pmatrix}\frac{p^*r}{\overline{y}_*^{r+1}}\left((r+1)\overline{\lambda}_0^{2k}-2\overline{\lambda}_0^k\right)\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}\\ &=\frac{1}{\kappa\left(\lambda_0^2\right)}\begin{pmatrix}J_{11}&\cdots&J_{k+1,1}\\ J_{12}&\cdots&J_{k+1,2}\\ J_{13}&\cdots&J_{k+1,3}\\ \vdots&\vdots&\vdots\\ J_{1,k+1}&\cdots&J_{k+1,k+1}\end{pmatrix}\begin{pmatrix}\frac{p^*r}{\overline{y}_*^{r+1}}\left((r+1)\overline{\lambda}_0^{2k}-2\overline{\lambda}_0^k\right)\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix},\end{aligned}$$

i.e.,

$$\mathbf{k}_{20}=\frac{p^*r\left((r+1)\overline{\lambda}_0^{2k}-2\overline{\lambda}_0^k\right)}{\kappa\left(\lambda_0^2\right)\overline{y}_*^{r+1}}\begin{pmatrix}J_{11}\\ J_{12}\\ J_{13}\\ \vdots\\ J_{1,k+1}\end{pmatrix}=\frac{p^*r\left((r+1)\overline{\lambda}_0^{2k}-2\overline{\lambda}_0^k\right)}{\kappa\left(\lambda_0^2\right)\overline{y}_*^{r+1}}\begin{pmatrix}\lambda_0^{2k}\\ \lambda_0^{2(k-1)}\\ \vdots\\ \lambda_0^2\\ 1\end{pmatrix}, \tag{28}$$

where J_{lm} $(l, m \in \{1, 2, ..., k+1\})$ are the corresponding algebraic complements of the matrix $\lambda_0^2 I_{k+1} - J_G(\mathbf{0}, p^*)$ and $\det\left(\lambda_0^2 I_{k+1} - J_G(\mathbf{0}, p^*)\right) = \kappa\left(\lambda_0^2\right)$ ($\kappa(\lambda)$ is a characteristic polynomial in (15)).

Analogously,

$$\begin{aligned}
\mathbf{k}_{11} &= \left(I_{k+1} - J_G(\mathbf{0}, p^*)\right)^{-1} \mathbf{h}_{11} \\
&= \begin{pmatrix} 1 - \frac{p^*}{\overline{y}_*^r} & 0 & 0 & \cdots & 0 & \frac{p^* r}{\overline{y}_*^r} \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{p^* r}{\overline{y}_*^{r+1}}\left((r+1) - \overline{\lambda}_0^k - \lambda_0^k\right) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&= \frac{p^* r\left((r+1) - \overline{\lambda}_0^k - \lambda_0^k\right)}{\kappa(1)\,\overline{y}_*^{r+1}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.
\end{aligned} \tag{29}$$

Note that $\mathbf{k}_{02} = \overline{\mathbf{k}}_{20}$.

If we set $\xi_n = e^{i\theta_n}$, then the normal form (27) can be written as the polar form

$$\begin{cases} \rho_{n+1} = |\lambda(\nu)|\,\rho_n + a(\nu)\,\rho_n^3 + O\left(\rho_n^4\right), \\ \theta_{n+1} = \theta_n + \arg\lambda(\nu) + b(\nu)\,\rho_n^2 + O\left(\rho_n^3\right), \end{cases} \tag{30}$$

where $a(\nu) = |\lambda(\nu)|\,\mathfrak{Re}\left(\frac{c(\nu)}{\lambda(\nu)}\right)$ and $b(\nu) = \mathfrak{Im}\left(\frac{c(\nu)}{\lambda(\nu)}\right)$. The Taylor expansion of the coefficients of the first equation in (30) about $\nu = 0$ gives

$$\rho_{n+1} = (1 + d_*\nu)\,\rho_n + a(0)\,\rho_n^3 + O\left(\rho_n^4\right),$$

where $d_* = \left.\frac{d|\lambda(p)|}{dp}\right|_{p=p^*}$ (and note that $p = p^*$ when $\nu = 0$).

If we set

$$H_0\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix} + \frac{1}{2}\mathbf{k}_{20} z^2 + \mathbf{k}_{11} z\overline{z} + \frac{1}{2}\mathbf{k}_{02}\overline{z}^2\right) = \sum_{2 \le l+m \le 3} \frac{1}{l!m!}\zeta_{lm} z^l \overline{z}^m + O\left(|z|^4\right),$$

then using (28) and (29), we get

$$\zeta_{21} = \left.\frac{\partial^3}{\partial z^2 \partial \overline{z}} H_0\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix} + \frac{1}{2}\mathbf{k}_{20} z^2 + \mathbf{k}_{11} z\overline{z}\right)\right|_{z=0} = \begin{pmatrix} \frac{\partial^3}{\partial z^2 \partial \overline{z}}\zeta_{21}^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where

$$\begin{aligned}
\zeta_{21}^1 &= 1 - \overline{y}_* + p^* \frac{A(z,\overline{z})}{B^r(z,\overline{z})} + \frac{p^*}{\overline{y}_*^r}\left(r\left(B(z,\overline{z}) - \overline{y}_*\right) - \left(A(z,\overline{z}) - \overline{y}_*\right)\right), \\
A(z,\overline{z}) &= z + \overline{z} + \frac{1}{2} z^2 \lambda_0^{2k} K + z\overline{z} L + \overline{y}_*, \\
B(z,\overline{z}) &= z\overline{\lambda}_0^k + \overline{z}\lambda_0^k + \frac{1}{2} z^2 K + z\overline{z} L + \overline{y}_*,
\end{aligned}$$

and

$$K = \frac{p^* r\left((r+1)\,\overline{\lambda}_0^{2k} - 2\overline{\lambda}_0^{k}\right)}{\kappa\left(\lambda_0^2\right)\overline{y}_*^{r+1}},\ L = \frac{p^* r\left((r+1) - \overline{\lambda}_0^{k} - \lambda_0^{k}\right)}{\kappa\left(1\right)\overline{y}_*^{r+1}}.$$

Since $\frac{\partial^3}{\partial z^2\partial\overline{z}}A\left(z,\overline{z}\right) = \frac{\partial^3}{\partial z^2\partial\overline{z}}B\left(z,\overline{z}\right) = 0$, we have

$$\frac{\partial^3}{\partial z^2\partial\overline{z}}\zeta_{21}^1 = p^*\frac{\partial^3}{\partial z^2\partial\overline{z}}\left(\frac{A\left(z,\overline{z}\right)}{B^r\left(z,\overline{z}\right)}\right).$$

This implies that

$$\frac{\partial^3}{\partial z^2\partial\overline{z}}\zeta_{21}^1 = p^*\left(U+V+W\right),$$

where

$$U = \frac{p^* r^2\left(2\lambda_0^{2k} - (r+1)\,\lambda_0^{k} - (r+1)\,\overline{\lambda}_0^{2k} + \left(r^2+2r+3\right)\overline{\lambda}_0^{k} - 2\,(r+1)\right)}{\kappa\left(\lambda_0^2\right)\overline{y}_*^{2(r+1)}},$$

$$V = \frac{2p^* r^2\left(-r\overline{\lambda}_0^{2k} + \left(r^2+r+1\right)\overline{\lambda}_0^{k} + \lambda_0^{k} - 2r - 1\right)}{\kappa\left(1\right)\overline{y}_*^{2(r+1)}},$$

$$W = \frac{r\,(r+1)\left(2 + \overline{\lambda}_0^{2k} - (r+2)\,\overline{\lambda}_0^{k}\right)}{\overline{y}_*^{r+2}},$$

and

$$\zeta_{21} = \begin{pmatrix} p^*\left(U+V+W\right) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Finaly, based on Theorem 2.1 in [19], we obtain

$$c\left(0\right) = \frac{1}{2}\mathbf{p}\zeta_{21} = \frac{p^*}{2\left(1 + k - \frac{kp^*}{\overline{y}_*^r}\overline{\lambda}_0\right)}\left(U+V+W\right)$$

and (by Corollary 2.1 in [19])

$$a\left(0\right) = \mathfrak{Re}\left\{c\left(0\right)\overline{\lambda}_0\right\} = \mathfrak{Re}\left\{\frac{p^*\overline{\lambda}_0}{2\left(1 + k - \frac{kp^*}{\overline{y}_*^r}\overline{\lambda}_0\right)}\left(U+V+W\right)\right\}.$$

Remark 6. Calculating the first Lyapunov coefficient $a(0)$, i.e., determining its sign, is very complex and takes a lot of time. We performed many calculations using the Dynamica 4 software for different pairs (p,r) along the NS bifurcation curves for $k = 2,3,5,7$, which were constructed parametrically using (18) and (20). In all cases, we observed a supercritical NS bifurcation, and in some examples in Section 6, we performed a more detailed analysis with graphical visualizations. Unfortunately, we did not encounter a single case of subcritical bifurcation. However, because of this, we cannot conclude that we will always have a supercritical bifurcation, although in Theorem 4 we will show that this occurs when $k = 2$.
Therefore, based on Corollary 2.1 in [19] (see also [7]), [18], [24]), and in view of the fact that $d_* > 0$, we can state the following:

Assume that $k \in \{2,3,\ldots\}$, $r > 0$, $\nu > 0$, and $p = p^* + \nu$. If $a\left(0\right) \neq 0$, then Equation (1) undergoes a Neimark–Sacker bifurcation at the equilibrium point $\overline{y}$ with parameter p which varies in a small neighborhood of p^*. More specifically, if $a\left(0\right) < 0$, then there exists a neighborhood $\mathcal{V}$ of equilibrium point $\overline{y}$ such

*that for $|p-p^*|<\nu$ and initial conditions $y_0, y_{-1}, ..., y_{-k} \in \mathcal{V}$ the ω-limit set of solution of Equation (1) is the equilibrium point $\overline{y}$ if $p<p^*$ and belongs to a closed and attracting curve $\Gamma(p)$ if $p>p^*$ (**supercritical** N-S bifurcation). If $a(0)>0$, then there exists a neighborhood $\mathcal{V}$ of equilibrium point $\overline{y}$ such that for $|p-p^*|<\nu$ and initial conditions $y_0, y_{-1}, ..., y_{-k} \in \mathcal{V}$ the α-limit set of solution of Equation (1) is the equilibrium point $\overline{y}$ if $p<p^*$ and belongs to a closed and unstable curve $\Gamma(p)$ if $p>p^*$ (**subcritical** N-S bifurcation).*

Theorem 3. *The invariant curve $\Gamma(p)$ mentioned in Remark 6 can be approximated by*

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} \overline{y} \\ \overline{y} \end{pmatrix} + 2\rho_0 \mathfrak{Re}\left\{ \begin{pmatrix} 1 \\ \overline{\lambda}_0^k \end{pmatrix} e^{i\theta} \right\} + \rho_0^2 \left\{ \mathfrak{Re}\left[\frac{p^* r\left((r+1)\overline{\lambda}_0^{2k} - 2\overline{\lambda}_0^k\right)}{\kappa\left(\lambda_0^2\right)\overline{y}^{r+1}} \begin{pmatrix} \lambda_0^{2k} \\ 1 \end{pmatrix} e^{2i\theta} \right] + \frac{p^* r\left((r+1) - \overline{\lambda}_0^k - \lambda_0^k\right)}{\kappa(1)\,\overline{y}^{r+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \tag{31}$$

where

$$\rho_0 = \sqrt{-\frac{d_*}{a(0)}\nu} \quad \text{and} \quad \theta \in [0, 2\pi].$$

Proof. By the proof of Corollary 2.1 in [19], we see that invariant curve $\Gamma(p)$ can be approximated by

$$2\rho_0 \mathfrak{Re}\left\{\mathbf{q} e^{i\theta}\right\} + \rho_0^2 \left\{\mathfrak{Re}\left[\mathbf{k}_{20} e^{2i\theta}\right] + \mathbf{k}_{11}\right\}. \tag{32}$$

Setting $y_n = u - \overline{y}$, $y_{n-k} = v - \overline{y}$, and substituting (28) and (29) in (32), we obtain (31).

□

It is indeed difficult to determine which parameter values make the coefficient $a(0)$ positive or negative. Fortunately, as the following theorem shows, it is still possible to do so for $k=2$, while for $k>2$ it becomes very difficult.

Theorem 4. *Assume that $k=2$, $r>\frac{\sqrt{5}-1}{2}$, $p^* = \frac{1}{\sqrt{r(r+1)}}\left(\frac{\sqrt{r(r+1)}}{\sqrt{r(r+1)}-1}\right)^r$, $\nu>0$, and $p=p^*+\nu$. Then $a(0)<0$, and Equation (1) undergoes a Neimark–Sacker bifurcation at the equilibrium point $\overline{y}_* = \frac{\sqrt{r(r+1)}}{\sqrt{r(r+1)}-1}$ with parameter p which varies in a small neighborhood of p^*. More specifically, then there exists a neighborhood $\mathcal{V}$ of equilibrium point $\overline{y}_*$ such that for $|p-p^*|<\nu$ and initial conditions $y_0, y_{-1}, ..., y_{-k} \in \mathcal{V}$ the ω-limit set of solution of Equation (1) is the equilibrium point $\overline{y}$ if $p<p^*$ and belongs to a closed and attracting curve $\Gamma(p)$ if $p>p^*$ (**supercritical** N-S bifurcation).*

Proof. For $k=2$, from (18) it follows that $\cos\varphi^*>0$ and $\sin 3\varphi^*>0$, i.e, $\varphi^* \in \left(0, \frac{\pi}{3}\right)$ and

$$\sin\varphi^* = \frac{1}{2}\sqrt{\frac{3r-1}{r}}, \quad \cos\varphi^* = \frac{1}{2}\sqrt{\frac{r+1}{r}}, \quad r>\frac{1}{3}.$$

Introducing the notations

$$a = \frac{1}{\sqrt{r(r+1)}}, \quad b = \sqrt{(3r-1)(r+1)},$$

and using (17), we obtain

$$r\frac{p^*}{\overline{y}^r} = \frac{1}{2\cos\varphi^*} = \frac{1}{\sqrt{\frac{r+1}{r}}} = ra, \quad \frac{p^*}{\overline{y}^r} = a.$$

Also,

$$\lambda_0 = \frac{1}{2}\left(\sqrt{\frac{r+1}{r}} + i\sqrt{\frac{3r-1}{r}}\right) = \frac{a}{2}(r+1+bi),$$

and

$$\begin{aligned}
\lambda_0^2 &= \frac{a^2}{4}(r+1+bi)^2 = \frac{-r+1+ib}{2r},\\
\overline{\lambda}_0^2 &= \frac{-r+1-ib}{2r},\\
\lambda_0^4 &= \frac{-r^2-2r+1+ib(-r+1)}{2r^2},\\
\overline{\lambda}_0^4 &= \frac{-r^2-2r+1-ib(-r+1)}{2r^2}.
\end{aligned}$$

The characteristic polynomial in this case has the form

$$\kappa(\lambda_0) = \lambda_0^3 - a\lambda_0^2 + ar,$$

so

$$\kappa(1) = 1 - a + ar$$

and

$$\kappa\left(\lambda_0^2\right) = \frac{ar\left(2r^3+r^2+2r-1\right)+(r-1)\left(2r^2+2r-1\right)}{2r^3} + \frac{(ar(r-1)+1-2r)b}{2r^3}i,$$

from which it follows that

$$\frac{1}{\kappa\left(\lambda_0^2\right)} = \frac{(r+1)}{(r-1)(r+1)\left(2r^2+3r-1\right)a+2r^3+2r^2+3r-1}\left(\frac{ar\left(2r^3+r^2+2r-1\right)+(r-1)\left(2r^2+2r-1\right)}{2r} - \frac{(ar(r-1)+1-2r)b}{2r}i\right).$$

Since

$$\frac{p^*r^2}{\overline{y}_*^{2(r+1)}} = ar^2(1-a)^{r+2},$$

we have

$$U = \frac{-a(1-a)^{r+2}(r+1)}{2r\left(\left(r^2-1\right)\left(3r+2r^2-1\right)a+2r^3+2r^2+3r-1\right)}(U_1+iU_2),$$

where

$$\begin{aligned}
U_1 &= (r+1)\left(ar\left(2r+r^2-1\right)\left(r^4+2r^3+5r^2-5r+1\right)+r^6-r^5-9r^4-12r^3+20r^2-8r+1\right),\\
U_2 &= b\left(ar\left(r^6+4r^5+8r^4+3r^3+4r^2-5r+1\right)+r^6+5r^5+7r^4-6r^3-8r^2+6r-1\right).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
V &= 2\frac{p^*r^2}{\overline{y}_*^{2(r+1)}}\frac{\left(-r\overline{\lambda}_0^4+\left(r^2+r+1\right)\overline{\lambda}_0^2+\lambda_0^2-2r-1\right)}{\kappa(1)}\\
&= \frac{2ar^2(1-a)^{r+2}}{(1-a+ar)}\left(-\frac{1}{2r}(ib+r+1)\left(r^2+2r-1\right)\right)\\
&= -\frac{ar\left(r^2+2r-1\right)(1-a)^{r+2}}{(1-a+ar)}(r+1+ib)
\end{aligned}$$

and

$$\begin{aligned}
W &= \frac{r(r+1)}{\overline{y}_*^{r+2}}\left(2+\overline{\lambda}_0^4-(r+2)\overline{\lambda}_0^2\right)\\
&= \frac{(r+1)(1-a)^{r+2}}{2r}\left(r^3+4r^2-4r+1+ib\left(r^2+3r-1\right)\right).
\end{aligned}$$

Since

$$p^* = \frac{a}{(1-a)^r} \quad \text{and} \quad \overline{y}_* = \frac{1}{1-a}, \tag{33}$$

we obtain

$$\begin{aligned}\frac{p^*\overline{\lambda}_0}{2\left(3-2\frac{p^*}{\overline{y}_*^r}\overline{\lambda}_0\right)} &= \frac{a}{2\left(1-a\right)^r}\frac{\frac{1}{2}a\left((r+1)-ib\right)}{\left(3-2a\left(\frac{1}{2}a\left((r+1)-ib\right)\right)\right)}\\ &= \frac{1}{4\left(3r+2\right)\left(3r-1\right)\left(1-a\right)^r}\left(3r-1-3bi\right).\end{aligned}$$

Finally,

$$a\left(0\right) = \mathfrak{Re}\left\{\frac{p^*\overline{\lambda}_0}{2\left(3-2\frac{p^*}{\overline{y}_*^r}\overline{\lambda}_0\right)}\left(U+V+W\right)\right\} = -\frac{\left(1-a\right)^2\left(r+1\right)}{4\left(3r+2\right)r\left(a\left(r-1\right)+1\right)}\cdot\mathcal{K},$$

where

$$\mathcal{K} = \frac{r\left(ar\left(4r^5+r^4+17r^3+24r^2+13r-7\right)+4r^5-r^4-9r^3-36r^2+9r+1\right)}{a\left(r-1\right)\left(r+1\right)\left(2r^2+3r-1\right)+2r^3+2r^2+3r-1},$$

i.e.,

$$a\left(0\right) = -\frac{\sqrt{r(r+1)}\left(\sqrt{r(r+1)}-1\right)^2\left(r\left(4r^5+r^4+17r^3+24r^2+13r-7\right)+\left(4r^5-r^4-9r^3-36r^2+9r+1\right)\sqrt{r(r+1)}\right)}{4r\left(3r+2\right)\left(r-1+\sqrt{r(r+1)}\right)\left(\left(r-1\right)\left(r+1\right)\left(2r^2+3r-1\right)+\left(2r^3+2r^2+3r-1\right)\sqrt{r(r+1)}\right)}.$$

For a unique equilibrium to exist, it follows from (33) that $a < 1$, i.e., $r > \frac{\sqrt{5}-1}{2}$ $\left(> \frac{1}{3}\right)$.

In addition, since $r^2+r-1 > 0$, it holds

(i) $r-1+\sqrt{r\left(r+1\right)} > 0$ (obviously),

(ii) $2r^2+3r-1 = \left(r^2+r-1\right)+r^2+2r > 0$, which implies

$$\left(r-1\right)\left(r+1\right)\left(2r^2+3r-1\right)+\left(2r^3+2r^2+3r-1\right)\sqrt{r\left(r+1\right)} > 0.$$

Since $13r-7 > 0$ for $r > \frac{\sqrt{5}-1}{2}$, we have $r\left(4r^5+r^4+17r^3+24r^2+13r-7\right) > 0$, so

$$r\left(4r^5+r^4+17r^3+24r^2+13r-7\right)+\left(4r^5-r^4-9r^3-36r^2+9r+1\right)\sqrt{r\left(r+1\right)} > 0,$$

if $4r^5-r^4-9r^3-36r^2+9r+1 \geq 0$.

If not, that is, if $4r^5-r^4-9r^3-36r^2+9r+1 < 0$, then

$$r\left(4r^5+r^4+17r^3+24r^2+13r-7\right)+\left(4r^5-r^4-9r^3-36r^2+9r+1\right)\sqrt{r\left(r+1\right)} > 0$$

if

$$r^2\left(4r^5+r^4+17r^3+24r^2+13r-7\right)^2 > \left(4r^5-r^4-9r^3-36r^2+9r+1\right)^2 r\left(r+1\right),$$

that is, if

$$r\left(r^2+r-1\right)\left(24r^5+55r^4+73r^3+34r^2-23r+1\right)\left(3r-1\right)^2 > 0,$$

which is satisfied for $r > \frac{\sqrt{5}-1}{2}$.

Hence, we have $a\left(0\right) < 0$ for $r > \frac{\sqrt{5}-1}{2}$.

□

4. Oscillations of solutions

The following results about the oscillations of the solutions of Equation (1) are expected; it is simple to prove, but it has some importance in understanding the behavior of the solutions of Equation (1).

Lemma 7. *Let $\{y_n\}$ be a non-equilibrium solution of Equation (1). Then, every solution of Equation (1) has semi-cycles with at most $2k+1$ terms.*

Proof. Now, we will prove the lemma for the negative semi-cycle. The proof for a positive semi-cycle is similar and hence omitted. Suppose y_{n_0} is the first term in a negative semi-cycle with $2k+1$ terms. Then,

$$y_{n_0}, y_{n_0+1}, \cdots, y_{n_0+2k} < \bar{y}.$$

Using that and the facts $\bar{y} > max\{1, p^{\frac{1}{r}}\}$ and $\frac{p}{\bar{y}^r} = 1 - \frac{1}{\bar{y}}$, we obtain

$$\begin{aligned} y_{n_0+k+i} &= 1 + p\frac{y_{n_0+k+i-1}}{y^r_{n_0+i-1}} > 1 + p\frac{y_{n_0+k+i-1}}{\bar{y}^r} \\ &= 1 + \left(1 - \frac{1}{\bar{y}}\right) y_{n_0+k+i-1} = 1 - \frac{y_{n_0+k+i-1}}{\bar{y}} + y_{n_0+k+i-1} \\ &> y_{n_0+k+i-1}, \end{aligned}$$

for $i = 1, 2, \cdots, k$. Therefore

$$y_{n_0+k} < y_{n_0+k+1} < \cdots < y_{n_0+2k-1} < y_{n_0+2k}.$$

This then implies the following:

$$y_{n_0+2k+1} = 1 + p\frac{y_{n_0+2k}}{y^r_{n_0+k}} > \bar{y} = 1 + \frac{p}{y^{r-1}_{n_0+k}}\frac{y_{n_0+2k}}{y_{n_0+k}} > 1 + \frac{p}{y^{r-1}_{n_0+k}} > 1 + \frac{p}{\bar{y}^{r-1}} = \bar{y},$$

which means that a semi-cycle can have at most $2k+1$ terms. □

Remark 7. The proof of the previous lemma implies that the extreme term in a semi-cycle occurs in the first $k+1$ terms. Also, every solution oscillates around the equilibrium point $\bar{y}$.

Lemma 8. *If every solution of Equation (1) has a semi-cycle of length at least k, then the following semi-cycle has at least $k+1$ terms.*

Proof. Suppose that Equation (1) has a semi-cycle of length at least k. Then, we get a negative semi-cycle with k terms. The positive semi-cycle is similar and omitted. We also assume that y_{n_0} is the first following positive semi-cycle. Thus, we have

$$y_{n_0-k}, y_{n_0-k+1}, \cdots, y_{n_0-1} < \bar{y} < y_{n_0}.$$

Hence, from Equation (1), for all $i = 1, 2, ..., k$, and by using $\bar{y} = 1 + p\frac{\bar{y}}{\bar{y}^r}$, we obtain

$$y_{n_0+i} = 1 + p\frac{y_{n_0+i-1}}{y^r_{n_0+i-1-k}} > 1 + p\frac{\bar{y}}{\bar{y}^r} = \bar{y},$$

where $i = 1, 2, ..., k$. Therefore, this positive semi-cycle has at least $k+1$ terms. Also, every semi-cycle after that must have at least $k+1$ terms. □

5. Rate of convergence

The result on the rate of convergence of the solutions of Equation (1) is significant in the case of global stability (also, see [16]). Specific problems with this investigation arise from the nature of the exponent r, which can be an arbitrary positive number (not necessarily an integer).

Now, we overcome the rate of convergence of Equation (1).

Theorem 5. *Under assumptions of Theorem 1 all solutions of Equation (1) which are eventually different from the equilibrium $\bar{y}$ satisfy the following:*

$$\lim_{n\to\infty}\left|\frac{y_{n+1}-\bar{y}}{y_n-\bar{y}}\right| = \left|\lambda_j\right|,$$

or

$$\lim_{n\to\infty}\sup\left(|y_n-\bar{y}|\right)^{1/n} = \left|\lambda_j\right|,$$

where $j\in\{1,\cdots,k\}$ and λ_j are the roots of the characteristic equation

$$\lambda^{k+1} - p\frac{1}{\bar{y}^r}\lambda^k + \frac{rp}{\bar{y}^r} = 0$$

of Equation (1).

Proof. By using Equation (1), we obtain

$$y_{n+1}-\bar{y} = \frac{py_n}{y_{n-k}^r} - \frac{p\overline{y}}{\overline{y}^r} = \frac{p}{y_{n-k}^r}(y_n-\bar{y}) + p\overline{y}\left(\frac{1}{y_{n-k}^r}-\frac{1}{\overline{y}^r}\right).$$

Set $p_n = -\frac{p}{y_{n-k}^r}$.

We consider the following cases in the form of r.

(i) r is a positive integer

Then

$$\frac{1}{y_{n-k}^r}-\frac{1}{\overline{y}^r} = \frac{\overline{y}^r - y_{n-k}^r}{y_{n-k}^r\overline{y}^r} = -\frac{\left(\overline{y}^{r-1}+\overline{y}^{r-2}y_{n-k}+\ldots+y_{n-k}^{r-1}\right)}{y_{n-k}^r\overline{y}^r}(y_{n-k}-\overline{y})$$

and set

$$q_n = p\overline{y}\frac{\left(\overline{y}^{r-1}+\overline{y}^{r-2}y_{n-k}+\ldots+y_{n-k}^{r-1}\right)}{y_{n-k}^r\overline{y}^r}.$$

(ii) $r=\frac{s}{t}$, s and t are positive integers such that $\gcd(s,t)=1$, i.e., r is a positive rational number

Then

$$\begin{aligned}\frac{1}{y_{n-k}^r}-\frac{1}{\overline{y}^r} &= -\frac{\left(y_{n-k}^{\frac{1}{t}}\right)^s-\left(\overline{y}^{\frac{1}{t}}\right)^s}{(y_{n-k}\overline{y})^{\frac{s}{t}}} = -\frac{\left(y_{n-k}^{\frac{1}{t}}-\overline{y}^{\frac{1}{t}}\right)\left(y_{n-k}^{\frac{s-1}{t}}+y_{n-k}^{\frac{s-2}{t}}\overline{y}^{\frac{1}{t}}+\ldots+\overline{y}^{\frac{s-1}{t}}\right)}{(y_{n-k}\overline{y})^{\frac{s}{t}}}\\ &= -\frac{\left(y_{n-k}^{\frac{1}{t}}-\overline{y}^{\frac{1}{t}}\right)\left(y_{n-k}^{\frac{s-1}{t}}+y_{n-k}^{\frac{s-2}{t}}\overline{y}^{\frac{1}{t}}+\ldots+\overline{y}^{\frac{s-1}{t}}\right)}{(y_{n-k}\overline{y})^{\frac{s}{t}}}\cdot\frac{y_{n-k}^{\frac{t-1}{t}}+y_{n-k}^{\frac{t-2}{t}}\overline{y}^{\frac{1}{t}}+\ldots+\overline{y}^{\frac{t-1}{t}}}{y_{n-k}^{\frac{t-1}{t}}+y_{n-k}^{\frac{t-2}{t}}\overline{y}^{\frac{1}{t}}+\ldots+\overline{y}^{\frac{t-1}{t}}}\\ &= -\frac{\left(y_{n-k}^{\frac{s-1}{t}}+y_{n-k}^{\frac{s-2}{t}}\overline{y}^{\frac{1}{t}}+\ldots+\overline{y}^{\frac{s-1}{t}}\right)}{(y_{n-k}\overline{y})^{\frac{s}{t}}\left(y_{n-k}^{\frac{t-1}{t}}+y_{n-k}^{\frac{t-2}{t}}\overline{y}^{\frac{1}{t}}+\ldots+\overline{y}^{\frac{t-1}{t}}\right)}(y_{n-k}-\overline{y})\end{aligned}$$

and set

$$q_n = p\overline{y}\frac{\left(y_{n-k}^{\frac{s-1}{t}} + y_{n-k}^{\frac{s-2}{t}}\overline{y}^{\frac{1}{t}} + \ldots + \overline{y}^{\frac{s-1}{t}}\right)}{(y_{n-k}\overline{y})^{\frac{s}{t}}\left(y_{n-k}^{\frac{t-1}{t}} + y_{n-k}^{\frac{t-2}{t}}\overline{y}^{\frac{1}{t}} + \ldots + \overline{y}^{\frac{t-1}{t}}\right)}.$$

(iii) r is a positive irrational number

Then there exists a sequence of rational numbers $\{v_m\}_{m=0}^{\infty}$, $v_m = \frac{s_m}{t_m}$, $\gcd(s_m, t_m) = 1$, such that $r = \lim_{m\to\infty} v_m$. We have that

$$\begin{aligned}\frac{1}{y_{n-k}^r} - \frac{1}{\overline{y}^r} &= \frac{\overline{y}^r - y_{n-k}^r}{y_{n-k}^r\overline{y}^r} = \frac{\overline{y}^{\lim_{m\to\infty} v_m} - y_{n-k}^{\lim_{m\to\infty} v_m}}{(y_{n-k}\overline{y})^{\lim_{m\to\infty} v_m}} = \frac{\lim_{m\to\infty}\overline{y}^{v_m} - \lim_{m\to\infty} y_{n-k}^{v_m}}{\lim_{m\to\infty}(y_{n-k}\overline{y})^{v_m}} \\ &= \lim_{m\to\infty}\frac{\overline{y}^{v_m} - y_{n-k}^{v_m}}{(y_{n-k}\overline{y})^{v_m}} = -\lim_{m\to\infty}\frac{\overline{y}^{\frac{s_m}{t_m}} - y_{n-k}^{\frac{s_m}{t_m}}}{(y_{n-k}\overline{y})^{\frac{s_m}{t_m}}} \\ &= -\lim_{m\to\infty}\frac{\left(y_{n-k}^{\frac{s_m-1}{t_m}} + y_{n-k}^{\frac{s_m-2}{t_m}}\overline{y}^{\frac{1}{t_m}} + \ldots + \overline{y}^{\frac{s_m-1}{t_m}}\right)}{(y_{n-k}\overline{y})^{\frac{s_m}{t_m}}\left(y_{n-k}^{\frac{t_m-1}{t_m}} + y_{n-k}^{\frac{t_m-2}{t_m}}\overline{y}^{\frac{1}{t_m}} + \ldots + \overline{y}^{\frac{t_m-1}{t_m}}\right)}(y_{n-k} - \overline{y}),\end{aligned}$$

and set

$$q_n = p\overline{y}\lim_{m\to\infty}\frac{\left(y_{n-k}^{\frac{s_m-1}{t_m}} + y_{n-k}^{\frac{s_m-2}{t_m}}\overline{y}^{\frac{1}{t_m}} + \ldots + \overline{y}^{\frac{s_m-1}{t_m}}\right)}{(y_{n-k}\overline{y})^{\frac{s_m}{t_m}}\left(y_{n-k}^{\frac{t_m-1}{t_m}} + y_{n-k}^{\frac{t_m-2}{t_m}}\overline{y}^{\frac{1}{t_m}} + \ldots + \overline{y}^{\frac{t_m-1}{t_m}}\right)}.$$

If we set $e_n = y_n - \bar{y}$, then we get that

$$e_{n+1} + p_n e_n + q_n e_{n-k} = 0. \tag{34}$$

Since the equilibrium point $\bar{y}$ is globally asymptotically stable, and since

$$\begin{aligned}\lim_{m\to\infty}\left(y_{n-k}^{\frac{s-1}{t}} + y_{n-k}^{\frac{s-2}{t}}\overline{y}^{\frac{1}{t}} + \ldots + \overline{y}^{\frac{s-1}{t}}\right) &= s\bar{y}^{\frac{s-1}{t}}, \\ \lim_{m\to\infty}\left(y_{n-k}^{\frac{t-1}{t}} + y_{n-k}^{\frac{t-2}{t}}\overline{y}^{\frac{1}{t}} + \ldots + \overline{y}^{\frac{t-1}{t}}\right) &= t\bar{y}^{\frac{t-1}{t}},\end{aligned}$$

we obtain that

$$\lim_{n\to\infty} p_n = -\frac{p}{\bar{y}^r}, \quad \lim_{n\to\infty} q_n = \frac{rp}{\bar{y}^r}.$$

Thus, the limiting equation of Equation (34) is the linearized equation of Equation (1) at $\bar{y}$ whose characteristic equation is (7). Now, the conclusion follows as an immediate consequence of Poincaré's Theorem and Perron's Theorem [20]. □

6. Numerical examples

Now, we introduce three numerical examples to verify our theoretical results. Bifurcation diagrams and phase portraits in all figures are generated by Dynamica 4 [11]. Also, based on the computational algorithm LCEs for discrete systems in [21], we provided graphs of Lyapunov maximal coefficients.

Example 1. Consider Equation (1) for $r = \frac{2}{3}$ and $k = 2$. Then, we obtain the following third-order difference equation:

$$y_{n+1} = 1 + p\frac{y_n}{y_{n-2}^{\frac{2}{3}}}, \tag{35}$$

where $y_{-2}, y_{-1}, y_0, p \in (0, \infty)$.

By solving Equation (18), we get that $\varphi^* = 2\ arctan\left(\frac{2\sqrt{6}-\sqrt{15}}{3}\right) \approx 0.65906$, and from (20), we get that $p^* \approx 6.8699$ is the Neimark–Sacker bifurcation value of the parameter p of Equation (1). Then, the equilibrium point, when $p = p^*$, is $\overline{y}^* \approx 19.487$ (p^* and $\overline{y}^*$ can be calculated by directly substituting $r = \frac{2}{3}$ into the formulas in Theorem 4). The corresponding characteristic equation of linearized equation of (35) about the equilibrium point is

$$\lambda^3 - \frac{6.8699}{19.487^{\frac{2}{3}}}\lambda^2 + \frac{2 \cdot 6.899}{3 \cdot 19.487^{\frac{2}{3}}} = 0. \tag{36}$$

Three roots of Equation (36) are

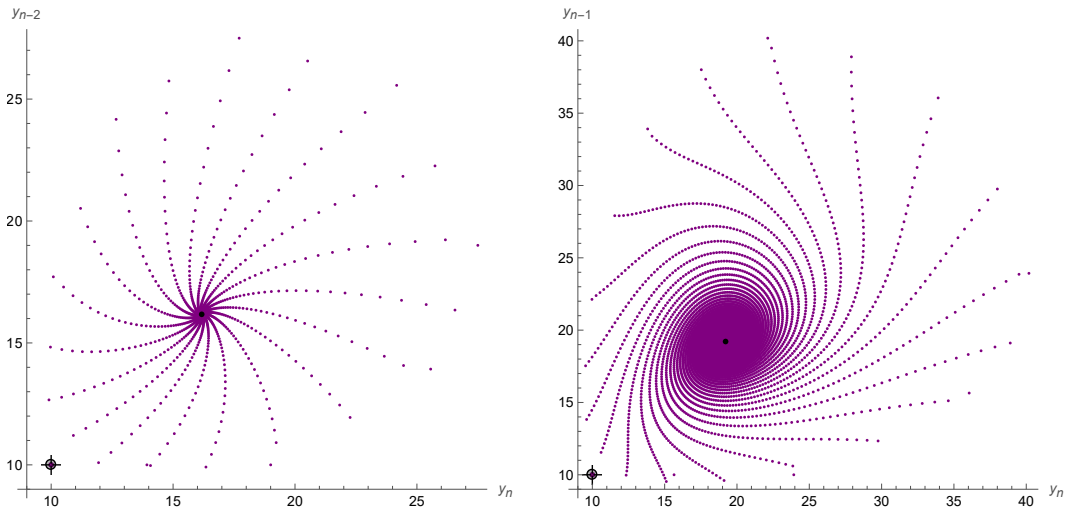
$$\lambda_1 = -0.63246, \quad \lambda_2 = 0.79057 - 0.61237i, \quad \lambda_3 = 0.79057 + 0.61237i,$$

where $|\lambda_{2,3}| = 1$. Figure 1, (a) for $p = 6 < p^* = 6.8699$ and (b) for $p = 6.8 < p^* = 6.8699$, in 2D projection onto the (y_n, y_{n-2}) plane shows that the positive equilibrium point $\bar{y}_*$ is asymptotically stable. After that, the positive equilibrium point loses its stability at the Neimark–Sacker bifurcation value $p = p^*$ and an invariant simple closed curve appears onto the (y_n, y_{n-2}) plane for $p = 6.9 > p^*$ (Figure 2) and for $p = 7 > p^*$ (Figure 3). This invariant closed curve is stable, which means that all orbits starting outside or inside the curve, except at the equilibrium point, tend to the attracting closed curve. A bifurcation diagram in the (p, x) plane (a) and the corresponding Lyapunov exponent (b) for $k = 2$ and $r = \frac{2}{3}$ are presented in Figure 4, which supports the existence of chaos after the occurrence of the Neimark–Sacker bifurcation.

Setting $k = 2$, $r = \frac{2}{3}$, $p = 6.9$, $\overline{y} = \overline{y}\,(p) = 19.6065$ in (31), we get an approximate expression for the invariant curve of the following form:

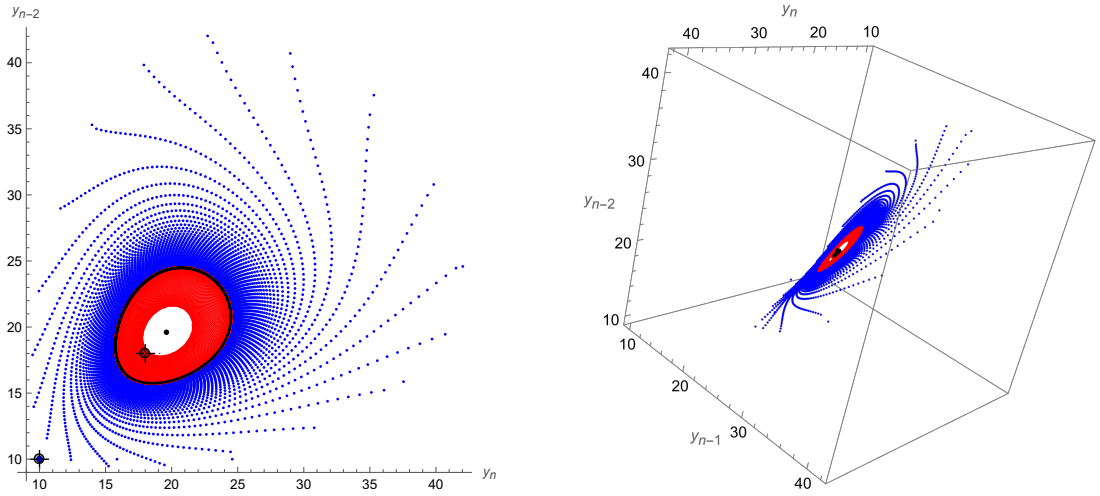
$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} 0.246\,75\cos 2\theta + 9.\,080\,9 \times 10^{-3}\sin 2\theta + 4.\,366\,4\cos\theta + 19.\,869 \\ 0.111\,51\sin 2\theta - 0.220\,3\cos 2\theta + 1.\,091\,6\cos\theta + 4.\,227\,7\sin\theta + 19.\,869 \end{pmatrix},$$

where $\theta \in [0, 2\pi]$ (Figure 2 a)). Also, $\rho_0 = 2.1832$, since $\nu = p - p^* = 0.0301$. Moreover, $d_* = 0.0045525 > 0$ and $a(0) = -0.000028749 < 0$, so the Neimark–Sacker bifurcation is supercritical.



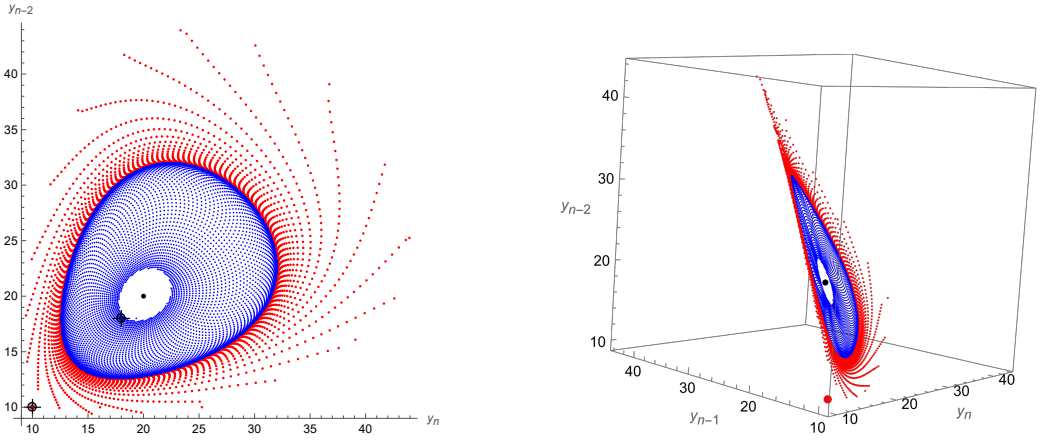
(a) (b)

Figure 1: LAS equilibrium point: 2D projection onto the (y_n, y_{n-2}) plane for $k = 2$, $r = \frac{2}{3}$, and (a) $p = 6$, with initial values $(10.0, 10.0, 10.0)$ and 5 000 iterations, (b) $p = 6.8$, with initial values $(10.0, 10.0, 10.0)$ and 15 000 iterations



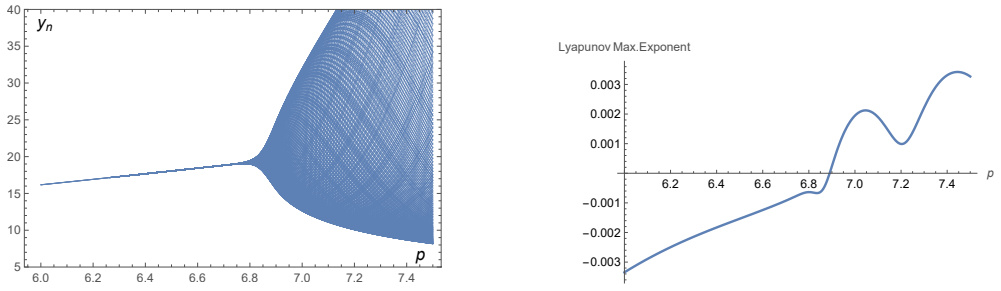
(a) (b)

Figure 2: Invariant curve - stable: (a) 2D projection onto the (y_n, y_{n-2}) plane (with the approximate curve - dark), (b) in 3D space (y_n, y_{n-1}, y_{n-2}), for $k = 2$, $r = \frac{2}{3}$ and $p = 6.9$, with initial values $(10.0, 10.0, 10.0)$ and 15 000 iterations for the blue orbit and 100 000 iterations for the red orbit



(a) (b)

Figure 3: Invariant curve - stable: (a) 2D projection onto the (y_n, y_{n-2}) plane, (b) in 3D space (y_n, y_{n-1}, y_{n-2}), for $k = 2$, $r = \frac{2}{3}$ and $p = 7$, with initial values $(10.0, 10.0, 10.0)$ and 15 000 iterations for the red orbit and 100 000 iterations for the blue orbit



(a) (b)

Figure 4: (a) Bifurcation diagram in the (p, y_n) plane and (b) the corresponding Lyapunov exponent in the (p, L_c) plane (for $k = 2$ and $r = \frac{2}{3}$)

Example 2. Consider Equation (1) for $r = \frac{3}{5}$ and $k = 3$. Then, we obtain the following fourth-order difference equation:

$$y_{n+1} = 1 + p\frac{y_n}{y_{n-3}^{\frac{3}{5}}},$$

where $y_{-3}, y_{-2}, y_{-1}, y_0, p \in (0, \infty)$.

By solving Equation (18), we get that

$$\varphi^* = \pi - 2\arctan\frac{17\sqrt{17}\sqrt[3]{\frac{32}{289}\sqrt{129} - \frac{6176}{4913}}}{\sqrt{-544\sqrt[3]{\frac{32}{289}\sqrt{129} - \frac{6176}{4913}} + 6647\sqrt[3]{-\frac{395\,264}{1419\,857}\sqrt{129} + \frac{76\,318\,720}{24\,137\,569}} + 544\sqrt{129} - 6176}},$$

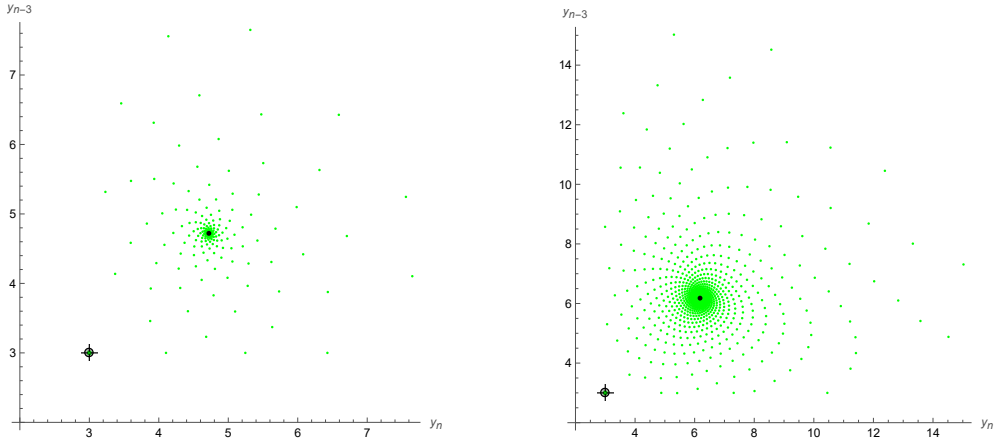
i.e., $\varphi^* \approx 0.53293$, and from (20), we get that $p^* \approx 2.6139$ is the Neimark–Sacker bifurcation value of the parameter p of Equation (1). Then, the equilibrium point, when $p = p^*$, is $\overline{y}^* \approx 6.5402$, and the corresponding characteristic equation of the linearized equation of (35) about the equilibrium point is

$$\lambda^4 - \frac{2.6139}{6.5402^{\frac{3}{5}}}\lambda^3 + \frac{3 \cdot 2.6139}{5 \cdot 6.5402^{\frac{3}{5}}} = 0. \tag{37}$$

The characteristic values as the roots of Equation (37) are

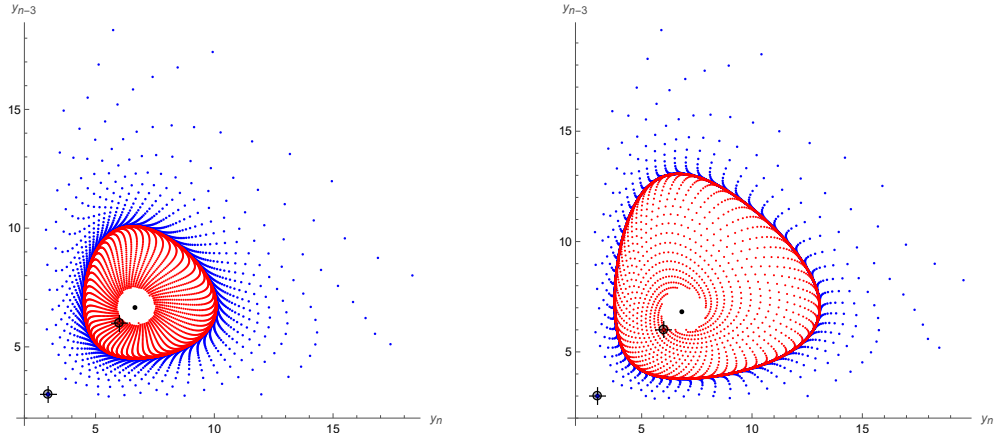
$$\lambda_1 = -0.43777 - 0.56268i, \quad \lambda_2 - 0.43777 + 0.56268i, \quad \lambda_3 = 0.86133 - 0.5006i, \quad \lambda_4 = 0.86133 + 0.5006i,$$

where $|\lambda_{1,2}| < 1$ and $|\lambda_{3,4}| = 1$.



(a) (b)

Figure 5: LAS equilibrium point for (a) $p = 2$ and (b) $p = 2.4$ (with inital values $(3.0, 3.0, 3.0, 3.0)$ and 1 000 iterations)



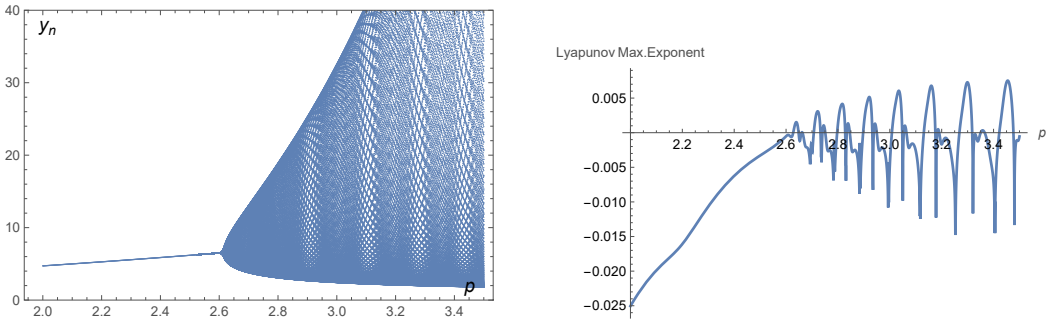
(a) (b)

Figure 6: Invariant curve - stable for (a) $p = 2.65$ and (b) $p = 2.7$ (with initial values $(3.0, 3.0, 3.0, 3.0)$ and 10 000 iterations for the blue orbit, and with initial values $(6.0, 6.0, 6.0, 6.0)$ and 10 000 iterations for the red orbit)

Figure 5 shows that the positive equilibrium point $\bar{y}_*$ is asymptotically stable for (a) $p = 2 < p^* = 2.6139$, (b) $p = 2.4 < p^* = 2.6139$, and changes its stability at a Neimark–Sacker bifurcation value $p = p^*$ when an invariant simple closed curve appears on the (y_n, y_{n-3}) plane for (a) $p = 2.65 > p^*$, (b) $p = 2.7 > p^*$ in Figure 6. This invariant closed curve is stable, which means that all orbits starting outside or inside the curve, except at the equilibrium point, tend to the attracting closed curve.

Here, $d_* = 0.032388$ and $a(0) = -0.00067264 < 0$, so the Neimark–Sacker bifurcation is supercritical.

A bifurcation diagram in the (p, x) plane (a), and the corresponding Lyapunov exponent (b) for $k = 3$ and $r = \frac{3}{5}$ are presented in Figure 7, which supports the existence of chaos after the occurrence of the Neimark–Sacker bifurcation.



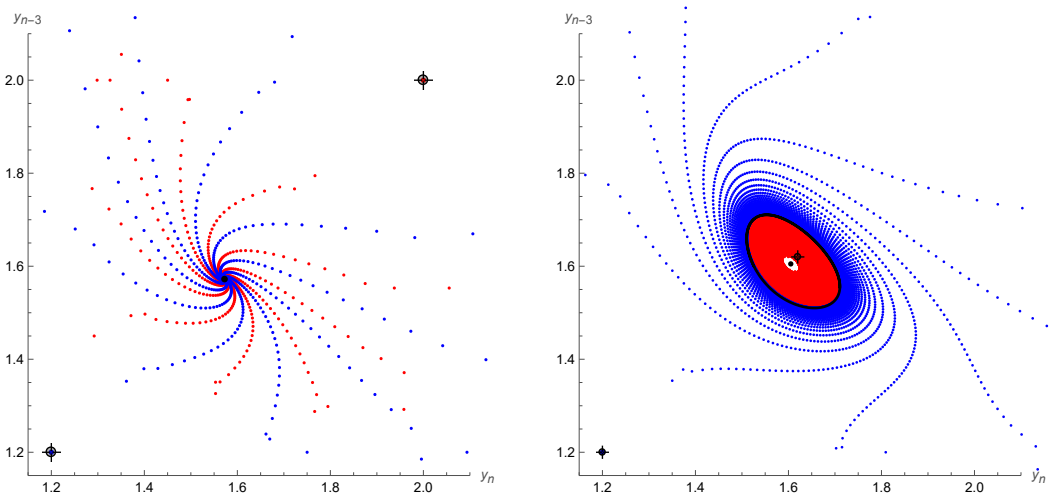
(a) (b)

Figure 7: (a) Bifurcation diagram in the (p, y_n) plane and (b) the corresponding Lyapunov exponent in the (p, L_c) plane (for $k = 3$ and $r = \frac{3}{5}$)

Example 3. Consider Equation (1) for $r = 2$ and $k = 3$. Then, we obtain the following fourth-order difference equation:

$$y_{n+1} = 1 + p\frac{y_n}{y_{n-3}^2},$$

where $y_{-3}, y_{-2}, y_{-1}, y_0, p \in (0, \infty)$.



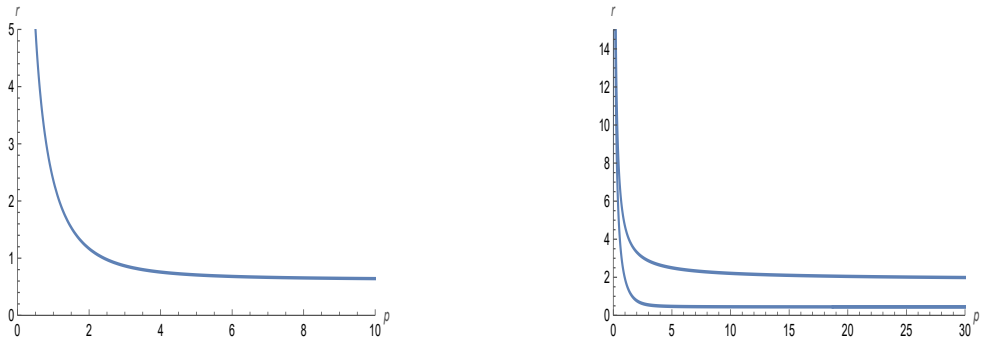
(a) (b)

Figure 8: (a) LAS equilibrium point for $p = 0.9$ 2D projection onto the (y_n, y_{n-2}) plane (with initial values $(1.2; 1.2; 1.2; 1.2)$ for the blue orbit and with initial values $(2, 2, 2, 2)$) for the red orbit) and 10 000 iterations; (b) Invariant curve - stable for $p = 0.971$ 2D projection onto the (y_n, y_{n-2}) plane (with initial values $(1.2, 1.2, 1.2, 1.2)$ for the blue orbit and with initial values $(1.62, 1.62, 1.62, 1.62)$) for the red orbit) and 10 000 iterations

In the case where the value of the parameter p is around the bifurcation value p_2^* the equilibrium point is always a repeller because all eigenvalues are greater than 1 in modulus (only for $p = p_2^*$ are two equal in modulus to 1). A closed, stable invariant curve also appears, but it covers a very large area (which is difficult to draw well).

By solving Equation (18), we get that $\varphi_1^* \approx 0.703097$ and $\varphi_2^* \approx 2.256887$. From (20), we get that $p_1^* \approx 0.968966$ and $p_2^* \approx 27.2582$ are the Neimark–Sacker bifurcation values of the parameter p of Equation (1).

Then, the equilibrium point is $\overline{y}_1^* \approx 1.604988$ when $p = p_1^*$, and $\overline{y}_2^* \approx 5.7448$ when $p = p_2^*$. Figure 8 shows that the positive equilibrium point $\overline{y}_1^*$ (a) is asymptotically stable for $p = 0.9 < p_1^* = 0.968966$, and changes its stability at a Neimark–Sacker bifurcation value $p = p_1^*$, when an invariant simple closed curve appears on the (y_n, y_{n-3}) plane for $p = 0.971 > p_1^*$. This invariant closed curve is stable, which means that all orbits starting outside or inside the curve, except at the equilibrium point, tend to the attracting closed curve.

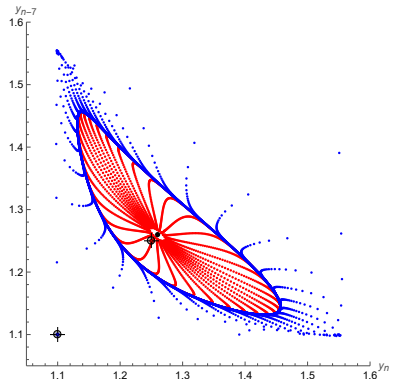


(a) $k = 2$ (b) $k = 3$

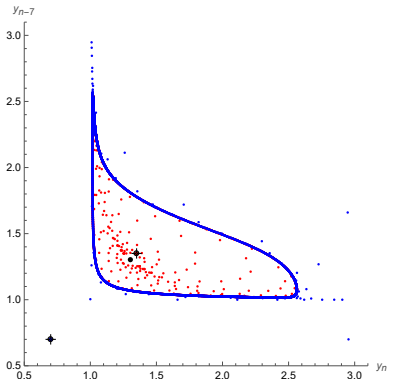
(c) $k = 5$ (d) $k = 7$

Figure 9: NS bifurcation curves for different values of k

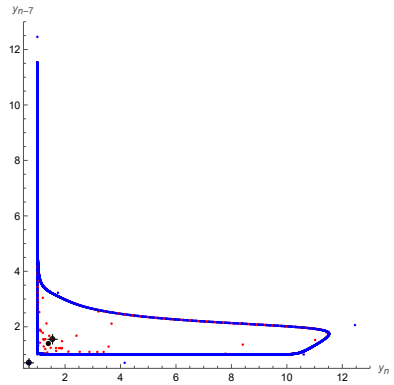
Figure 9 shows NS bifurcation curves for: (a) $k = 2$, (b) $k = 3$, (c) $k = 5$, and (d) $k = 7$, with the restriction stated in Remark 4 (b). Figure 10 presents four possible situations for the invariant curves in the case of the supercritical Neimark–Sacker bifurcation for $k = 7$ and $r = 4$, using Figure 9 (d). Our calculations show that the unstable manifold dimension is: (a) $N_{un} = 0$, (b) $N_{un} = 5$, (c) $N_{un} = 4$, and (d) $N_{un} = 6$. Moreover, case (d) shows that even when there is a supercritical Neimark–Sacker bifurcation ($a(0) = -0.28605$), if $N_{un} > 0$, we may not observe invariant curves in the 2D projection (in our case, it looks like clusters are formed along some imaginary curve).



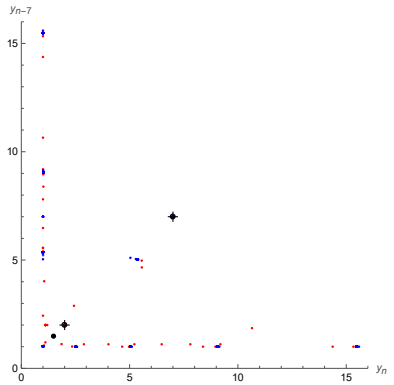
(a) $p^* \approx 0.506$, $p = 0.52$, $\bar{y}_* = 1.26$, with initial values $y_i = 1.25$ for the red orbit, and $y_i = 1.1$ for the blue orbit $(i = -7, ..., -1, 0)$ and 5 000 iterations



(b) $p^* \approx 0.669$, $p = 0.67$, $\bar{y}_* = 1.303$, with initial values $y_i = 1.35$ for the red orbit, and $y_i = 0.7$ for the blue orbit $(i = -7, ..., -1, 0)$ and 5 000 iterations



(c) $p^* \approx 1.0705$, $p = 1.08$, $\bar{y}_* = 1.3965$, with initial values $y_i = 1.55$ for the red orbit, and $y_i = 0.7$ for the blue orbit $(i = -7, ..., -1, 0)$ and 5 000 iterations



(d) $p^* \approx 1.5887$, $p = 1.6$, $\bar{y}_* = 1.468$, with initial values $y_i = 2.0$ for the red orbit, and $y_i = 7.0$ for the blue orbit $(i = -7, ..., -1, 0)$ and 1 000 iterations

Figure 10: Invariant curves for $k = 7$ and $r = 4$ as 2D projections onto the (y_n, y_{n-7}) plane

7. Conclusion

Investigating the global dynamics of a specific rational difference equation with delay, particularly one involving terms raised to arbitrary exponents, presents a very complex problem. In this paper, we studied the dynamical properties of the solutions of the difference equation (1). We proved that the solutions of this equation are eventually bounded, thereby establishing an invariant and attracting interval. Due to the complexity of the equation, it is not possible to explicitly determine the equilibrium points; however, we demonstrated the existence of a unique positive equilibrium. Using Theorem 1.3.7 from [9], we derived the conditions for the parameters p and r under which the equilibrium point is locally asymptotically stable for an arbitrary fixed value of the parameter k. Furthermore, after showing that the given equation (1) does not exhibit periodic solutions of the minimum period-two, we applied the so-called *"m-M theorem"* to obtain a condition on the parameters p and r for global asymptotic stability of the equilibrium point.

We also focused on the existence of the Neimark–Sacker bifurcation at the equilibrium point $\overline{y}_*$. It was essential to verify the non-degeneracy condition of the non-hyperbolic equilibrium point. Determining the sign of the coefficient $a(0)$ requires complex calculations for different values of k and r. However, for $k = 2$ and under certain parameter conditions, we showed that the Neimark–Sacker bifurcation is supercritical. In some cases, for $k > 2$, our examples showed that $a(0)$ is negative, indicating that the Neimark–Sacker bifurcation is supercritical. However, we cannot conclude that we will always have a supercritical bifurcation.

We also showed that -1 can be an eigenvalue of the characteristic equation (15), for even k and certain

parameter conditions, as well as the existence of a minimal period-two solution. It hints at the possibility of a period-doubling bifurcation and possibly other types of bifurcation. We leave these topics for future research.

We explored the oscillatory behavior and the rate of convergence of the solutions to Equation (1). Investigating the rate of convergence required considerable effort, as the parameter r does not necessarily have to be an integer.

Finally, we gave several numerical examples and visualizations to confirm our theoretical results. Numerical simulations confirmed the complex dynamics of Equation (1) for different values of $k \in \{2, 3, ...\}$ and $r > 0$.

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