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Damping optimization of discrete mechanical systems – rod/string model

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Abstract. This paper investigates two optimization criteria for damping optimization in a multi-body oscillator system with arbitrary degrees of freedom (n), resembling free vibrations of a string or rod. These criteria are the total average energy and the total average displacement, both evaluated over all possible initial data. Our first result shows that both criteria are equivalent to the trace minimization of the solution of the Lyapunov equation with different right-hand sides. As the second result, we prove that in the case of damping with one damper, for the discrete system, the minimal trace for each criterion can be expressed as a linear or cubic function of the dimension n. Consequently, the optimal damping position is determined solely by the number of dominant eigenfrequencies and the optimal viscosity, independent of the dimension n, offering efficient damping optimization in discrete systems. The paper concludes with numerical examples illustrating the presented theoretical framework and results.

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1. Introduction

One of the crucial problems when attempting to control vibrational systems is the optimal positioning of dampers and the determination of their respective optimal damping values. However, apart from some special cases, the resulting scientific challenges are still open in different disciplines such as e.g. mathematics, engineering, and physics. The problem arises from considering the wave-like equations, where after an appropriate discretization we consider the dynamics of the second order system of ordinary differential equations, cf. e. g. [4]. The restriction to 1D vibrating systems (or networks thereof) is natural, since meaningful engineering applications mostly deal with only such kind of systems using a finite number of point-dampers.

In this paper, we will answer the question of what the best position is for one damper in a given vibrational system (like n-mass oscillator in Figure 1). The best position will be determined by some optimality in the solution of the corresponding system of ODEs, independent of the initial data, such as like initial position and velocity.

For the sake of clarity, let us consider a mechanical system described by the system of ordinary differential equations (ODE):

$$\begin{aligned} M\ddot{\mathbf{x}}(t) + v \cdot D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) &= \mathbf{0}, \\ \mathbf{x}_0 = \mathbf{x}(0), \mathbf{v}_0 &= \dot{\mathbf{x}}(0), \end{aligned} \tag{1}$$

where the mass and the stiffness matrices M and K are symmetric positive definite real matrices of order $n \times n$. The damping is defined as a rank 1 matrix $D = \mathbf{e}_k \mathbf{e}_k^T$, where $\mathbf{e}_k \in \mathbb{R}^n$ is the k-th canonical basis vector giving the dampers' positions and a parameter v is called viscosity. The vectors $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$, and $\mathbf{x}$ belong to $\mathbb{R}^n$ and denote the acceleration, velocity, and displacement, respectively.

In particular, the ODE system (1) can be applied on the vibration chain of masses and springs shown in Figure 1.

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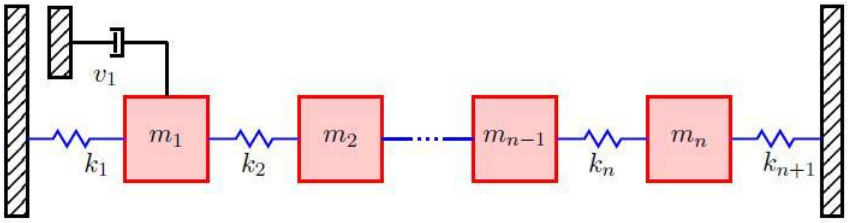


Figure 1: The n-mass vibration chain with one damper

Here, the mass matrix is defined as

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n), \quad m_i > 0, \tag{2}$$

and the stiffness matrix is

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix}. \tag{3}$$

In the setting shown in Figure 1, the damping matrix is

$$v \cdot D = v_1 \cdot \mathbf{e}_1 \mathbf{e}_1^T, \qquad v_1 > 0,$$

with $\mathbf{e}_1 \in \mathbb{R}^n, \mathbf{e}_1^T = (1, 0, \ldots, 0)$.

Extensive research spanning four decades has been focused on the optimal design, placement, and sizing of fluid viscous dampers. An efficient and systematic procedure for finding the optimal damper positioning to minimize the amplitude of a transfer function of a cantilever beam has been considered in [22, 2, 19].

For the problem of finding the "best damping", one of the crucial choices is the optimization criterion, which is a choice of a penalty function that has to be minimized. Once the optimization criterion has been defined (or determined), in most (existing) cases one runs through "all possible positions" and then the best position is one with the smallest penalty function. Our goal is to determine how efficiently one can calculate the best position of an (additional) external damper for two different criteria (penalty function).

The first optimization criterion will be:

i) **The total average energy over all possible initial data.**

The second criterion will be

ii) **The total average displacement over all possible initial data.**

The problem of finding the optimal positioning of a viscous damper for a linear conservative mechanical system based on an energy criterion has been studied in many papers, such as [11, 16, 22, 26, 27], or in the last couple of decades, in [10, 29, 15, 24, 20, 9, 23, 25].

On the other hand, to the best of our knowledge, the total average displacement over all possible initial data is a novel concept. Usually, the average displacement (amplitude) optimization is connected with problems treated in e. g. [21, 13]. Recently, an overview of optimal damper placement methods in different structures has been given in [14].

A comprehensive study of the controllability and stability of second order infinite dimensional systems coming from elasticity can be found in [12]. Besides many results like stabilization of second order evolution equations by a class of unbounded feedbacks, stabilization of second order evolution equations

with unbounded feedback with delay or systems without delay in [12] and [1] authors present results on the optimal location of one damper which calms down the whole undamped spectrum of the corresponding pair (M, K). The optimization criterion used in [12] or in [1] is similar to the minimization of the total average energy (criterion i)).

To enhance comprehension of the main motivation behind this paper, explain the first criterion i) *the mean value of the total energy*, will be briefly explained in more detail in Section 2.1. The minimization of the mean value of the total energy is equivalent to the minimization of the trace

$$\operatorname{trace}(ZX(p, v)) \to \min,$$

where X is a solution of the Lyapunov equation

$$A^T X(p, v) + X(p, v)A = -I_{2n},$$

where A is defined using M, D and K (see equation (8) below), and $v > 0$ and p are the damper's viscosity and its position, respectively. Let $\omega_1, \ldots, \omega_n$ be the undamped eigenfrequencies, that is, the square roots of the eigenvalues of the matrix pair (K, M).

The matrix $Z = Z_s \oplus Z_s$ depends on the part of the spectrum that we try to calm. For example, if we are interested in the best way to calm the first s undamped eigenfrequencies, that is, $0 < \omega_1 < \omega_2 < \ldots < \omega_s$ of the undamped system, the matrix Z will have the following form:

$$Z = Z_s \oplus Z_s \qquad Z_s = \begin{bmatrix} I_s & \\ & 0_{(n-s)} \end{bmatrix}.$$

Although numerical experiments suggest that the main ideas presented in this paper hold for general M and K, at present we can only provide rigorous proofs for models whose eigensystems admit closed-form expressions.

Throughout this paper we consider the mechanical system (1), where all $m_1 = \ldots = m_n = 1$, $k_1 = \ldots = k_{n+1} = 1$ in (2) and (3), respectively. This means that

$$M = I_n, \qquad K = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \tag{4}$$

which corresponds with the discretization of the system of ordinary differential equations (1).

So far the optimization of damper positions has primarily been approached using heuristic methods, without rigorous proofs. Examples of such approaches can be found in [5], [6], [7].

Numerical experiments indicate that the main ideas presented in this paper hold for a general definite pair (M, K), where M and K can be simultaneously diagonalized with a positive spectrum. However, at the moment, for the discrete systems, we can only prove results for M and K, as in (4).

The main result of this paper shows that, for discrete system, both criteria - i) total average energy and ii) the total average displacement, the optimal position $\mathbf{p}_{\text{opt}} = p_{\text{opt}}/n$, where $1 \le p_{\text{opt}} \le n$ depends only on the number of dominant eigenfrequencies s and not on the dimension n.

Example 1. To highlight the difference between the damping of the whole spectrum and of its part for string/rod vibrations, let us consider the system of ODE (1) of dimension $n = 600$. Further, let M and K be as in (4).

As the first case, we consider the problem of calming down the whole spectrum (all undamped eigenfrequencies $\omega_1 < \ldots < \omega_n$), that is, the case when $Z = I_{2n}$. The first illustration presents all $n = 600$ minimal traces, $\operatorname{trace}(X(v_{\text{opt}}, p))$, v_{opt} is optimal viscosity and p is position $1 \le p \le n$ and $\mathbf{p}=p/n \in \{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ (see Figure 2).

As one can see from Figure 2, the optimal position is located at the middle of the string, consistent with the result presented in [12] or [1].

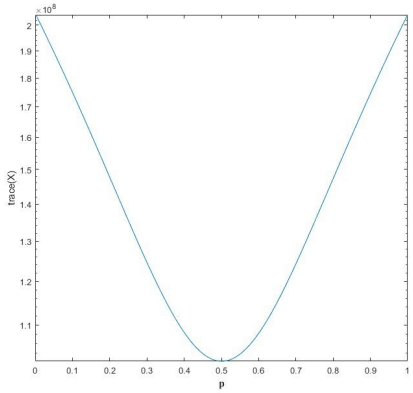


Figure 2: The optimal trace as a function of positions; damping of the whole spectrum, ω_k, $k = 1, \ldots, n$

Although the main result from [1] and Figure 2 shows that the optimal location of just one damper which calms the whole spectrum uniformly has been solved and thus is not a challenge anymore, we will show that the problem of calming down a part of the spectrum is completely different and still far away from a general solution.

Thus, Figure 3 shows all $n = 600$ minimal trace$(X(v_{\text{opt}}, p))$, v_{opt} is optimal viscosity and p is position $1 \leq p \leq n$ for the problem when one tries to calm the first 20 undamped eigenfrequencies $\omega_1 < \ldots < \omega_{20}$. This means that we have chosen a model in which the undamped eigenfrequencies $\omega_1 < \ldots < \omega_{20}$ are, in a certain sense, dominant. A similar problem has been considered in [23, 24, 25]. Here $Z = Z_s \oplus Z_s$, and Z_s has rank $s = 20$.

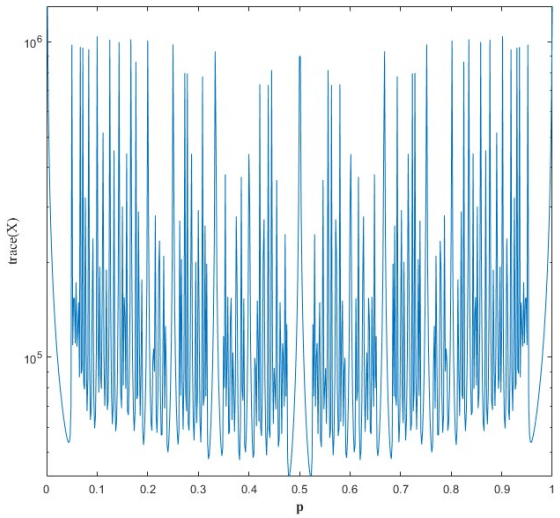


Figure 3: The optimal trace as a function of positions; damping of the part of the spectrum, for ω_k, $k = 1, \ldots, 20$

As one can see from Figure 3, the optimal location of the damper is at 0.48 (or due to the symmetry at 0.52) with several different local minima.

Moreover, if one is interested in calming down some small part of the undamped spectrum $\omega_{i+1} < \ldots < \omega_{i+s}$, with $s \ll n$ we will show that the optimal position depends on s and i.

In that sense, Figure 4 shows the optimal traces when one tries to calm down spectrum $\omega_{51} < \ldots < \omega_{70}$.

The best position is at 0.0083.

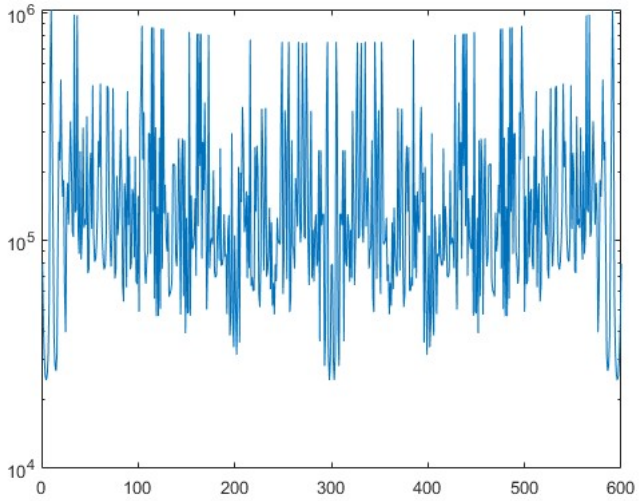


Figure 4: The optimal trace as a function of positions; damping of the part of the spectrum, for ω_k, $k = 51, \ldots, 70$

The above illustrations show that the calculation of the optimal locations together with the corresponding viscosity may be a very demanding task.

In this paper, we will present several new results connected with the calculation of the optimal location together with the corresponding viscosity for one damper which calms down a part of the spectrum of the string or rod model described by ODE system (1), discretized such that (4) holds.

The first result is a new approach to defining the *total average displacement criterion over all possible initial data*, criterion ii). Based on the formula for the trace of the solution of Lyapunov equation from [27] or [8], we develop a new, very efficient numerical procedure for calculating the optimal viscosity for the criterion of the total average displacement over all possible initial data.

The second result, which has already been mentioned, claims that the optimal position of the damper depends only on the number of dominant eigenfrequencies s and their location i ($\omega_{i+1} < \ldots < \omega_{i+s}$) and does not depend on the dimension n. It holds for both criteria.

The paper is organized as follows. In Section 2, we present two optimization criteria. The *average total energy* criterion is briefly presented in Section 2.1. In Section 2.2, we present a novel approach in more detail for the *average total displacement* criterion. Section 3 contains a precise analysis of the string or rod free vibrations, with exact expressions for undamped eigenfrequencies as well as corresponding eigenvectors. Sections 3.1 and 3.2 contain some auxiliary results and notations. The main results can be found in Sections 3.3 and 3.4, where we have shown that the minimal trace for the *average total energy* criterion is a linear function, while for the *average total displacement* criterion, it is a cubic function of dimension n. Finally, in Section 4, we present several numerical examples which illustrate results from previous sections.

Throughout the paper, M denotes the mass matrix, K represents the stiffness matrix, and c is a vector used to define the corresponding damping matrix $C = vcc^T$, where $v > 0$ is a real parameter. The values of M, K, and c may vary depending on the specific model (e.g., a string, rod, or discrete mass–spring system).

2. Optimization criteria

Although the system depicted in Figure 1 assumes that the mass matrix is diagonal, our approach is more general and we can treat any definite pair (M, K).

In fact, in our approach, as a preprocessing step, an eigenvalue decomposition of the pair (M, K) is done. Since M and K are symmetric positive definite, there exists a nonsingular matrix Φ such that

$$\Phi^T K \Phi = \Omega^2 = \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I_n, \tag{5}$$

$0 < \omega_1 < \omega_2 < \ldots < \omega_n$ (ω_i are the eigenfrequencies of the undamped system). Now, multiplying equation (1) from the left-hand side by Φ^T and using equation (5) we get

$$\Phi^{-1}\ddot{\mathbf{x}}(t) + v \cdot \Phi^T D \Phi \Phi^{-1}\dot{\mathbf{x}}(t) + \Omega^2 \Phi^{-1}\mathbf{x}(t) = \mathbf{0},$$

where

$$D = v \cdot \mathbf{e}_k \mathbf{e}_k^T \tag{6}$$

represents a damper located at the k-th position ($k \in \{1, 2, \ldots, n\}$) with viscosity v. By introducing the substitution

$$\mathbf{y}_1(t) = \Omega \Phi^{-1}\mathbf{x}(t), \quad \mathbf{y}_2(t) = \Phi^{-1}\dot{\mathbf{x}}(t), \tag{7}$$

we get the corresponding first order ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{y}_1(t)\\ \mathbf{y}_2(t)\end{bmatrix} = \begin{bmatrix}0 & \Omega\\ -\Omega & -C\end{bmatrix}\begin{bmatrix}\mathbf{y}_1(t)\\ \mathbf{y}_2(t)\end{bmatrix} =: A\begin{bmatrix}\mathbf{y}_1(t)\\ \mathbf{y}_2(t)\end{bmatrix}, \tag{8}$$

where the matrix C is a representation of the damping matrix (6) in the eigenvector basis, that is,

$$C = v\, \Phi^T D \Phi.$$

2.1. Optimization criterion – average total energy

Note that if we write

$$\mathbf{y} = \begin{bmatrix}\mathbf{y}_1(t) & \mathbf{y}_2(t)\end{bmatrix}^T,$$

from equation (7) there follows

$$\mathbf{y}(t)^T\mathbf{y}(t) = \|\mathbf{y}_1(t)\|^2 + \|\mathbf{y}_2(t)\|^2 = \mathbf{x}^T K \mathbf{x} + \dot{\mathbf{x}}^T M \dot{\mathbf{x}} = 2E(t).$$

In other words, the Euclidean norm of this phase-space representation equals twice the total energy of the system. From this, it follows that *all phase-space matrices are unitarily equivalent*. Thus, for all total-energy relevant considerations, we may choose any of these representations at our convenience.

For the first optimization criterion, we will use a minimization of the mean value of the total energy. In [17, 28], it was shown that this optimization criterion is equivalent to the minimization of the trace

$$\operatorname{trace}(Z_\Delta X) \to \min, \tag{9}$$

where X is a solution of the Lyapunov equation

$$A^T X + X A = -I, \tag{10}$$

where A is defined in equation (8) and $Z_\Delta = Z_s \oplus Z_s$ depends on the part of the spectrum that we try to calm.

For example, as already shown, if we are interested in the best way to calm the first s eigenfrequencies $0 < \omega_1 < \omega_2 < \ldots < \omega_s$ of the undamped system, the matrix Z_Δ for $\Delta = 1$ will have the following form:

$$Z_1 = Z_s \oplus Z_s \qquad Z_s = \begin{bmatrix}I_s & \\ & 0_{(n-s)}\end{bmatrix}.$$

For more details on the construction of the matrix Z, see, for example, [17].

It is easy to show that the trace minimization (9) with Lyapunov equation (10) is equivalent to the trace minimization of the solution of the so-called dual Lyapunov equation of the form

$$AY + YA^T = -Z_\Delta,$$

since

$$\operatorname{trace}(Y) = \operatorname{trace}(Z_\Delta X).$$

2.2. Optimization criterion – total average displacement

Since, to the best of our knowledge, a theory for total average displacement similar to the one given above (for the average total energy) has not been presented yet, in this section we will present the basic results on Lyapunov theory for lowering displacements, which will be our second criterion.

Recall that in equation (7) we have defined

$$\mathbf{y}_1(t) = \Omega\Phi^{-1}\mathbf{x}(t), \quad \mathbf{y}_2(t) = \Phi^{-1}\dot{\mathbf{x}}(t),$$

which together with equation (8) gives

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = A\mathbf{y},$$

with

$$A = \begin{bmatrix} \mathbf{0} & \Omega \\ -\Omega & -C \end{bmatrix}, \tag{11}$$

which is solved by

$$\mathbf{y} = e^{At}\begin{bmatrix} \mathbf{y}_{10} \\ \mathbf{y}_{20} \end{bmatrix},$$

where $\mathbf{y}_0 = \begin{bmatrix}\mathbf{y}_{10} & \mathbf{y}_{20}\end{bmatrix}^T$ contains the initial data.

Let

$$K = U_K \Lambda_K U_K^T$$

be the eigenvalue decomposition of the matrix K. Note that

$$\Omega^{-1}\Phi^T U_K \Lambda_K U_K^T \Phi\Omega^{-1} = I_n,$$

which means that $\Omega^{-1}\Phi^T U_K$ is unitarily similar to the diagonal matrix $\Lambda_K^{-1/2}$, which further implies that we can write the singular value decomposition

$$U_K^T \Phi\Omega^{-1} = U\Lambda_K^{-1/2}V^T,$$

or

$$\Omega^{-1}\Phi^T\Phi\Omega^{-1} = V\Lambda_K^{-1}V^T \doteq \widehat{K}^{-1}, \tag{12}$$

which means that $\widehat{K}$ is unitarily similar to K; here, U and V are orthogonal matrices.

The quantity to be minimized is the mean displacement $\hat{\mathbf{x}}$ given by

$$\begin{aligned}\hat{\mathbf{x}}^2 &= \int_0^\infty \|\mathbf{x}(t)\|^2 \mathrm{d}t = \int_0^\infty \mathbf{y}_1(t)^T \Omega^{-1}\Phi^T\Phi\Omega^{-1}\mathbf{y}_1(t)\mathrm{d}t \\ &= \int_0^\infty \left(P_1 e^{At}\mathbf{y}_0\right)^T \widehat{K}^{-1} P_1 e^{At}\mathbf{y}_0 \mathrm{d}t,\end{aligned}$$

where $\widehat{K}$ is defined in equation (12) and

$$P_1 = \begin{bmatrix} I_n & \mathbf{0} \end{bmatrix}.$$

So,

$$\int_0^\infty \|\mathbf{x}(t)\|^2 \mathrm{d}t = \mathbf{y}_0^T \hat{X}\mathbf{y}_0, \tag{13}$$

$$\hat{X} = \int_0^\infty e^{A^T t} Z e^{At}\mathrm{d}t, \quad Z = P_1^T \widehat{K}^{-1} P_1 = \begin{bmatrix} \widehat{K}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{14}$$

where $\hat{X}$ solves the Lyapunov equation

$$A^T\hat{X} + \hat{X}A = -Z.$$

Proposition 1. *The matrix $\hat{X}$ is symmetric positive definite.*

Proof. Z is positive semidefinite, so is $\hat{X}$. Assume $\hat{X}z = 0$; then by equation (13) and the positive definiteness of $\widehat{K}$,

$$\mathbf{x}(t) = \widehat{K}^{-1/2} P_1 e^{At} z = 0, \quad \forall t \geq 0.$$

Hence also $\dot{\mathbf{x}}(t) = \mathbf{0}$ for all $t \geq 0$ and thus $z = 0$.

□

The average over the unit sphere

$$\operatorname{trace}(\hat{X}) = \int_0^\infty \operatorname{trace}\left(e^{At} e^{A^T t} Z\right) \mathrm{d}t = \operatorname{trace}(JXJZ) = \operatorname{trace}(XZ)$$

where the matrix J is defined as

$$J = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & -I_n \end{bmatrix},$$

and X solves the standard Lyapunov equation

$$A^T X + XA = -I.$$

We have used the J-symmetry of A and the fact that J and Z commute.

Frequency-cut average with a frequency-cut projection P can be obtained as

$$\begin{aligned} \operatorname{trace}(\hat{X}P) &= \operatorname{trace} \int_0^\infty \left(e^{A^T t} Z e^{At} P\right) \mathrm{d}t \\ &= \operatorname{trace}\left(Z \int_0^\infty e^{At} P e^{A^T t} \mathrm{d}t\right) = \operatorname{trace}(Z\hat{Y}), \end{aligned}$$

where

$$A\hat{Y} + \hat{Y}A^T = -P,$$

and since $P,\ J$ commute as well as $Z,\ J$

$$\operatorname{trace}(\hat{X}P) = \operatorname{trace}(Z\tilde{Y}),$$

where $\tilde{Y}$ solves the standard Lyapunov equation

$$A^T\tilde{Y} + \tilde{Y}A = -P.$$

Note that in both criteria one needs to minimize the trace of the corresponding Lyapunov equation, with the same system matrix but a different right-hand side.

As an illustration that the optimal viscosities are different for each criterion, we present an example.

Example 2. Consider a one-dimensional oscillator

$$m\ddot{x} + c\dot{x} + kx = 0,$$

and the Lyapunov equation

$$\begin{bmatrix} 0 & -\sqrt{k/m} \\ \sqrt{k/m} & -c/m \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{k/m} \\ -\sqrt{k/m} & -c/m \end{bmatrix}$$

$$= \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix},$$

with $a, b \geq 0,\ a + b > 0$. This gives the equations

$$-2\sqrt{k/m}x_{12} = -a \qquad \sqrt{k/m}(x_{11} - x_{22}) - x_{12}c/m = 0$$

and

$$x_{12}\sqrt{k/m} - x_{22}c/m + x_{12}\sqrt{k/m} - x_{22}c/m = -b.$$

Hence

$$x_{12} = \frac{a}{2\sqrt{k/m}} = \frac{a}{2}\sqrt{m/k},$$

$$(x_{22} - x_{11})\sqrt{k/m} + \frac{ac}{2\sqrt{km}} = 0,$$

$$x_{22} = \left(\frac{a}{2} + \frac{b}{2}\right) m/c,$$

$$x_{11} = x_{22} + \frac{ac}{2k} = \frac{a+b}{2} m/c + \frac{ac}{2k}.$$

For $a = b = 1$, we obtain the trace for the average energy criterion

$$\operatorname{trace}(X_1) = 2m/c + \frac{c}{2k},$$

with the known minimum at the critical damping $c = 2\sqrt{mk}$. In the case of the average displacement criterion, we have $a = 1/k,\ b = 0$, giving the trace

$$\operatorname{trace}(X_2) = m/(kc) + c/(2k^2),$$

again with a unique minimum $c = \sqrt{2}\sqrt{mk}$, obviously different from the one given above.

This shows that a further study of the properties of both criteria will be an interesting issue, especially a comparison between them in the sense of the quality of the solution and the complexity of the calculation. However, in this paper, we will not further consider the optimization criteria, this will be a subject of our future studies. In the rest of the paper, we will concentrate only on the position optimization for both criteria for the vibrations of the structured models.

3. Vibrations of rod/string and discrete mass–spring model

Our main result concerns string or rod vibrations described by the following partial differential equation:

$$\rho(x)u(x,t)_{tt} + c(x)u(x,t)_t - (k(x)u(x,t)_x)_x = 0, u(0,t) = u(1,t) = 0, \quad x \in (0,1), \tag{15}$$

with point damping $c(x) = \delta(x - y)$ concentrated in the position $x = y$.

The main advantage of equation (15) is that it depends continuously on the damper position y, which allows for an analytical minimization. This property can be naturally carried over to the discretized setting, where the damper position is likewise treated as a continuous variable. For simplicity, we assume $\rho(x) = 1$ and $k(x) = 1$. Using separation of variables or, equivalently, a Fourier method, one obtains the undamped system

$$\omega^2 u + u(x,t)_{xx} = 0,$$

which is explicitly diagonalized with the undamped frequencies $\omega = j\pi$ and the orthonormal eigenvectors

$$u_k(x) = \frac{1}{\sqrt{2}} \sin k\pi x.$$

In the weak formulation of the formal equation (15), the delta damper is represented by the quadratic form

$$\omega(u,v) = du(y)v(y), d \geq 0,$$

to which there corresponds the matrix $C(y)$ in the basis u_k, given as

$$C_{kj} = \frac{d}{2} \sin(k\pi y) \sin(j\pi x).$$

Now, discretization consists of cutting a finite matrix out of these.

This leads to the finite dimensional system with

$$K = \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2), M = I_n, C = C(y) = dc(y)c(y)^T,$$

with

$$c(y) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin(\pi y) & \sin(2\pi y) & \ldots & \sin(n\pi y) \end{bmatrix}^T.$$

More details about the properties of the vibrating system, described by the partial differential equation (15), can be found in [12, 4.1 The Wave Equation].

If ρ and k are not constant, we may still use the Fourier decomposition above and by common integrals compute the coefficients of the matrices M and K as follows:

$$M_{kj} = \frac{1}{2} \int_0^1 \sin(k\pi x \rho(x)) \sin(j\pi x) dx,$$

$$K_{kj} = \frac{kj\pi^2}{2} \int_0^1 \cos(k\pi x k(x)) \cos(j\pi x) dx,$$

and C as above. The only difference is that now the matrices M and K are not diagonal anymore. Note that choosing other boundary conditions would lead to other trigonometric functions. Thus, we end up with minimizing the Lyapunov trace for one-dimensional damping as described in [27], which is given by an explicit formula as a function of y. This can again be minimised either analytically or numerically or just by plotting.

To do this, we must first normalise $c(y)$ to a constant norm that is independent of y. We have

$$\begin{aligned} \|c(y)\|^2 &= \frac{1}{2} \sum_{k=1}^{n} \sin^2(k\pi y) = \frac{n}{4} - \frac{1}{4} \sum_{k=1}^{n} \cos(2k\pi y) \\ &= \frac{n}{4} - \frac{1}{4} Re \left(\sum_{k=1}^{n} e^{2k\pi \imath y} \right) \\ &= \frac{n}{4} - \frac{1}{4} Re \frac{e^{2\pi \imath y}(1 - e^{2\pi \imath n y})}{1 - e^{2\pi \imath n y}}, 0 < y < 1. \end{aligned} \tag{16}$$

The same approach can be applied to the rod model

$$\begin{aligned} \rho(x)u(x,t)_{tt} + c(x)u(x,t)_t - (k(x)u(x,t)_x)_x + (a(x)u(x,t)_{xx})_{xx} &= 0, \\ u(0,t) = u(x,t)_{xx}(0,t) = u(1,t) = u(x,t)_{xx}(1,t) &= 0. \end{aligned}$$

If the material is homogeneous, say, $\sigma(x) = \sigma_0 > 0$, $\rho(x) = 1$, $k(x) = k_0$ and $a(x) = a_0$, then the stationary undamped equation

$$-\omega^2 u - k_0 u(x,t)_{xx} + a_0 u(x,t)_{xxxx} = 0$$

has again the eigenfunctions

$$u_k(x) = \frac{1}{\sqrt{2}} \sin k\pi x,$$

with the eigenfrequencies

$$\omega_k = k\pi \sqrt{k^2\pi^2 a_0 + k_0}.$$

This was the spectral discretization of the continuous system. Another way is to start from the finite dimensional case as in (2) and (3), that is,

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n), \quad m_i > 0,$$
$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix}.$$

For $k_i = m_i = 1$, the eigenfrequencies are

$$\omega_l = 2 \sin \frac{l\pi}{2(n+1)}, l = 1, \ldots, n,$$

and the eigenvector matrix $Q = (Q_{rl})$ given by

$$Q_{rl} = \sqrt{\frac{2}{n+1}} \sin \frac{rl\pi}{n+1}.$$

That is, the k-th eigenvector is given by

$$u_k = \sqrt{\frac{2}{n+1}} \left[\sin \tfrac{1k\pi}{n+1} \quad \sin \tfrac{2k\pi}{n+1} \quad \ldots \quad \sin \tfrac{nk\pi}{n+1} \right]^T \tag{17}$$

This case is interesting both in itself and also because it can be understood as a discretization of the continuous string. Recall that we have denoted the eigenvector matrix by $Q = (Q_{rl})$. This matrix also happens to be symmetric, so its k-th column, that is, the vector u_k from (17) is at the same time the representation of the canonical unit vector e_k in the orthonormal basis $u_1, \ldots, u_n$. Our damped system with the damping matrix $C = ve_k e_k^T$ is now equivalent to the system

$$M = I_n, K = \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2), C = vu_k u_k^T,$$

which describes the damper at the position k. We now make the discrete variable k continuous by setting

$$C(z) = vc(z)c(z)^T,$$

with

$$c(z) = \sqrt{\frac{2}{n+1}} \left[\sin \tfrac{1zn\pi}{n+1} \quad \sin \tfrac{2zn\pi}{n+1} \quad \ldots \quad \sin \tfrac{nzn\pi}{n+1} \right]^T, 0 < z \leq 1, \tag{18}$$

such that

$$c\left(\frac{k}{n}\right) = u_k.$$

Now, this is strikingly analogous to the spectral discretization considered previously. Moreover, for large n, the vectors $c(z)$ coincide here and there. The normalization formulae here are quite analogous to those in (16). Here too, this construction starts from the special stiffness matrix K above, but it can be used with any 1D vibrational system provided the boundary conditions are taken accordingly.

3.1. Optimal dampers' position – discrete mass spring mechanical system

In this section, we present the main result concerning the optimal position of a single damper in a discrete mass–spring mechanical system. It is shown that the optimal position can be expressed as a simple function

of the system dimension n, which allows the optimal location to be computed for small system sizes and then extrapolated to the general case.

We focus on a single damper due to the fact that only for one damper there exists an explicit expression for the trace of the solution to the corresponding Lyapunov equation.

The initial criterion to be applied is the total average energy criterion. As outlined in Section 2, we will consider the Lyapunov equation

$$AX + XA^T = -Z_{\mathcal{A}},$$

where A is defined in (11) and for $\mathcal{A} = 1$

$$Z_1 = Z_s \oplus Z_s \qquad Z_s = \begin{bmatrix} I_s & \\ & 0_{(n-s)} \end{bmatrix}. \tag{19}$$

Our goal is to find the minimal trace(X), which corresponds with the minimization of the total average energy. Further,

$$\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_n), \omega_j = 2 \sin \frac{j\pi}{2(n+1)}.$$

For a given position p_k, the damping matrix C is defined as

$$C = cc^T,$$

where for $p_k \in \{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$

$$c(p_k, n) = \sqrt{\frac{2}{n+1}} \left[\sin \tfrac{1p_k n\pi}{n+1} \quad \sin \tfrac{2p_k n\pi}{n+1} \quad \ldots \quad \sin \tfrac{np_k n\pi}{n+1}\right]^T.$$

Note that $p_k = k/n$ is a discrete set of points defined by a continuous variable $0 < z \leq 1$ from (18).

Since

$$\omega_j = \omega_j(n),$$

we see that the trace(X) is a function of dimension n, positions p_k, and viscosity v, that is

$$\operatorname{trace}(X) = \operatorname{trace}(X(p_k, v, n)).$$

We will show that for a fixed position $\mathbf{p}$ (which does not depend on n) defined as

$$\mathbf{p} \in (0, 1), \tag{20}$$

the optimal trace$(X(\mathbf{p}_{\text{opt}}, v_{\text{opt}}))$ is a simple function (linear for the "energy criterion" or cubic for the "displacement" criterion) of dimension n. Here the optimal trace$(X(\mathbf{p}_{\text{opt}}, v_{\text{opt}}, n))$ is defined by the optimal position and corresponding viscosity

$$(\mathbf{p}_{\text{opt}}, v_{\text{opt}}) = \operatorname*{argmin}_{\mathbf{p}, v} \operatorname{trace}(X(\mathbf{p}, v)).$$

3.2. Optimal position of one damper for the discrete mass–spring model

As previously highlighted, determining the optimal placement of several dampers poses a significant challenge. As illustrated below, even in the case of the rod/string model, where explicit expressions for all undamped eigenfrequencies and eigenvectors are available, positioning multiple dampers remains an unresolved issue.

Thus, in the next section, we will present our main results regarding the properties of the trace function of the solution to the Lyapunov equation in the case of one damper. These findings will enable us to identify the optimal position for a single damper in the discrete mass–spring model, applicable across all dimensions.

The main tool will be a modified formula obtained by VESELIĆ in [27] in the form documented in [8] for the trace of the Lyapunov equation

$$A^T X + XA = -I_{2n},$$

where A is defined as

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -v\mathbf{c}\mathbf{c}^T \end{bmatrix}, \tag{21}$$

and $v > 0$ is the viscosity parameter which has to be optimized and for a given position $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^T$, $c_i \neq 0$.

The trace of the solution X multiplied by diagonal matrix Z_Δ is given by

$$\operatorname{trace}(Z_\Delta X) = \frac{a}{v} + bv, \tag{22}$$

where

$$a = \sum_{k=1}^{n} \frac{2z_k}{c_k^2}\,, \tag{23}$$

$$b = \sum_{k=1}^{n} \frac{z_k c_k^2}{2\omega_k^2} + z_k \left(\sum_{j \neq k}^{n} \frac{3\omega_k^2 c_j^2 + \omega_k^2 c_k^2 + \omega_j^2 c_j^2 + \omega_j^2 c_k^2}{(\omega_k^2 - \omega_j^2)^2} + 2\frac{\omega_k^2}{c_k^2} \left(\sum_{j \neq k}^{n} \frac{c_j^2}{\omega_k^2 - \omega_j^2} \right)^2 \right), \tag{24}$$

and Z_Δ is any diagonal matrix, i.e.

$$Z_\Delta = \operatorname{diag}(z_1, \dots, z_n, z_1, \dots, z_n). \tag{25}$$

From (22) we get the optimal viscosity "for free"

$$v_{\text{opt}} = \sqrt{\frac{a}{b}}.$$

This means that the optimal trace for a damper in a fixed position is

$$\operatorname{trace}(X_{\text{opt}}) = 2\sqrt{ab}. \tag{26}$$

In what follows, we will show that the optimal trace (26) for the position defined as in (20), ($\mathbf{p} = k/n$), is asymptotically a linear function of the dimension n, that is,

$$\operatorname{trace}(X_{\text{opt}}) = \operatorname{trace}(X_{\text{opt}}(\mathbf{p}, n)) \sim \mathcal{O}(n).$$

More precisely, we will show that for the optimal trace (26) at position $\mathbf{p}$, the corresponding optimal coefficients a and b are linear functions of the dimension n, i.e., we will show that for n large enough ($n \to \infty$)

$$a(n) \sim \alpha_1 n + \alpha_0, b(n) \sim \beta_1 n + \beta_0,$$

for some functions α_0, α_1 and β_0, β_1, which do not depend on n.

3.3. Optimal trace for the average total energy

As mentioned above, using expression (26) and the structure of a and b from (23) and (24), respectively, we will prove that for the rod or string model for n large enough, the optimal trace is asymptotically a linear function of n. This means that for n large enough, the optimal trace divided by n is a function that does not depend on n.

More precisely, we will show that

$$\lim_{n\to\infty} \frac{a(n)}{n} = \alpha_1, \quad \lim_{n\to\infty} \frac{b(n)}{n} = \beta_1,$$

where α_1 and β_1 do not depend on n; thus,

$$\lim_{n\to\infty} (\operatorname{trace}(X(\mathbf{p},n)) = \eta(\mathbf{p},n) \sim \eta_1 n + \eta_0,$$

where η_1 and η_0 are some functions that are independent of n.

For the criterion of the average total energy, frequency-cut projection is defined as in (19):

$$Z_1 = Z_s \oplus Z_s, \qquad Z_s = \begin{bmatrix} I_s & \\ & 0_{(n-s)} \end{bmatrix},$$

which means that all $z_k = 1$ from (25), for $k = 1, \ldots, s$, while $z_k = 0$, for $k = s+1, \ldots, n$.

Recall that ω_j and $c_j(\mathbf{p}, n)$ for the string/rod model are given by

$$\omega_j(n) = 2 \sin \frac{j\pi}{2(n+1)},$$

and

$$c_j(\mathbf{p}, n) = \sqrt{\frac{2}{n+1}} \sin \frac{j\mathbf{p}n\pi}{n+1}.$$

If we insert these into a and b from (23) and (24), respectively, we get

$$a(n) = \sum_{k=1}^{s} \frac{(n+1)}{\sin^2\left(\frac{\pi k n \mathbf{p}}{n+1}\right)}, \tag{27}$$

$$b(n) = b_1(n) + b_2(n) + b_3(n),$$

$$b_1(n) = \sum_{k=1}^{s} \frac{\csc^2\left(\frac{\pi k}{2(n+1)}\right) \sin^2\left(\frac{\pi k n \mathbf{p}}{n+1}\right)}{4(n+1)}, \tag{28}$$

$$b_2(n) = \sum_{k=1}^{s} \sum_{j\neq k}^{n} \frac{\omega_k^2(3c_j^2 + c_k^2) + \omega_j^2(c_j^2 + c_k^2)}{(\omega_k^2 - \omega_j^2)^2}, \tag{29}$$

$$b_3(n) = \sum_{k=1}^{s} \sum_{j\neq k}^{n} 2\frac{\omega_k^2}{c_k^2} \left(\sum_{j\neq k}^{n} \frac{c_j^2}{\omega_k^2 - \omega_j^2} \right)^2, \tag{30}$$

which, after additional simplification, yields

$$\begin{aligned} b_2(n) = &\sum_{k=1}^{s} \sum_{j\neq k}^{n} \frac{2\sin^2\left(\frac{\pi j}{2n+2}\right)\left(\sin^2\left(\frac{\pi j n \mathbf{p}}{n+1}\right) + \sin^2\left(\frac{\pi k n \mathbf{p}}{n+1}\right)\right)}{(n+1)\left(\cos\left(\frac{\pi j}{n+1}\right) - \cos\left(\frac{\pi k}{n+1}\right)\right)^2} \\ &+ \frac{2\sin^2\left(\frac{\pi k}{2n+2}\right)\left(3\sin^2\left(\frac{\pi j n \mathbf{p}}{n+1}\right) + 2\sin^2\left(\frac{\pi k n \mathbf{p}}{n+1}\right)\right)}{(n+1)\left(\cos\left(\frac{\pi j}{n+1}\right) - \cos\left(\frac{\pi k}{n+1}\right)\right)^2} \\ b_3(n) = &\sum_{k=1}^{s} 4 \cdot (n+1) \sin^2\left(\frac{\pi k}{2n+2}\right) \csc^2\left(\frac{\pi k n \mathbf{p}}{n+1}\right) \\ &\times \left(\sum_{j\neq k}^{n} \frac{\sin^2\left(\frac{\pi j n \mathbf{p}}{n+1}\right)}{(n+1)\left(\cos\left(\frac{\pi j}{n+1}\right) - \cos\left(\frac{\pi k}{n+1}\right)\right)} \right)^2. \end{aligned}$$

Thus, we proceed with the calculation of the limits of functions $a(n)/n$ and $b(n)/n$.

Lemma 1. *Let $a(n)$ be defined as in (27). Then the limit*

$$\lim_{n\to\infty}\frac{a(n)}{n}=\sum_{k=1}^{s}\csc^2(\pi k\mathbf{p})$$

exists. In particular, the function $a(n)$ has an oblique asymptote with the slope

$$\alpha_1=\sum_{k=1}^{s}\csc^2(\pi k\mathbf{p}). \tag{31}$$

Proof. By definition,

$$\frac{a(n)}{n}=\sum_{k=1}^{s}\frac{(n+1)}{n\sin^2\left(\frac{\pi kn\mathbf{p}}{n+1}\right)}.$$

Passing to the limit $n\to\infty$ yields

$$\lim_{n\to\infty}\frac{a(n)}{n}=\sum_{k=1}^{s}\csc^2(\pi k\mathbf{p}),$$

which proves the claim. □

Lemma 2. *(a) Let $b_1(n)$ be defined as in (28). Then the limit*

$$\lim_{n\to\infty}\frac{b_1(n)}{n}=\sum_{k=1}^{s}\frac{2\sin^2(\pi k\mathbf{p})}{\pi^2k^2}\doteq\beta_{11} \tag{32}$$

exists.

(b) Let $b_2(n)$ be defined as in (29). Then the limit

$$\lim_{n\to\infty}\frac{b_2(n)}{n}\doteq\beta_{12}\quad\textit{with}\quad\beta_{12}\le s\quad\textit{exists}. \tag{33}$$

Proof. (a) The ratio $\dfrac{b_1(n)}{n}$ is given by

$$\beta_{11}=\frac{b_1(n)}{n}=\sum_{k=1}^{s}\frac{\csc^2\left(\frac{\pi k}{2(n+1)}\right)\sin^2\left(\frac{\pi kn\mathbf{p}}{n+1}\right)}{4n(n+1)}.$$

Passing to the limit $n\to\infty$ yields

$$\lim_{n\to\infty}\frac{b_1(n)}{n}=\sum_{k=1}^{s}\frac{2\sin^2(\pi k\mathbf{p})}{\pi^2k^2},$$

which proves the claim.

(b) Further,

$$\frac{b_2(n)}{n}=\sum_{k=1}^{s}\sum_{j\neq k}^{n}\Gamma(j,k,\mathbf{p}),$$

where

$$\Gamma(j,k,\mathbf{p}) = \frac{2\sin^2\left(\frac{\pi j}{2n+2}\right)\left(\sin^2\left(\frac{\pi jn\mathbf{p}}{n+1}\right)+\sin^2\left(\frac{\pi kn\mathbf{p}}{n+1}\right)\right)}{n(n+1)\left(\cos\left(\frac{\pi j}{n+1}\right)-\cos\left(\frac{\pi k}{n+1}\right)\right)^2} + \frac{\sin^2\left(\frac{\pi k}{2n+2}\right)\left(3\sin^2\left(\frac{\pi jn\mathbf{p}}{n+1}\right)+2\sin^2\left(\frac{\pi kn\mathbf{p}}{n+1}\right)\right)}{n(n+1)\left(\cos\left(\frac{\pi j}{n+1}\right)-\cos\left(\frac{\pi k}{n+1}\right)\right)^2}.$$

Let us use simple properties of the limit of trigonometric functions and some trigonometric identities, like

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right),$$

and

$$0 \le \sin^2 x \le x^2 \quad \text{for all } x \in \mathbb{R}.$$

Using these estimates, the following upper bound holds for $\Gamma(j,k,\mathbf{p})$:

$$0 < \Gamma(j,k,\mathbf{p}) \le \frac{2\left(2(j\pi)^2+4(k\pi)^2\right)}{n(n+1)(2n+2)^2\left(\sin\left(\frac{(j+k)\pi}{2(n+1)}\right)\sin\left(\frac{(j-k)\pi}{2(n+1)}\right)\right)^2},$$

which further implies

$$0 < \Gamma(j,k,\mathbf{p}) \le \frac{4(n+1)\left(j^2+2k^2\right)}{\pi^2 n(j-k)^2(j+k)^2}.$$

For a fixed $k \in \mathbb{N}$, the sum

$$S(k) = \frac{4}{\pi^2}\sum_{j\neq k}^{\infty}\frac{(n+1)\left(j^2+2k^2\right)}{n(j-k)^2(j+k)^2} \tag{34}$$

is convergent. Indeed, note that for a fixed k

$$\begin{aligned}\sum_{j\neq k}^{\infty}\frac{(n+1)\left(j^2+2k^2\right)}{n(j-k)^2(j+k)^2} &= \sum_{j\neq k}^{\infty}\frac{\left(j^2+2k^2\right)}{(j^2-k^2)^2}\\ &= \sum_{j\neq k}^{\infty}\frac{1}{(j^2-k^2)} + 3\sum_{j\neq k}^{\infty}\frac{k^2}{(j^2-k^2)^2}\\ &= \sum_{j=1}^{k-1}\frac{1}{(j^2-k^2)} + \sum_{j=k+1}^{\infty}\frac{1}{(j^2-k^2)} + 3\sum_{j=1}^{k-1}\frac{k^2}{(j^2-k^2)^2} + 3\sum_{j=k+1}^{\infty}\frac{k^2}{(j^2-k^2)^2}.\end{aligned}$$

Since for $j > k$ we have

$$\frac{k^2}{(j^2-k^2)^2} \le \frac{k^2}{j^2-k^2},$$

we will use

$$\sum_{j=k+1}^{\infty}\frac{1}{j^2-k^2} = \frac{1}{2k}H_{2k},$$

where H_{2k} are generalized harmonic numbers (see [3, Theorem 3.4]). Combining this with the numerical illustration (for $n \to \infty$)

$$S(1) = 0.468064, S(2) = 0.867021, S(3) = 0.940901, S(4) = 0.96676, \ldots$$
$$S(10) = 0.994687, \ldots, S(100) = 0.999953, \ldots, S(\infty) = 1,$$

we can conclude that the sum $S(k)$ from (34) is convergent.

Thus, the limit of $\dfrac{b_2(n)}{n}$ is bounded by

$$\lim_{n\to\infty} \frac{b_2(n)}{n} \le s \cdot \lim_{n\to\infty} \sum_{j\neq k}^{n} \frac{4(n+1)\left(j^2+2k^2\right)}{\pi^2 n(j-k)^2(j+k)^2} \le s.$$

Therefore, the limit $\beta_{12} = \lim_{n\to\infty} b_2(n)/n$ exists and satisfies $\beta_{12} \le s$, which proves (33).

□

Lemma 3. *Let $b_3(n)$ be as defined in (30). Then*

$$\beta_{13} \doteq \lim_{n\to\infty} \frac{b_3(n)}{n} \le \sum_{k=1}^{s} \csc^2(\pi k \mathbf{p})\, T(k, \mathbf{p}), \tag{35}$$

where

$$T(k, \mathbf{p}) = \sum_{j=1}^{k} \frac{1}{k^2 - j^2} + \frac{1}{2k} H_{2k}. \tag{36}$$

Proof. For the last limit, consider

$$\frac{b_3(n)}{n} = \sum_{k=1}^{s} \frac{4(n+1)}{n} \sin^2\left(\frac{\pi k}{2n+2}\right) \csc^2\left(\frac{\pi k n \mathbf{p}}{n+1}\right) \times \left(\sum_{j\neq k}^{n} \frac{\sin^2\left(\frac{\pi j n \mathbf{p}}{n+1}\right)}{\left(\cos\left(\frac{\pi j}{n+1}\right) - \cos\left(\frac{\pi k}{n+1}\right)\right)} \right)^2.$$

Note that

$$\lim_{n\to\infty} \frac{b_3(n)}{n} = \lim_{n\to\infty} \sum_{k=1}^{s} \frac{\left((n+1)\pi^2 k^2\right) \csc^2\left(\frac{\pi k n \mathbf{p}}{n+1}\right)}{n} \left(\sum_{j\neq k}^{n} \frac{2\sin^2\left(\frac{\pi j n \mathbf{p}}{n+1}\right)}{\pi^2\left(k^2 - j^2\right)} \right)^2$$
$$= \sum_{k=1}^{s} \csc^2(\pi k \mathbf{p}) \left(\sum_{j\neq k}^{\infty} \frac{2k \sin^2(\pi j \mathbf{p})}{\pi\left(k^2 - j^2\right)} \right)^2.$$

Note for fixed k,

$$\left| \frac{\sin^2(\pi j \mathbf{p})}{\left(k^2 - j^2\right)} \right| \le \left| \frac{1}{j^2 - k^2} \right|,$$

which uses

$$\sum_{j=k+1}^{\infty} \frac{1}{j^2 - k^2} = \frac{1}{2k} H_{2k},$$

where H_{2k} are generalized harmonic numbers, implies that the sums

$$\sum_{k=1}^{s} \csc^2(\pi k\mathbf{p}) \left(\sum_{j=1}^{\infty} \frac{2k \sin^2(\pi j\mathbf{p})}{\pi \left(k^2 - j^2\right)} \right)^2$$

are convergent. We have

$$\lim_{n\to\infty} \frac{b_3(n)}{n} \leq \sum_{k=1}^{s} \csc^2(\pi k\mathbf{p})\, T(k,\mathbf{p}),$$

where

$$T(k,\mathbf{p}) = \sum_{j=1}^{k} \frac{1}{k^2 - j^2} + \frac{1}{2k} H_{2k}.$$

Note that the limit in (35) depends only on s and $\mathbf{p}$. □

Combining Lemmas 2–3, namely (32), (33), and (35), respectively, we conclude that

$$\lim_{n\to\infty} \frac{b(n)}{n} = \beta_{11} + \beta_{12} + \beta_{13} = \beta_1(s,\mathbf{p}), \tag{37}$$

and from (31) there follows

$$\lim_{n\to\infty} \frac{a(n)}{n} = \alpha_1(s,\mathbf{p}). \tag{38}$$

To summarize, if one is interested in the calculation of the best damping, i.e. the best position and the corresponding optimal viscosity using the average total energy criterion, this is equivalent to the minimization

$$\operatorname{trace}(X(\mathbf{p},v,s)) \to \min,$$

where $X(\mathbf{p},v,s)$ is the solution of the Lyapunov equation

$$AX(\mathbf{p},v,s) + X(\mathbf{p},v,s)A^T = -Z, \tag{39}$$

where A is defined in (21) and $v > 0$ is the viscosity parameter which has to be optimized for a damper $\mathbf{c} = c(\mathbf{p})$ in a given position $\mathbf{p} = k/n$.

If one tries to calm down the first s undamped eigenfrequencies $\omega_1 < \ldots < \omega_s$, the matrix Z_1 is defined as

$$Z_1 = Z_s \oplus Z_s, Z_s = \begin{bmatrix} I_s & 0 \\ 0 & 0_{n-s} \end{bmatrix}.$$

Thus, let

$$v_{\text{opt}} = \operatorname*{argmin}_{v} \operatorname{trace}(X(\mathbf{p},v,s)) \tag{40}$$

be the optimal viscosity obtained for the position $\mathbf{p}$ and given s.

Based on the calculation above, we can state the following theorem.

Theorem 1. *Let $X(\mathbf{p}, v_{\text{opt}}, s)$ be the solution of the Lyapunov equation (39), with optimal viscosity v_{opt} as in (40). Then it holds*

$$\lim_{n\to\infty} \frac{\operatorname{trace}(X(\mathbf{p}, v_{\text{opt}}, s))}{n} = \eta_1(\mathbf{p}, s),$$

where $\eta_1(\mathbf{p}, s)$, is a function that is independent of n.

Proof. Using (26), one can see that

$$\frac{\operatorname{trace}(X(\mathbf{p}, v_{\text{opt}}, s))}{n} = 2\sqrt{\frac{a(n)}{n}\frac{b(n)}{n}}.$$

Now, from (38) and (37) there follows

$$\lim_{n\to\infty} \frac{\operatorname{trace}(X(\mathbf{p}, v_{\text{opt}}, s))}{n} = 2\sqrt{\alpha_1(s,\mathbf{p})\beta_1(s,\mathbf{p})} \doteq \eta_1(\mathbf{p}, s),$$

which completes the proof. □

The main benefit from the result of Theorem 1 is that, for a given set of dominant eigenfrequencies which we try to calm, i.e. for a given matrix Z_1, one can calculate the best position with the corresponding optimal viscosity for some modest dimension n, which then holds for general (large) n. Note that the optimal position $\mathbf{p}$ will change if we change the number of dominant eigenfrequencies s which we try to calm.

All this will be illustrated in the section with numerical examples.

3.4. Optimal trace for the total average displacement

To minimize the total average displacement within the rod or string model, it is necessary to minimize the trace of the solution to the Lyapunov equation. This involves using the same system matrix but with a distinct right-hand side.

Recall that from (12) and (14) it follows that the right-hand side of the Lyapunov equation contains

$$\widehat{K}^{-1} = \Omega^{-1}\Phi^T\Phi\Omega^{-1} = \Omega^{-2}.$$

Here we have used that Φ is an orthogonal matrix.

Thus, the projection matrix Z_Δ will be denoted by $\Delta = 2$, and from (22) and (14) it follows that it is a diagonal given by

$$Z_2 = \operatorname{diag}(\Omega_s^{-2}, 0_{n-s}) \oplus 0_n,$$

where

$$\Omega_s = \operatorname{diag}(\omega_1, \ldots, \omega_s).$$

This configuration enables us (once again) to use a and b from (23) and (26) to determine the optimal trace.

Note if we are interested in calming down s dominant eigenfrequencies $\omega_1, \ldots, \omega_s$, the non-zero diagonal entries of the matrix are given by

$$z_k = \frac{1}{\omega_k^2}, \qquad k = 1, \ldots, s.$$

Since for the string/rod model ω_k is given by

$$\omega_k = 2\sin\frac{\pi k}{2n+2},$$

we see that ω_k^2 has the same rate of convergence as $\dfrac{1}{n^2}$, i.e.

$$\lim_{n\to\infty} n^2\omega_k^2 = k^2\pi^2. \tag{41}$$

Thus, in the case of the minimization of the total average displacement, for n large enough, the optimal trace function divided by n^3 is a function which does not depend on n.

More precisely, from (26), follows

$$\operatorname{trace}(X(\mathbf{p}, v_{\text{opt}}, s)) = 2\sqrt{a_K(n)b_K(n)}, \tag{42}$$

where

$$a_K(n) = \sum_{k=1}^{s} \frac{2}{\omega_k^2 c_k^2},$$
$$b_K(n) = \sum_{k=1}^{s} \frac{c_k^2}{2\omega_k^4} + \frac{1}{\omega_k^2}\left(\sum_{j\neq k}^{n} \frac{2\omega_k^2 c_j^2 + \omega_k^2 c_k^2}{(\omega_k^2-\omega_j^2)^2} + \frac{\omega_k^2}{c_k^2}\left(\sum_{j\neq k}^{n} \frac{c_j^2}{\omega_k^2-\omega_j^2}\right)^2\right).$$

Now, using (31) and (41), it is easy to show

$$\lim_{n\to\infty} \frac{a_K(n)}{n^3} = \sum_{k=1}^{s} \frac{1}{k^2\pi^2} \csc^2(\pi k\mathbf{p}). \tag{43}$$

Further, let b_{K1}, b_{K2} and b_{K3} correspond to b_1, b_2 and b_3 from (27), such that b_{Ki} is obtained from b_i by multiplying its elements from the first sum by $1/\omega_k^2$, $i = 1, 2, 3$.

Now again using (41) and (32), it is easy to get

$$\lim_{n\to\infty} \frac{b_{K1}(n)}{n^3} = \sum_{k=1}^{s} \frac{2\sin^2(\pi k\mathbf{p})}{\pi^4 k^4}. \tag{44}$$

Similarly, using (31) and (33), we get

$$\lim_{n\to\infty} \frac{b_{K2}(n)}{n^3} \leq \sum_{k=1}^{s} \frac{1}{k^2\pi^2}, \tag{45}$$

and using (41) and (35), we have

$$\lim_{n\to\infty} \frac{b_{K3}(n)}{n^3} \leq \sum_{k=1}^{s} \frac{1}{k^2\pi^2} \csc^2(\pi k\mathbf{p}) T(k,\mathbf{p}), \tag{46}$$

where $T(k,\mathbf{p})$ is defined as in (36), i.e.

$$T(k,\mathbf{p}) = \sum_{j=1}^{k} \frac{1}{k^2-j^2} + \frac{1}{2k} H_{2k}.$$

Now, similarly to the previous section from (43), and (44), (45) and (46), we know that limits exist; thus we can write

$$\lim_{n\to\infty} \frac{a(n)}{n^3} = \gamma_1(s,\mathbf{p}), \tag{47}$$

and

$$\lim_{n\to\infty} \frac{b_K(n)}{n^3} = \gamma_2(s,\mathbf{p}). \tag{48}$$

To summarize, for the total average displacement, if one tries to calm down the first s undamped eigenfrequencies $\omega_1 < \ldots < \omega_s$, the matrix Z_2 is defined as

$$Z_2 = \operatorname{diag}(\Omega_s^{-2}, 0_{n-s}) \oplus 0_n, \tag{49}$$

where

$$\Omega_s = \operatorname{diag}(\omega_1, \ldots, \omega_s).$$

Thus, let

$$v_{\text{opt}} = \underset{v}{\operatorname{argmin}} \operatorname{trace}(X(\mathbf{p}, v, s))$$

be the optimal viscosity obtained for the position $\mathbf{p}$ and given s using the total average displacement criterion.

We can state the following theorem.

Theorem 2. *Let $X(\mathbf{p}, v_{\text{opt}}, s)$ be the solution of the Lyapunov equation (39), with Z from (49) and optimal viscosity v_{opt} as in (40). Then it holds*

$$\lim_{n\to\infty} \frac{\operatorname{trace}(X(\mathbf{p}, v_{\text{opt}}, s))}{n^3} = \xi(\mathbf{p}, s),$$

where $\xi(\mathbf{p}, s)$, is a function that is independent of n.

Proof. Using (42), one can see that

$$\frac{\operatorname{trace}(X(\mathbf{p}, v_{\text{opt}}, s))}{n^3} = 2\sqrt{\frac{a_K(n)}{n^3}\frac{b_K(n)}{n^3}}.$$

Now, from (47) and (48) there follows

$$\lim_{n\to\infty} \frac{\operatorname{trace}(X(\mathbf{p}, v_{\text{opt}}, s))}{n^3} = 2\sqrt{\gamma_1(s, \mathbf{p})\gamma_2(s, \mathbf{p})} \doteq \xi(\mathbf{p}, s),$$

which completes the proof. □

Typically, the optimal position achieved through minimizing the total average energy may not align with the optimal position attained through minimizing the total average displacement. However, there are instances where they can be close or even identical. This will be illustrated in the next section with numerical examples.

The main benefit of Theorem 2 is similar to the benefit of Theorem 1, i.e. for a given set of dominant eigenfrequencies which we try to calm, that is for a given matrix Z, one can calculate the best position with the corresponding optimal viscosity for some modest dimension n, which then holds for general (large) n. Note, that the optimal position $\mathbf{p}$ will change if we change the number of dominant eigenfrequencies s which we try to calm.

Remark 1. It is important to note that in the case of string or rod oscillations, the eigenfrequencies and the corresponding eigenvectors do not depend on the dimension n.

For the string, which is explicitly diagonalized with the undamped frequencies

$$\omega_k = k\pi,$$

the orthonormal eigenvectors are given by

$$u_k(x) = \frac{1}{\sqrt{2}} \sin(k\pi x).$$

For the rod, the eigenvectors are also

$$u_k(x) = \frac{1}{\sqrt{2}} \sin(k\pi x),$$

but the eigenfrequencies are

$$\omega_k = k\pi\sqrt{k^2\pi^2 a_0 + k_0}.$$

Since the summations in equations (23) and (24) run from $k = 1$ to $k = s$ for a fixed s, the quantities a and b remain equal for any $n > s$. This is due to the fact that the first s eigenpairs are independent of the total dimension n.

Remark 2. Regarding the relationship between the discretization size s and the accuracy of the numerical solution, a classical engineering example presented in Reddy [18] demonstrates that for linear finite elements applied to the one-dimensional wave equation with fixed ends, the L^2-norm error of the solution converges with order $\mathcal{O}(h^2)$, where $h = 1/n$ is the uniform mesh size. This result is based on the comparison between the analytical eigenmode solution and the finite element approximation using a consistent mass matrix.

Therefore, if we aim for an accuracy of approximately $\varepsilon = 10^{-8}$, it is sufficient to choose the number of elements such that $n > 10000$, under the assumption that $s < n$. This ensures that the number of retained modes remains below the spatial resolution limit imposed by the mesh, maintaining both stability and accuracy of the FEM approximation.

4. Numerical illustration of the main results

As an illustration of the presented results, we will investigate the mechanical system from Section 3, where all masses and stiffness are equal to 1.

Example 3. Consider the mechanical system from Figure 1 with all $m_i = k_i = 1 = k_{n+1}$, $i = 1, \ldots, n$. This means that we are looking for the best position for the damper with optimal viscosity v for the mechanical system described by equation (1), where

$$M = I_n, \qquad K = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

We consider the optimal placing of just one damper for both criteria. In the first part of this section, we will calculate the optimal position for dimensions

$$n \in \{2000, 3000, \ldots, 10000\}.$$

Criterion: the total average energy

We will show that, when damping the first $s = 100$ dominant frequencies $(\omega_1, \omega_2, \ldots, \omega_s)$, the optimal position with respect to the total average energy criterion is

$$\mathbf{p} = 0.495.$$

More precisely, Table 1 shows all optimal positions for all dimensions $n \in \{2000, 3000, \ldots, 10000\}$.

dim (n)	2000	3000	4000	5000	6000	7000	8000	9000	10000
opt. pos.	991	1487	1982	2477	2973	3468	3963	4458	4950

Table 1: Optimal positions for $\omega_1 < \ldots < \omega_{100}$

On the other hand, for the case when one tries to calm down $s = 100$ frequencies starting from 100, i.e. $\omega_{101}, \omega_{102}, \ldots, \omega_{200}$, the matrix Z will be defined as

$$Z_1 = Z_{(100,s)} \oplus Z_{(100,s)} \qquad Z_{(100,s)} = \begin{bmatrix} 0_{100} & & \\ & I_s & \\ & & 0_{(n-s-100)} \end{bmatrix}.$$

The optimal position is:

$$\mathbf{p} = 0.0032.$$

Table 2 below shows all optimal positions for all dimensions $n \in \{2000, 3000, \ldots, 10000\}$.

dim (n)	2000	3000	4000	5000	6000	7000	8000	9000	10000
opt. pos.	6	10	13	16	19	23	26	28	32

Table 2: Optimal positions for $\omega_{101} < \ldots < \omega_{200}$

We note that for the first case, the optimal position is close to the center of the "structure" (or chain) while in the second case, the optimal position is close to one of the edges.

A similar conclusion (or property) holds if we consider the first s dominant frequencies for $s = 20$, i.e. $\omega_1, \omega_2, \ldots, \omega_{20}$. The best position is:

$$\mathbf{p} = 0.4785,$$

while for the eigenfrequencies starting from 100-th $(\omega_{101}, \omega_{102}, \ldots, \omega_{120})$, the best position is:

$$\mathbf{p} = 0.0045.$$

Criterion: the total average displacement

For the case when one tries to calm down the first $s = 100$ dominant frequencies, $(\omega_1, \omega_2, \ldots, \omega_s)$, using the total average displacement, the optimal position is:

$$\mathbf{p} = 0.419,$$

while optimal positions for all for all $n \in \{2000, 3000, \ldots, 10000\}$ can be seen in Table 3.

dim (n)	2000	3000	4000	5000	6000	7000	8000	9000	10000
opt. pos.	838	1260	1658	1940	2481	2893	3359	3779	4199

Table 3: Optimal positions for $\omega_1 < \ldots < \omega_{100}$

On the other hand, for the case when one tries to calm down $s = 100$ frequencies starting from 100, i.e. $(\omega_{101}, \omega_{102}, \ldots, \omega_{200})$, the optimal position is:

$$\mathbf{p} = 0.0034.$$

Table 4 contains all optimal positions for all $n \in \{2000, 3000, \ldots, 10000\}$.

dim (n)	2000	3000	4000	5000	6000	7000	8000	9000	10000
opt. pos.	7	10	14	18	21	24	28	32	34

Table 4: Optimal positions for $\omega_{101} < \ldots < \omega_{200}$

Note that here, similarly to the "energy criterion", the optimal position for the first case, is closer to the center of the "structure" (or chain), although positions do not coincide, while for the second case, the optimal position is close to one of the edges.

Similarly to the "energy criterion", if we consider the first $s = 20$ dominant eigenfrequencies $(\omega_1, \omega_2, \ldots, \omega_{20})$, the optimal position is:

$$\mathbf{p} = 0.3813,$$

while for the eigenfrequencies starting from 100-th $(\omega_{101}, \omega_{102}, \ldots, \omega_{120})$, the optimal position is

$$\mathbf{p} = 0.0045.$$

We can see that for a certain set of dominant eigenfrequencies, each criterion gives a different optimal position. Thus, as mentioned in one of the previous sections, the investigation of properties of both criteria and a mutual comparison will be addressed in our future studies.

Just to illustrate a for mutual comparison of both criteria, let us consider one more example where we will change the range of the dominant eigenfrequencies $\omega_{i+1}, \omega_{i+2}, \ldots, \omega_{i+s}$.

Thus, the tables below contain the optimal position for both criteria for a different set of the dominant eigenfrequencies.

From Tables 5 and 6, one can see that in most of the cases the optimal positions coincide or they are close. The major difference is for the cases when one considers the first part of the spectrum.

$(i+1)-(i+s)$	$1-20$	$1-50$	$1-100$	$101-120$	$101-150$	$201-220$
p; Av. En.	0.479	0.491	0.495	0.0045	0.004	0.0024
p; Av. Dis.	0.381	0.382	0.42	0.0045	0.004	0.0024

Table 5: Comparison between the "energy" and "displacement" criterion

$(i+1)-(i+s)$	$1-5$	$1-10$	$11-15$	$11-20$	$31-35$	$31-40$
p; Av. En.	0.435	0.459	0.0385	0.0317	0.045	0.014
p; Av. Dis.	0.424	0.417	0.039	0.035	0.0455	0.0143

Table 6: Comparison between the "energy" and "displacement" criterion

5. Conclusion and outlook

The paper's main contribution is the study of two distinct optimization criteria for damping optimization in a multi-body oscillator system with arbitrary degrees of freedom (n), which corresponds to the model of string/rod free vibrations. As the first result, we have shown that both criteria are equivalent to the trace minimization of the solution of the Lyapunov equation with different right-hand sides. The second result proves that the minimal trace for each criterion can be expressed as a simple (linear or cubic) function of dimension n. In other words, the optimal position solely depends on the number of dominant eigenfrequencies and the optimal viscosity and does not depend on the dimension n. This highlights a simplified approach to damping optimization in such systems.

Conjectures and future work:

1. Numerical experiments show that similar results about the optimal positions of several dampers hold as well. Namely, the optimal positions of $r \geq 2$ dampers depend only on the number of dominant eigenfrequencies s and their location i ($\omega_{i+1} < \ldots < \omega_{i+s}$) and do not on the dimension n. It is true for both criteria.

We need to prove this, and we intend to tackle this issue in the near future.

2. Study the system of ODEs of the more general structure like the vibration chain of masses and springs shown in Figure 1 with general m_i and k_i.

3. Some general theoretical results about the stability of infinite dimensional systems with a bounded spectrum. For example, a deeper understanding of the spectral properties with one-, two- or multi-dimensional damping.

4. Further study of the properties of both optimization criteria: i) **The total average energy over all possible initial data** and, ii) **The total average displacement over all possible initial data**, especially a comparison between them in the sense of the quality of the solution and the complexity of the calculation.

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