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ON COMMUTATIVITY OF σ-PRIME RINGS

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Abstract. Let R be a 2-torsion free σ-prime ring having a σ-square closed Lie ideal U and an automorphism T centralizing on U. We prove that if there exists u_0 in $Sa_\sigma(R)$ with $Ru_0 \subset U$ and if T commutes with σ on U, then U is contained in the center of R. This result is then applied to generalize the result of J. Mayne for centralizing automorphisms to σ-prime rings. Finally, for a 2-torsion free σ-prime ring possessing a nonzero derivation, we give suitable conditions under which the ring must be commutative.

1. Introduction

A linear mapping T from a ring to itself is called centralizing on a subset S of the ring if $[x, T(x)]$ is in the center of the ring for every x in S. In particular, if T satisfies $[x, T(x)] = 0$ for all x in S then T is called commuting on S. In [6] Posner showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative. In [2] the same result is proved for a prime ring with a nontrivial centralizing automorphism. A number of authors have generalized these results by considering mappings which are only assumed to be centralizing on an appropriate ideal of the ring. In [1] Awtar considered centralizing derivations on Lie and Jordan ideals. In the Jordan case, he proved that if a prime ring of characteristic not two has a nontrivial derivation which is centralizing on a Jordan ideal, then the ideal must be contained in the center of the ring. This result is extended in [3] where it is shown that if R is any prime ring with a nontrivial centralizing automorphism or derivation on a nonzero ideal or (quadratic) Jordan ideal,

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then R is commutative. For prime rings Mayne, in [4], also showed that a nontrivial automorphism which is centralizing on a Lie ideal implies that the ideal is contained in the center if the ring is not of characteristic two. In this paper, the corresponding result for σ-prime rings with σ-square closed Lie ideals is proved, where σ is an involution, Theorem 2.4. An immediate consequence of Theorem 2.4 and the fact that a σ-ideal is a $\sigma-$square closed Lie ideal is Theorem 2.5 which extends the result of [3] for centralizing automorphisms to σ-prime rings of characteristic not two. To end this paper, for a 2-torsion free σ-prime ring having a nonzero derivation we give suitable conditions under which the ring must be commutative, Theorem 3.2 and Theorem 3.3.

Throughout, R will represent an associative ring with center $Z(R)$. We say R is 2-torsion free if for $x \in R$, $2x = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$. An involution σ of a ring R is an anti-automorphism of order 2 (i.e. σ is an additive mapping satisfying $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$ for all $x, y \in R$). If R is equipped with an involution σ, we set $Sa_\sigma(R) := \{r \text{ in } R \text{ such that } \sigma(r) = \pm r\}$. Recall that R is σ-prime if $aRb = aR\sigma(b) = 0$ implies that either $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$. A Lie ideal U of R is called a square closed Lie ideal if $u^2 \in U$ for all $u \in U$ and a σ-square closed Lie ideal if U is invariant under σ. The fact that $(u + v)^2 \in U$ together with $[u, v] \in U$ yield $2uv \in U$ for all $u, v \in U$.

2. Automorphisms centralizing on σ-square closed Lie ideals

Throughout this section R will denote a 2-torsion free σ-prime ring, where σ is an involution of R.

Lemma 2.1. *If T is an homomorphism of R which is centralizing on a square closed Lie ideal U, then T is commuting on U.*

Proof. By linearization $[x, T(y)] + [y, T(x)]$ is in $Z(R)$ for all x and y in U. Thus, $[x, T(x^2)] + [x^2, T(x)]$ in $Z(R)$ and therefore

$$T(x)[x, T(x)] + [x, T(x)]x + x[x, T(x)] + [x, T(x)]x = 2(x + T(x))[x, T(x)]$$

in $Z(R)$. Since R is 2-torsion free, then

$$(x + T(x))[x, T(x)] \text{ in } Z(R).$$

For r in R, we then get

$$r(x + T(x))[x, T(x)] = (x + T(x))[x, T(x)]r = (x + T(x))r[x, T(x)].$$

Hence $[r, x + T(x)][x, T(x)] = 0$ for all r in R. In particular,

$$0 = [x, x + T(x)][x, T(x)] = [x, T(x)]^2.$$

Since $[x, T(x)]$ in $Z(R)$, then

$$[x, T(x)]R[x, T(x)] = 0.$$

Therefore,

$$[x, T(x)]R[x, T(x)]\sigma([x, T(x)]) = 0$$

and since $[x, T(x)]\sigma([x, T(x)])$ is invariant under σ, the σ-primeness of R yields $[x, T(x)] = 0$ or $[x, T(x)]\sigma([x, T(x)]) = 0$. If $[x, T(x)]\sigma([x, T(x)]) = 0$ then

$$[x, T(x)]R\sigma([x, T(x)]) = 0, \text{ because } [x, T(x)] \in Z(R)$$

and consequently

$$[x, T(x)]R[x, T(x)] = [x, T(x)]R\sigma([x, T(x)]) = 0.$$

Once again using the σ-primeness of R, we then get $[x, T(x)] = 0$ for all x in U, hence T is commuting on U. $\square$

From now on assume that T is an *automorphism centralizing on a σ-square closed Lie ideal U which contains an element u_0 in $Sa_\sigma(R)$ such that* $Ru_0 \subset U$. Since T is centralizing on U, Lemma 2.1 implies $[x, T(x)] = 0$ for all x in U.

Lemma 2.2. *If a, b in R are such that $aUb = aU\sigma(b) = 0$, then $a = 0$ or $b = 0$.*

Proof. Suppose $a \neq 0$. We have to distinguish two cases:

1) u_0 in $Z(R)$. Let r in R. From $aru_0b = aru_0\sigma(b) = 0$ it follows that

$$aRu_0b = aRu_0\sigma(b) = aR\sigma(u_0b) = 0$$

so that $u_0b = 0$. Since u_0 is central, then $u_0Rb = \sigma(u_0)Rb = 0$ proving $b = 0$.

2) $u_0 \notin Z(R)$. If $a[t, u_0] = 0$ for all t in R, then

$$a[tr, u_0] = at[r, u_0] = 0 \text{ so that } aR[r, u_0] = 0 = aR\sigma([r, u_0])$$

proving $[r, u_0] = 0$ for all r in R which contradicts $u_0 \notin Z(R)$. Thus there exists t in R such that $a[t, u_0] \neq 0$. From

$$a[t, u_0]rb = a[t, u_0]r\sigma(b) = 0,$$

it follows that

$$a[t, u_0]Rb = a[t, u_0]R\sigma(b) = 0$$

and the σ-primeness of R yields $b = 0$. $\square$

Lemma 2.3. *Suppose that T commutes with σ on U. If x in $U \cap Sa_\sigma(R)$ satisfies $T(x) \neq x$, then x in $Z(R)$.*

Proof. Let x in $U \cap Sa_\sigma(R)$ with $T(x) \neq x$. From $[t, T(t)] = 0$, for all t in U, we conclude $[t, T(y)] = [T(t), y]$ for all t, y in U. In particular $[x, T(2xy)] = [T(x), 2xy]$, because $2xy$ in U. Since R is 2-torsion free, thus

$$T(x)[x, T(y)] - x[T(x), y] = 0$$

and therefore

$$(T(x) - x)[T(x), y] = 0 \text{ for all } y \text{ in } U.$$

For u in U, as $2uy$ in U and once again using the fact that R is 2-torsion free we obtain

$$0 = (x - T(x))[T(x), uy] = (x - T(x))u[T(x), y].$$

Hence

$$(x - T(x))U[T(x), y] = (x - T(x))U\sigma([T(x), y]) = 0.$$

Applying Lemma 2.2, since $T(x) \neq x$, then $[T(x), y] = 0$ for all y in U. Whence

$$[T(x), tru_0] = [T(x), t]ru_0 = 0 \quad \text{ for all } r, t \text{ in } R.$$

Thus $[T(x), t]Ru_0 = 0$, which proves $[T(x), t] = 0$ so that $T(x)$ in $Z(R)$. Since T is an automorphism then x in $Z(R)$. □

Theorem 2.4. *Let R be a 2-torsion free σ-prime ring having an automorphism $T \neq 1$ centralizing on a $\sigma-$square closed Lie ideal U. If T commutes with σ on U and there exists u_0 in $Sa_\sigma(R)$ with $Ru_0 \subset U$, then U is contained in $Z(R)$.*

Proof. Suppose that T is identity on U, hence for all t, r in R we then get

$$T(tru_0) = tru_0 = T(t)T(ru_0) = T(t)ru_0.$$

Thus

$$(T(t) - t)ru_0 = 0 \text{ so that } (T(t) - t)Ru_0 = 0.$$

Since R is σ-prime this yields $T(t) = t$ for all t in R which is impossible. Thus T is nontrivial on U. Since R is 2-torsion free, the fact that $x + \sigma(x)$ and $x - \sigma(x)$ are in $U \cap Sa_\sigma(R)$ for all x in U assures that T is nontrivial on $U \cap Sa_\sigma(R)$. Therefore, there must be an element x in $U \cap Sa_\sigma(R)$ such that $x \neq T(x)$ and x is then in $Z(R)$ by Lemma 2.3. Let $0 \neq y$ be in $U \cap Sa_\sigma(R)$ and not be in $Z(R)$. Once again using Lemma 2.3, we obtain $T(y) = y$. But then

$$T(xy) = T(x)y = xy \text{ so that } (T(x) - x)y = 0$$

and therefore

$$(T(x) - x)Ry = (T(x) - x)R\sigma(y) = 0, \text{ because } x \text{ in } Z(R).$$

As R is σ-prime this yields $y = 0$. Hence for all y in $U \cap Sa_\sigma(R)$, y must be in $Z(R)$. Now let x in U. The fact that $x - \sigma(x)$ and $x + \sigma(x)$ are elements in $U \cap Sa_\sigma(R)$ gives $x - \sigma(x)$ and $x + \sigma(x)$ in $Z(R)$ and thus $2x$ in $Z(R)$. Consequently, x in $Z(R)$ which proves $U \subset Z(R)$. □

In [3] it is proved that if a prime ring has a nontrivial automorphism which centralizes on a nonzero ideal, then the ring is commutative. The purpose of the following theorem is to generalize this result to σ-prime rings with characteristic not two.

THEOREM 2.5. *Let R be a 2-torsion free σ-prime ring having an automorphism $T \neq 1$ which commutes with σ on a nonzero σ-ideal J of R. If T is centralizing on J, then R is a commutative ring.*

PROOF. Since a σ-ideal is a σ-square closed Lie ideal, from Theorem 2.4 it follows that J is contained in $Z(R)$. Now, if $x^2 = 0$ for all $x \in J$, then $(\sigma(x) + x)^2 = 0$. As $\sigma(x) + x$ is invariant under σ, the fact that $(\sigma(x) + x)R(\sigma(x) + x) = 0$ together with the σ-primeness of R yield $\sigma(x) = -x$. But $x^2 = 0$ implies $xRx = 0$ so that $x = 0$ which contradicts $J \neq 0$. Thus there exists an element $x \in J$ such that $x^2 \neq 0$. For all $r, s \in R$, we have

$$x^2 rs = x(xr)s = xrxs = x(rx)s = rxxs = xsrx = x^2 sr.$$

Hence $x^2(rs - sr) = 0$ so that $x^2R[r, s] = 0$ and similarly $x^2R\sigma([r, s]) = 0$. Since $x^2 \neq 0$, the σ-primeness of R gives $[r, s] = 0$ for all $r, s \in R$, proving the commutativity of R. ∎

3. DERIVATIONS IN σ-PRIME RINGS

Let R be a 2-torsion free σ-prime ring and let d be a nonzero derivation on R. Our aim in this section is to give suitable conditions under which the ring R must be commutative. We will make frequent and important uses of the following lemma.

LEMMA 3.1 ([5], 3) of Theorem 1). *Let I be a nonzero σ-ideal of R. If a, b in R are such that $aIb = 0 = aI\sigma(b)$, then $a = 0$ or $b = 0$.*

PROOF. Suppose $a \neq 0$, there exists some $x \in I$ such that $ax \neq 0$. Indeed, otherwise

$$aRx = 0 \text{ and } aR\sigma(x) = 0 \text{ for all } x \in I$$

and therefore $a = 0$. Since $aIRb = 0$ and $aIR\sigma(b) = 0$, we then obtain

$$axRb = axR\sigma(b) = 0.$$

In view of the σ-primeness of R this yields $b = 0$. ∎

THEOREM 3.2. *Let $0 \neq d$ be a derivation of R and let I be a nonzero σ-ideal of R. If r in $Sa_\sigma(R)$ satisfies $[d(x), r] = 0$ for all x in I, then r in $Z(R)$. Furthermore, if $d(I) \subset Z(R)$, then R is commutative.*

PROOF. Since $[d(uv), r] = 0$ for all u, v in I, it follows that

$$d(u)vr + ud(v)r - rd(u)v - rud(v) = 0.$$

Using $[d(u), r] = [d(v), r] = 0$, we obtain

$$d(u)[v, r] + [u, r]d(v) = 0 \quad \text{for all} \;\; u, v \in I. \tag{3.1}$$

Replacing v by vr in (3.1), we conclude that $[u, r]Id(r) = 0$. The fact that I is a σ-ideal together with r in $Sa_\sigma(R)$, give

$$\sigma([u, r])Id(r) = [u, r]Id(r) = 0.$$

Applying Lemma 3.1, either $d(r) = 0$ or $[u, r] = 0$. If $d(r) \neq 0$, then $[u, r] = 0$ for all u in I. Let t in R, from $[tu, r] = 0$ it follows that $[t, r]u = 0$. Let $0 \neq x_0$ in I, as

$$[t, r]Rx_0 = [t, r]R\sigma(x_0) = 0$$

then $[t, r] = 0$, since R is σ-prime, which proves r in $Z(R)$.

Now if $d(r) = 0$, then $d([u, r]) = [d(u), r] = 0$ and consequently

$$d([I, r]) = 0. \tag{3.2}$$

Replace v by $v\omega$ in (3.1), where ω in I, we have

$$d(u)v[\omega, r] + [u, r]vd(\omega) = 0. \tag{3.3}$$

Taking $[\omega, r]$ instead of ω in (3.3) and applying (3.2) we then get

$$d(u)v[[\omega, r], r] = 0 \quad \text{so that} \quad d(u)I[[\omega, r], r] = 0 = d(u)I\sigma([[\omega, r], r])$$

whence $d(I) = 0$ or $[[\omega, r], r] = 0$ for all ω in I, by Lemma 3.1.

If $d(I) = 0$, then for any t in R we get $d(tu) = d(t)u = 0$ for all u in I. Therefore

$$d(t)RI = d(t)R\sigma(I) = 0$$

and as $0 \neq I$, then $d(t) = 0$ in such a way that $d = 0$. Consequently,

$$[[\omega, r], r] = 0. \tag{3.4}$$

Replace ω by ωu in (3.4) we obtain

$$0 = [[\omega u, r], r] = [\omega, r][u, r] + [\omega, r][u, r]$$

in such a way that $[\omega, r][u, r] = 0$, because R is 2-torsion free. Hence

$$0 = [t\omega, r][u, r] = [t, r]\omega[u, r]$$

and consequently

$$[t, r]I[u, r] = 0 \;\; \text{for all} \;\; u \;\; \text{in} \;\; I.$$

Therefore

$$[t, r]I[u, r] = [t, r]I\sigma([u, r]) = 0,$$

once again using Lemma 3.1, we see that $[t, r] = 0$ or $[u, r] = 0$. If $[t, r] = 0$, then r in $Z(R)$. If $[u, r] = 0$ for all u in I, then for any $t \in R$

$$0 = [tu, r] = t[u, r] + [t, r]u = [t, r]u.$$

Hence

$$0 = [t, r]I = [t, r]I1 = [t, r]I\sigma(1).$$

Using Lemma 3.1 we conclude that $[t, r] = 0$, which proves that r in $Z(R)$.

Now suppose that $d(I) \subset Z(R)$ and let r in R. From the first part of the theorem we conclude $Sa_\sigma(R) \subset Z(R)$. Using the fact that

$$r + \sigma(r) \text{ and } r - \sigma(r) \text{ are elements of } Sa_\sigma(R)$$

we then obtain

$$r - \sigma(r) \in Z(R) \text{ and } r + \sigma(r) \in Z(R)$$

and hence $2r$ in $Z(R)$. Since R is 2-torsion free, then r in $Z(R)$ proving the commutativity of R. □

THEOREM 3.3. *Let d be a nonzero derivation of R and let a in $Sa_\sigma(R)$. If $d([R, a]) = 0$, then a in $Z(R)$. In particular, if $d(xy) - d(yx) = 0$, for all $x, y \in R$, then R is a commutative ring.*

PROOF. If $d(a) = 0$, from our hypothesis, we have for any r in R,

$$0 = d([r, a]) = d(r)a + rd(a) - d(a)r - ad(r) = d(r)a - ad(r) = [d(r), a].$$

Therefore

$$[d(r), a] = 0 \text{ for all } r \text{ in } R.$$

Applying Theorem 3.2, this yields a in $Z(R)$ and the proof is then complete. Now, assume that $d(a) \neq 0$. For all r in R,

$$0 = d([ar, a]) = d(a[r, a]) = d(a)[r, a] + ad([r, a])$$

and so,

$$d(a)[r, a] = 0. \tag{3.5}$$

Taking rs, s in R instead of r in (3.5), we obtain

$$0 = d(a)[rs, a] = d(a)r[s, a] + d(a)[r, a]s.$$

Using (3.5), this yields $d(a)r[s, a] = 0$ so that

$$d(a)R[s, a] = 0 \text{ for all } s \text{ in } R.$$

Since a in $Sa_\sigma(R)$, then

$$0 = d(a)R[s, a] = d(a)R\sigma([s, a])$$

and the σ-primeness of R yields $[s, a] = 0$ which proves a in $Z(R)$.

Now, assume that $d([x, y]) = 0$ for all $x, y \in R$. Applying the first part of our theorem, we then get $Sa_\sigma(R) \subset Z(R)$. For r in R, the fact that

$$r + \sigma(r) \text{ and } r - \sigma(r) \text{ are elements of } Sa_\sigma(R),$$

yields $2r$ in $Z(R)$. Since R is 2-torsion free, this yields r in $Z(R)$ which proves that R is a commutative ring. □

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