STUDENTSKA RUBRIKA

r-inscribable quadrilaterals

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Abstract. *In this paper we characterize convex quadrilaterals that are inscribable in a rectangle, i.e. they are r-inscribable. We also study the problem of finding the rectangle of minimum area and the one of maximum area, if the convex quadrilateral is r-inscribable.*

Key words: *quadrilaterals*

1. Introduction

In the Book IV of "Elements" Euclid gives the following definitions [3]:

Definition 1. *A rectilinear figure is said to be inscribed in a rectilinear figure when the respective (vertices) angles of the inscribed figure lie on the respective sides of the one in which it is inscribed.*

Definition 2. *A rectilinear figure is said to be circumscribed about a rectilinear figure when the respective sides of the circumscribed figure pass through the respective (vertices) angles of the one about which it is circumscribed.*

Anyway, in Book IV, devoted to the properties of polygons inscribed and circumscribed to a circle and to the construction of regular polygons, and in the whole work, Euclid does not use these definitions and does not study the problems concerning polygons circumscribed to the other polygons. The reason is, according to Artmann ([1]), that the content of the Book IV must have been written by another author and that Euclid inserted it in the "Elements", omitting some parts on topics that he thought were not fundamental.

In [5] an old problem concerning triangles, proposed in [3] and [6] has been studied and completely solved. It had been proved that for any given triangle $\boldsymbol{T}$ the set F of equilateral triangles circumscribed to $\boldsymbol{T}$ is not empty. Moreover, if A, B and C are the vertices of the triangle $\boldsymbol{T}$, such that $|AB| \geq |AC| \geq |BC|$, among the triangles of the set F there exists one of maximum area if and only if the median of the side $\overline{AB}$ with the side $\overline{BC}$ forms an angle smaller than $5\pi/6$.

For convex quadrilaterals we can study an analogous problem. Let $\boldsymbol{Q}$ be a convex quadrilateral. A rectangle $\boldsymbol{R}$ is said to be *circumscribed* to $\boldsymbol{Q}$ if each side of $\boldsymbol{R}$ contains one and only one vertex of $\boldsymbol{Q}$. A quadrilateral is said to be *r-inscribable* if there exists a rectangle that is circumscribed to it (Figure 1).

In this paper we:

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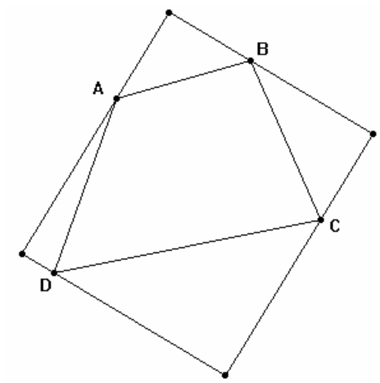

Figure 1.

a) give a necessary and sufficient condition for $\boldsymbol{Q}$ to be r-inscribable (Section 2);

b) when $\boldsymbol{Q}$ is r-inscribable, we prove that there does not exist a circumscribed rectangle to $\boldsymbol{Q}$ of minimum area and we find necessary and sufficient conditions so that there exists one of maximum area (Section 3).

The topic developed in this paper can be useful for teachers because they can get several elementary ideas for teaching. Teachers with the help of software (as Cabrì, Cinderella or Sketchpad, for example), may discover properties that are shown in the paper making the topic more interesting for the students.

2. r-inscribable quadrilaterals

Let $\boldsymbol{Q}$ be a convex quadrilateral with vertices A, B, C, D. In this section we find a necessary and sufficient condition for $\boldsymbol{Q}$ to be r-inscribable.

Let α and β be two consecutive angles of $\boldsymbol{Q}$ so that their sum is greater than or equal to the sum of any two consecutive angles of $\boldsymbol{Q}$, and let $\overline{\mathrm{AB}}$ be their common side so that $\measuredangle\mathrm{BAD} = \alpha$ and $\measuredangle\mathrm{ABC} = \beta$. If $\alpha \geq \beta$, then α is an obtuse angle or a right angle, so we get $\measuredangle\mathrm{BCD} = \gamma \leq \alpha$ and $\measuredangle\mathrm{ADC} = \delta \leq \beta$. If α is a right angle, $\boldsymbol{Q}$ is a rectangle and, of course, it is r-inscribable. Therefore we shall consider only the case when α is an obtuse angle.

Let us prove that: $\boldsymbol{Q}$ *is r-inscribable if and only if* $\alpha + \beta < 3\pi/2$.

If $\boldsymbol{R}$ is a circumscribed rectangle to $\boldsymbol{Q}$, one vertex P of $\boldsymbol{R}$ lies on the semicircle Γ with the diameter $\overline{\mathrm{AB}}$, being external to $\boldsymbol{Q}$. If $\measuredangle\mathrm{PAB} = \alpha'$ and $\measuredangle\mathrm{PBA} = \beta'$, then $\alpha + \alpha' + \beta + \beta' < 2\pi$, and since $\alpha' + \beta' = \pi/2$, we get $\alpha + \beta < 3\pi/2$.

Vice versa, let $\boldsymbol{Q}$ be a convex quadrilateral with $\alpha + \beta < 3\pi/2$.

The point P of semicircle Γ is the vertex of a circumscribed rectangle to $\boldsymbol{Q}$ if and only if the straight lines PA and PB and the perpendiculars to PA and PB passing through C and D respectively, are all external to $\boldsymbol{Q}$.

Let A$'$ and B$'$ be the intersection points of Γ with the straight lines AD and BC, respectively. Let A$''$ and B$''$ be the intersection points of Γ with the perpendiculars to DC passing through A and B, respectively (Figure 2). $\mathrm{A}' \neq \mathrm{A}$ because α is an obtuse angle, while $\mathrm{B}' \neq \mathrm{B}$ if and only if β is an obtuse angle. Moreover, $\mathrm{A}'' = \mathrm{A}$ and $\mathrm{B}'' = \mathrm{B}$ if and only if AB and CD are parallel to each other; if A” $\neq$ A, then $\mathrm{B}'' = \mathrm{B}$ and if $\mathrm{B}'' \neq \mathrm{B}$, then $\mathrm{A}'' = \mathrm{A}$, because the perpendiculars to DC passing through A and B are parallel to each other (Figure 2).

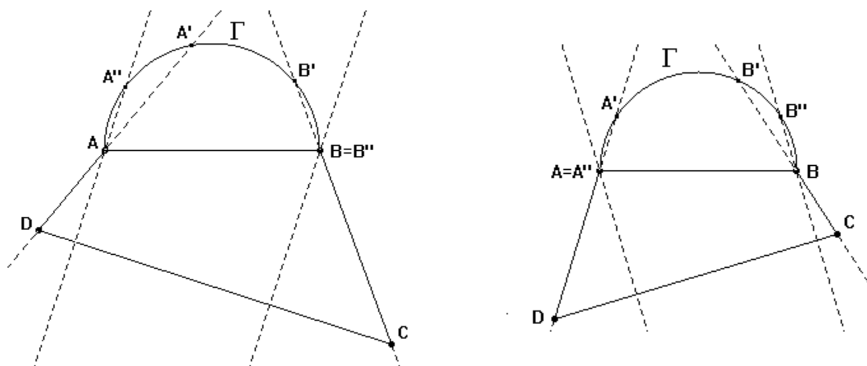

Figure 2.

Observe that the point P of Γ is such that the straight lines PA and PB are external to $\boldsymbol{Q}$ if and only if P does not belong either to the arc AA' or to the arc BB'.

Let us prove that a point P of Γ is such that the perpendicular to PA passing through D and the perpendicular to PB passing through C are external to $\boldsymbol{Q}$ if and only if P does not belong either to the arc AA'' or to the arc BB''.

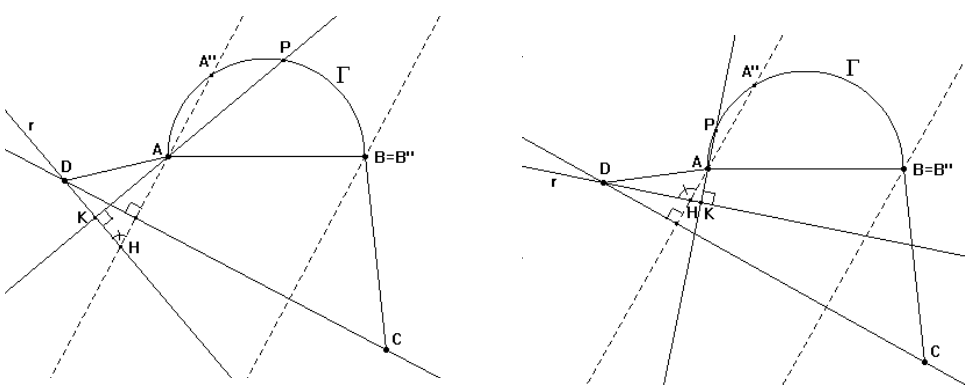

Figure 3. 3a. 3b

Let $\mathrm{A}'' \neq \mathrm{A}$ (the case $\mathrm{B}'' \neq \mathrm{B}$ is analogous to this one), then $\mathrm{B} = \mathrm{B}''$. Let P be a point of Γ, r be perpendicular to AP through D. Let H and K be the intersection points of r with the straight lines AA'' and AP, respectively. If $\mathrm{P} = \mathrm{A}''$, then r coincides with CD and it is not external to $\boldsymbol{Q}$. If $\mathrm{P} \neq \mathrm{A}''$, then H$\neq$K and A, H, K are the vertices of the right triangle, with $\measuredangle\mathrm{AKH} = \pi/2$; then $\measuredangle\mathrm{AHK} < \pi/2$. It follows that if P does not belong to the arc AA$''$, then $\measuredangle\mathrm{DHA} = \measuredangle\mathrm{AHK} < \pi/2$ and r is external to $\boldsymbol{Q}$ (Figure 3a); if P belongs to the arc AA$''$, then $\measuredangle\mathrm{DHA} = \pi - \measuredangle\mathrm{AHK} > \pi/2$ and r is not external to $\boldsymbol{Q}$ (Figure 3b).

Therefore, if $\overline{\Gamma}$ is the arc obtained from Γ without the points of the arcs AA$'$, BB$'$, AA$''$, BB$''$, the point P of Γ is the vertex of the rectangle circumscribed to $\boldsymbol{Q}$ if and only if $\mathrm{P} \in \overline{\Gamma}$ (Figure 4).

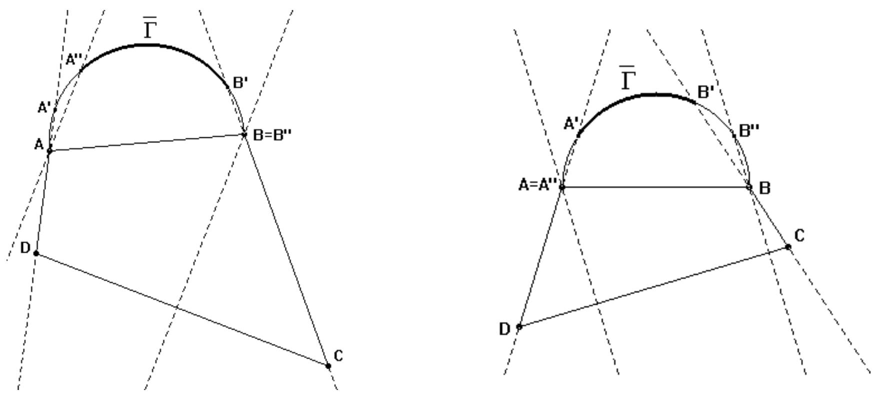

Figure 4.

Observe that $\overline{\Gamma}$ is the locus of the points P of Γ such that:

$$\max\{\measuredangle\mathrm{B'AB}, \measuredangle\mathrm{B''AB}\} < \measuredangle\mathrm{PAB} < \min\{\measuredangle\mathrm{A'AB}, \measuredangle\mathrm{A''AB}\}.$$

It is easy to prove that $\measuredangle\mathrm{A'AB} = \pi - \alpha$, $\measuredangle\mathrm{A''AB} = 3\pi/2 - \alpha - \delta$, $\measuredangle\mathrm{B'AB} = \beta - \pi/2$, $\measuredangle\mathrm{B''AB} = \beta + \gamma - \pi$, where $\measuredangle\mathrm{B'AB} = 0$ if $\beta < \pi/2$ and $\measuredangle\mathrm{B''AB} = 0$ if $\beta + \gamma < \pi$. In the end we have to prove that $\overline{\Gamma}$ is not empty. In fact:

$\beta - \pi/2 < \pi - \alpha$	because	$\alpha + \beta < 3\pi/2$;
$\beta - \pi/2 < 3\pi/2 - \alpha - \delta$	because	$\alpha + \beta + \delta < 2\pi$;
$\beta + \gamma - \pi < \pi - \alpha$	because	$\alpha + \beta + \gamma < 2\pi$;
$\beta + \gamma - \pi < 3\pi/2 - \alpha - \delta$	because	$\alpha + \beta + \gamma + \delta < 5\pi/2$.

Then, the arc $\overline{\Gamma}$ is not empty and it is open. Observe that if $\alpha + \beta < 3\pi/2$ then the sum of any two consecutive angles of $\boldsymbol{Q}$ is less than $3\pi/2$. It follows that:

Theorem 1. *A convex quadrilateral is r-inscribable if and only if the sum of any two of its consecutive angles is less than* $3\pi/2$.

Let us apply this result to a particular types of convex quadrilaterals:

- *Any **parallelogram** is r-inscribable* because the sum of any two consecutive angles of the parallelogram is equal to π;

- *Any quadrilateral with perpendicular diagonals, for example a **rhombus** or a **kite**, is r-inscribable* because the straight lines through the vertices of the quadrilateral and parallel to the diagonals determine a rectangle circumscribed to it.

- *Any* ***right trapezium*** *is r-inscribable* because the sum of the angles on the minor base is less than $3\pi/2$.

- *An* ***isosceles trapezium*** *is r-inscribable if and only if the angle that the minor base forms with an oblique side is less than* $3\pi/4$.

3. Area of circumscribed rectangles to an r-inscribable quadrilateral

Let $\boldsymbol{Q}$ be a convex quadrilateral r-inscribable. We study the problem of the existence of the rectangle of minimum area and the one of maximum area, among the circumscribed rectangles to $\boldsymbol{Q}$.

Let A, B, C, D be the vertices of $\boldsymbol{Q}$ so that the angle in A is bigger than or equal to any other angle of $\boldsymbol{Q}$ and, if O is the intersection point of the diagonals $\overline{\mathrm{AC}}$ and $\overline{\mathrm{BD}}$, the angle $\vartheta = \measuredangle\mathrm{AOB}$ is a right angle or an obtuse angle (Figure 5). Then, if $\alpha = \measuredangle\mathrm{DAB}$, it is $\alpha \geq \pi/2$.

First let us find the area of any rectangle $\boldsymbol{R}$ circumscribed to $\boldsymbol{Q}$.

Let $\mathrm{P_1}$, $\mathrm{P_2}$, $\mathrm{P_3}$, $\mathrm{P_4}$ be the vertices of $\boldsymbol{R}$ and suppose that $\mathrm{P_1}$ lies on the semicircle Γ with the diameter $\overline{\mathrm{AB}}$ being external to $\boldsymbol{Q}$. Moreover, let $\overline{\mathrm{P_1P_4}}$ and $\overline{\mathrm{P_1P_2}}$ be the sides of $\boldsymbol{R}$ that contain the vertices A and B of $\boldsymbol{Q}$, respectively.

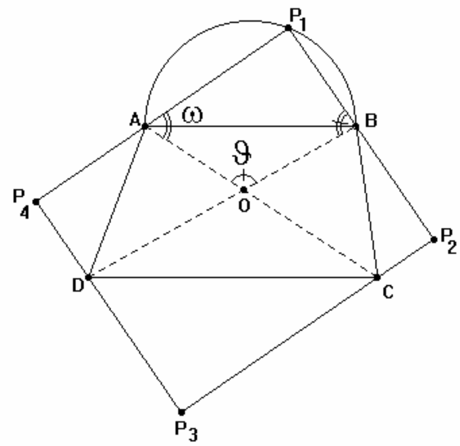

Figure 5.

If $\measuredangle\mathrm{P_1AO} = \omega$, then $\measuredangle\mathrm{P_1BO} = 3\pi/2 - \vartheta - \omega$. Note that when $\boldsymbol{R}$ varies, $\mathrm{P_1}$ describes an open arc Γ_1 of Γ and then ω describes the interval $<\omega', \omega''>$, with $\omega' = \measuredangle\mathrm{P'AO}$, $\omega'' = \measuredangle\mathrm{P''AO}$, and $\mathrm{P'}$, $\mathrm{P''}$ endpoints of Γ_1. Moreover, if $\mathrm{d} = |\mathrm{AC}|$ and $\mathrm{d'} = |\mathrm{BD}|$, then:

$$|\mathrm{P_1P_2}| = \mathrm{d}\sin\omega, \quad |\mathrm{P_1P_4}| = \mathrm{d'}\sin(3\pi/2 - \vartheta - \omega).$$

The area S of the rectangle $\boldsymbol{R}$ is

$$S = \mathrm{dd}' \sin\omega \cdot \sin(3\pi/2 - \vartheta - \omega) = \frac{1}{2}\mathrm{dd}'[\cos(2\omega - 3\pi/2 + \vartheta) - \cos(3\pi/2 - \vartheta)].$$

Now let us see if there exists a rectangle of minimum area and a rectangle of maximum area among the rectangles $\boldsymbol{R}$ circumscribed to $\boldsymbol{Q}$.

Among the rectangles $\boldsymbol{R}$ circumscribed to $\boldsymbol{Q}$ there exists one of minimum area S if and only if there exists $\omega \in <\omega', \omega''>$ such that $\cos(2\omega - 3\pi/2 + \vartheta) = -1$, which implies $\omega = 5\pi/4 - \vartheta/2$. But, since $5\pi/4 - \vartheta/2 \leq \pi$ because $\vartheta \geq \pi/2$, we can say that:

Theorem 2. *Among the rectangles circumscribed to* $\boldsymbol{Q}$ *there does not exist one of minimum area.*

Among the rectangles $\boldsymbol{R}$ circumscribed to $\boldsymbol{Q}$ there exists one of maximum area S if and only if there exists $\omega \in <\omega', \omega''>$ such that $\cos(2\omega - 3\pi/2 + \vartheta) = 1$, which implies $\omega = 3\pi/4 - \vartheta/2$. Note that for $\omega = 3\pi/4 - \vartheta/2$ we have

$$\measuredangle \mathrm{P_1BO} = 3\pi/2 - \vartheta - \omega = 3\pi/4 - \vartheta/2.$$

Then we can say that: Among the rectangles circumscribed to $\boldsymbol{Q}$, the one of maximum area, if it exists, is the one such that $\measuredangle \mathrm{P_1AO} = \measuredangle \mathrm{P_1BO} = 3\pi/4 - \vartheta/2$.

Observe that for $\omega = 3\pi/4 - \vartheta/2$ it is $|\mathrm{P_1P_2}| = \mathrm{d}\sin(3\pi/4 - \vartheta/2)$ and $|\mathrm{P_1P_4}| = \mathrm{d}'\sin(3\pi/4 - \vartheta/2)$, then $|\mathrm{P_1P_2}| = |\mathrm{P_1P_4}|$ if and only if $\mathrm{d} = \mathrm{d}'$. Therefore:

Theorem 3. *The rectangle circumscribed to* $\boldsymbol{Q}$ *of maximum area, if it exists, is a square if and only if* $\boldsymbol{Q}$ *is a quadrilateral with equal diagonals.*

Quadrilaterals with equal diagonals are, for example, the rectangles and the isosceles trapeziums.

Let us now try to construct the rectangle, which we'll denote with $\overline{R}$, such that $\measuredangle \mathrm{P_1AO} = \measuredangle \mathrm{P_1BO} = 3\pi/4 - \vartheta/2$.

There exists the point $\mathrm{P_1}$ of Γ, different from A and B, such that $\measuredangle \mathrm{P_1AO} = \measuredangle \mathrm{P_1BO} = 3\pi/4 - \vartheta/2$, if and only if $0 < 3\pi/4 - \vartheta/2 - \measuredangle \mathrm{BAC} < \pi/2$, i.e. if and only if:

$$\pi/4 - \vartheta/2 < \measuredangle \mathrm{BAC} < 3\pi/4 - \vartheta/2.$$

Since $\vartheta \geq \pi/2$, it is $\pi/4 - \vartheta/2 \leq 0$ and $3\pi/4 - \vartheta/2 \leq \pi/2$, if ϑ is a right angle the previous relation is still true, while if ϑ is an obtuse angle, the previous relation becomes:

$$\measuredangle \mathrm{BAC} < 3\pi/4 - \vartheta/2. \tag{1}$$

Then, *it is possible to construct the rectangle* $\overline{\boldsymbol{R}}$ *if and only if* ϑ *is a right angle or* ϑ *is an obtuse angle and (1) is true.*

If (1) is true, $\mathrm{P_1}$ is the intersection point of Γ with the ray with the origin in A, that lies on the half-plane with the origin in the straight line AB and not containing $\mathbf{Q}$, such that $\measuredangle \mathrm{P_1AO} = 3\pi/4 - \theta/2$. Moreover, $\overline{\mathbf{R}}$ is constructed drawing the perpendicular to $\mathrm{P_1B}$ through C, the perpendicular to $\mathrm{P_1A}$ through D, and determining the remaining vertices. Let $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$ be the sides of $\overline{\mathbf{R}}$ that contain the points B, C, D, respectively.

Let (1) be true. Let us now see if $\overline{\mathbf{R}}$ is circumscribed to $\boldsymbol{Q}$. We know that $\overline{\mathbf{R}}$ is circumscribed to $\mathbf{Q}$ if and only if the following straight lines are external to $\boldsymbol{Q}$: P_1A and P_1B; the perpendicular to P_1A passing through D and the perpendicular to P_1B passing through C.

Let us prove that P_1A and P_1B are external to $\mathbf{Q}$ if and only if ϑ is a right angle or ϑ is an obtuse angle and the following conditions are true:

$$\measuredangle ACB > \vartheta/2 - \pi/4 \tag{2}$$

$$\measuredangle ADB > \vartheta/2 - \pi/4. \tag{3}$$

In fact, if ϑ is a right angle, we have $\measuredangle P_1AO = \measuredangle P_1BO = \pi/2$; then P_1A and P_1B are respectively parallel to BD and AC and they are external to $\boldsymbol{Q}$.

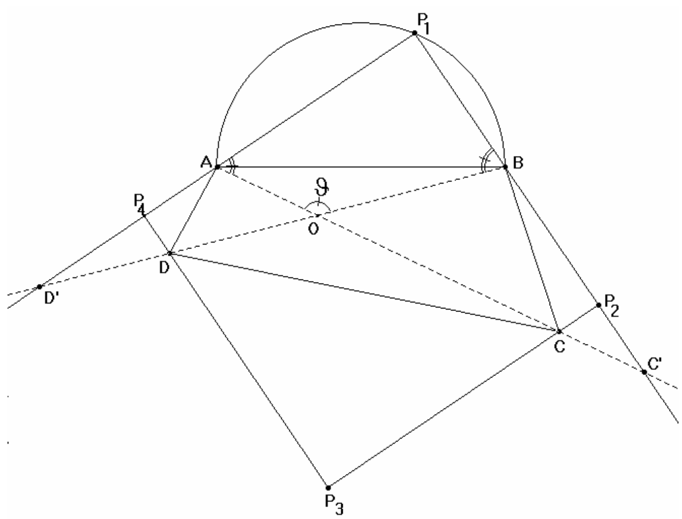

Figure 6.

If ϑ is an obtuse angle, the straight lines P_1A and BD have the common point D′ on the half-plane with the origin AB and containing $\boldsymbol{Q}$ because they form, with P_1B, the angles $\measuredangle BP_1A = \pi/2$ and $\measuredangle DBP_1 = 3\pi/4 - \vartheta/2 < \pi/2$ whose sum is less than π; analogously, P_1B and AC have their common point C′ on the half-plane with the origin AB containing $\boldsymbol{Q}$ (Figure 6). Since $\measuredangle P_1D'B = \measuredangle P_1C'A = \vartheta/2 - \pi/4$, P_1A and P_1B are external to $\boldsymbol{Q}$ if and only if the conditions (2) and (3) are true.

Let us consider now the straight line P_3P_4 passing trough D and perpendicular to P_1A. Let us prove that it is external to $\boldsymbol{Q}$.

In fact, P_3P_4 is external to $\boldsymbol{Q}$ if and only if $\measuredangle BDC < \measuredangle BDP_3$ and $\measuredangle DAP_4$ is an acute angle. Since $\measuredangle BDP_3 = \measuredangle DBP_1 = 3\pi/4 - \vartheta/2$, it is
$\measuredangle BDC < \measuredangle BDP_3$ if and only if $\measuredangle BDC < 3\pi/4 - \vartheta/2$. But, since
$\measuredangle ACD > \pi/4 - \vartheta/2$ because $\pi/4 - \vartheta/2 \leq 0$, then

$\measuredangle \mathrm{BDC} = \pi - \vartheta - \measuredangle \mathrm{ACD} < 3\pi/4 - \vartheta/2 = \measuredangle \mathrm{BDP}_3$. Moreover, $\measuredangle \mathrm{DAP}_4$ is an acute angle because $\measuredangle \mathrm{DAP}_1$ is an obtuse angle, since $\alpha \geq \pi/2$. Then $\mathrm{P}_3\mathrm{P}_4$ is external to $\boldsymbol{Q}$.

Finally, let us consider the straight line $\mathrm{P}_2\mathrm{P}_3$ passing through C and perpendicular to $\mathrm{P}_1\mathrm{B}$. Let us prove that it is external to $\boldsymbol{Q}$ if and only if:

$$\measuredangle \mathrm{ACB} < \vartheta/2 + \pi/4. \tag{4}$$

In fact, $\mathrm{P}_2\mathrm{P}_3$ is external to $\boldsymbol{Q}$ if and only if $\measuredangle \mathrm{ACD} < \measuredangle \mathrm{ACP}_3$ and $\measuredangle \mathrm{CBP}_2$ is an acute angle. Since $\measuredangle \mathrm{ACP}_3 = \measuredangle \mathrm{CAP}_1 = 3\pi/4 - \vartheta/2$, we have $\measuredangle \mathrm{ACD} < \measuredangle \mathrm{ACP}_3$ if and only if $\measuredangle \mathrm{ACD} < 3\pi/4 - \vartheta/2$. But, since $\measuredangle \mathrm{BDC} > \pi/4 - \vartheta/2$ because $\pi/4 - \vartheta/2 \leq 0$, then $\measuredangle \mathrm{ACD} = \pi - \vartheta - \measuredangle \mathrm{BDC} < 3\pi/4 - \vartheta/2 = \measuredangle \mathrm{ACP}_3$. Moreover, $\measuredangle \mathrm{CBP}_2$ is an acute angle if $\measuredangle \mathrm{P}_1\mathrm{BC}$ is an obtuse angle. But in the quadrilateral $\mathrm{P}_1\mathrm{BCA}$ we have $\measuredangle \mathrm{P}_1\mathrm{BC} + \pi/2 + 3\pi/4 - \vartheta/2 + \measuredangle \mathrm{ACB} = 2\pi$, which implies $\measuredangle \mathrm{P}_1\mathrm{BC} = 3\pi/4 + \vartheta/2 - \measuredangle \mathrm{ACB}$; then $\measuredangle \mathrm{P}_1\mathrm{BC} > \pi/2$ if and only if $3\pi/4 + \vartheta/2 - \measuredangle \mathrm{ACB} > \pi/2$, from where follows $\measuredangle \mathrm{ACB} < \vartheta/2 + \pi/4$.

It follows that $\mathrm{P}_2\mathrm{P}_3$ is external to $\boldsymbol{Q}$ if and only if (4) is true.

Observe that if ϑ is a right angle the condition (4) is true.

Then we can conclude that *the rectangle* $\overline{\boldsymbol{R}}$ *is circumscribed to* $\boldsymbol{Q}$ *if and only if* ϑ *is a right angle, or* ϑ *is an obtuse angle and the conditions* (2), (3), (4) *are true.*

So we have proved:

Theorem 4. *If the convex quadrilateral* $\boldsymbol{Q}$ *is r-inscribable (that is if the sum of any two consecutive angles of* $\boldsymbol{Q}$ *is less than* $3\pi/2$*), then there exists a rectangle of maximum area among those circumscribed to* $\boldsymbol{Q}$ *if and only if* ϑ *is a right angle or* ϑ *is an obtuse angle and the following conditions are true:*

$$\measuredangle \mathrm{BAC} < 3\pi/4 - \vartheta/2 \tag{5}$$

$$\measuredangle \mathrm{ACB} > \vartheta/2 - \pi/4 \tag{6}$$

$$\measuredangle \mathrm{ADB} > \vartheta/2 - \pi/4 \tag{7}$$

$$\measuredangle \mathrm{ACB} < \vartheta/2 + \pi/4. \tag{8}$$

Let us apply this result to particular cases of convex quadrilaterals.

- Let $\boldsymbol{Q}$ be a ***quadrilateral with perpendicular diagonals***. We know that $\boldsymbol{Q}$ is r-inscribable. Because of the previous theorem we can state that $\overline{\mathbf{R}}$ is the rectangle of maximum area circumscribed to $\boldsymbol{Q}$. In particular, this property is true for ***rhombi*** and ***kites***.

- Let $\boldsymbol{Q}$ be an ***isosceles trapezium***. We know that $\boldsymbol{Q}$ is r-inscribable if and only if $\alpha = \measuredangle \mathrm{DAB} < 3\pi/4$. Let us prove that if $\boldsymbol{Q}$ *is r-inscribable, then* $\overline{\boldsymbol{R}}$ *is the rectangle of maximum area that is circumscribed to it* (Figure 7b).

In fact, if $\measuredangle \mathrm{BAC} = \measuredangle \mathrm{DBA} = \pi/2 - \vartheta/2 < 3\pi/4 - \vartheta/2$, then (5) is true. Moreover, if $\measuredangle \mathrm{ACB} < \pi/2 < \vartheta/2 + \pi/4$, then (8) is also true. Since $\measuredangle \mathrm{ADB} = \measuredangle \mathrm{ACB} = \pi - \measuredangle \mathrm{BAC} - \measuredangle \mathrm{ABC}$, we have:

$\measuredangle \mathrm{ADB} = \pi/2 + \vartheta/2 - \alpha > \pi/2 + \vartheta/2 - 3\pi/4 = \vartheta/2 - \pi/4$,

and then (6) and (7) are also true.

Observe that the vertex P_1 of $\overline{\mathbf{R}}$ is such that

$$\measuredangle \mathrm{P_1AB} = \measuredangle \mathrm{P_1AO} - \measuredangle \mathrm{BAC} = 3\pi/4 - \vartheta/2 - (\pi/2 - \vartheta/2) = \pi/4,$$

and then it is the intersection point of Γ with the perpendicular bisector of the segment $\overline{\mathrm{AB}}$. Moreover, since an isosceles trapezium has equal diagonals, we can state that $\overline{\mathbf{R}}$ is a square.

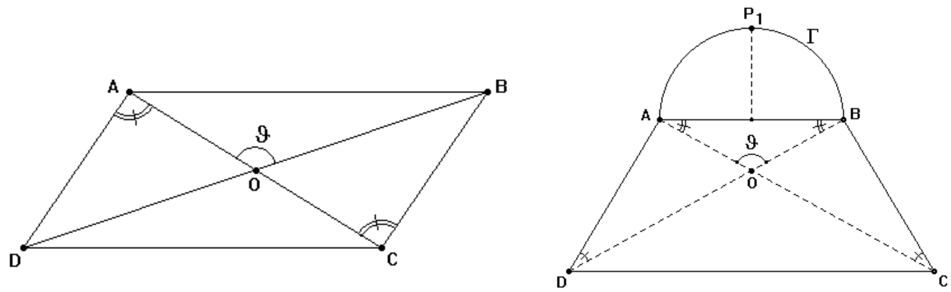

Figure 7. 7a. 7b

- Let $\boldsymbol{Q}$ be a parallelogram (Figure 7a). We know that $\boldsymbol{Q}$ is r-inscribable. Let us prove that $\overline{\mathbf{R}}$ *is the circumscribed rectangle to Q of maximum area if and only if*

$$\vartheta/2 - \pi/4 < \measuredangle \mathrm{DAC} < \vartheta/2 + \pi/4. \tag{9}$$

In fact, if $\measuredangle \mathrm{ACB} = \measuredangle \mathrm{DAC}$ and $\measuredangle \mathrm{ADB} = \vartheta - \measuredangle \mathrm{DAC}$, then $\measuredangle \mathrm{ADB} > \vartheta/2 - \pi/4$ if and only if $\measuredangle \mathrm{DAC} < \vartheta/2 + \pi/4$. Thus (8) is true if and only if (7) is true. Moreover, if $\pi/2 < \measuredangle \mathrm{DAB} = \measuredangle \mathrm{DAC} + \measuredangle \mathrm{BAC}$, then
$\measuredangle \mathrm{BAC} < 3\pi/4 - \vartheta/2$ if and only if $\measuredangle \mathrm{DAC} > \vartheta/2 - \pi/4$, i.e. (5) is true if and only if (6) is true.

Then, *there exists a rectangle of maximum area among those circumscribed to* $\boldsymbol{Q}$ *if and only if (9) is true.*

Note that *if* $\boldsymbol{Q}$ *is a rectangle, there exists a rectangle of maximum area among those circumscribed to* $\boldsymbol{Q}$, because $\measuredangle \mathrm{DAC} = \vartheta/2$ and then (9) is true.

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