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A CLASS OF NONABELIAN NONMETACYCLIC FINITE 2-GROUPS

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Abstract. Nonabelian nonmetacyclic finite 2-groups in which every proper subgroup is abelian or metacyclic and possessing at least one nonabelian and at least one nonmetacyclic proper subgroup have been investigated and classified. Using the obtained result and two previously known results one gets the complete classification of all nonabelian nonmetacyclic finite 2-groups in which every proper subgroup is abelian or metacyclic.

1. Introduction and preliminaries

The aim of this article is to prove the following

Theorem. *Let G be a nonabelian non-metacyclic finite 2-group with all proper subgroups being abelian or metacyclic and possessing at least one nonabelian and at least one nonmetacyclic proper subgroup. Then G is isomorphic to some of the groups:*

$$G = \langle a, b, c \mid a^{2^\mu} = b^{2^\nu} = c^2 = 1, a^b = a^{1+2^{\mu-1}}, a^c = a, b^c = b\rangle = \langle a, b\rangle \times \langle c\rangle,$$

$\mu \geq 2,\ \nu \geq 1$, that is, G is the direct product of a metacyclic minimal nonabelian group $\langle a, b\rangle$, distinct from Q_8, and the cyclic group $\langle c\rangle$ of order 2.

Using this Theorem and two previously known results:

Theorem 1 (Miller-Moreno, [2]). *A minimal nonabelian finite 2-group is isomorphic to some of the groups:*

(a) $G = \langle a, b \mid a^{2^\mu} = b^{2^\nu} = 1, a^b = a^{1+2^{\mu-1}}\rangle,\ \mu \geq 2,\ \nu \geq 1,$

(b) $G = \langle a, b, c \mid a^{2^\mu} = b^{2^\nu} = c^2 = 1, c = [a, b], [a, c] = [b, c] = 1\rangle,$
$\mu, \nu \geq 1,\ \mu + \nu > 2,$

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(c) $G \cong Q_8$.

THEOREM 2 (Blackburn, see Janko [1, Theorem 7.1]). *A minimal nonmetacyclic finite 2-group is isomorphic to some of the groups:*

(a) $G = \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = a^2b^2,\ a^b = a,\ a^c = a^3,\ b^c = a^2b^3\rangle$, *a special group of order 32,*
(b) $G \cong Q_8 \times Z_2$,
(c) $G \cong Q_8 * Z_4$, *the central product of* Q_8 *and* Z_4,
(d) $G \cong E_8$,

we get the following classification of the considered groups:

THEOREM 3. *Let* G *be a nonabelian nonmetacyclic finite 2-group with all proper subgroups being abelian or metacyclic. Then* G *is isomorphic to some of the groups:*

(a) $G = \langle a, b, c \mid a^{2^\mu} = b^{2^\nu} = c^2 = 1, c = [a, b], [a, c] = [b, c] = 1\rangle$, $\mu, \nu \geq 1,\ \mu + \nu > 2$,
(b) $G = \langle a, b, c \mid a^{2^\mu} = b^{2^\nu} = c^2 = 1, a^b = a^{1+2^{\mu-1}}, a^c = a, b^c = b\rangle$, $\mu \geq 2,\ \nu \geq 1$,
(c) $G = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, a^b = a, a^c = a^3, b^c = a^2b^3\rangle$,
(d) $G \cong Q_8 \times Z_2$,
(e) $G \cong Q_8 * Z_4$, *the central product of* Q_8 *and* Z_4.

PROOF. If all proper subgroups of G are abelian, then G is a nonmetacyclic minimal nonabelian group from the list in Theorem 1, which gives the case (a). If all proper subgroups of G are metacyclic, then G is a nonabelian group from the list in Theorem 2, which gives the cases (c),(d),(e). Otherwise, there must be in G some nonabelian metacyclic proper subgroup and some nonmetacyclic abelian proper subgroup, and applying our Theorem, we get the case (b). $\square$

2. Proof of the Theorem

Let A be an abelian maximal subgroup of G, which is not metacyclic and M a metacyclic maximal subgroup of G, which is not abelian. Denote $T = A \cap M$. The group T is both metacyclic and abelian.

We prove our Theorem in several steps:

(i) $M = \langle a, b \mid a^{2^\mu} = 1,\ b^{2^\nu} = a^{2^\rho},\ a^b = a^s\rangle,\ \mu \geq 2,\ \nu \geq 1,$ $1 \leq \rho \leq \mu,\ s > 1,\ 2 \nmid s.$

PROOF. Being metacyclic, M is of the form $M = \langle a, b \mid a^m = 1,\ b^n = a^r,\ a^b = a^s\rangle$. As M is a 2-group, obviously $m = 2^\mu,\ n = 2^\nu,\ r = r' \cdot 2^\rho$ with $2 \nmid r'$ and $2 \nmid s$. Replacing a by $a^{r'}$ we have $b^{2^\nu} = a^{2^\rho}$, the other relations remaining unchanged. As M is not abelian and $\langle a\rangle \lhd M$, it is $\mu \geq 2$, $\nu \geq 1$, $\rho \geq 1$ and $s > 1$. $\square$

(ii) *$d(A) = 3,\ d(T) = 2$ and $A = T \times \langle c\rangle$ for some involution $c \in A - T$. There are exactly three involutions in T.*

PROOF. $T = A \cap M$ is metacyclic and abelian. Therefore $d(T) \leq 2$. Since $A = \langle T, c\rangle$ for any $c \in A - T$, and A is not metacyclic, we have $d(A) \geq 3$ and $d(A) \leq d(T)+1$. It follows $d(T) = 2,\ d(A) = 3$, and so $\Omega_1(T) \cong E_4,\ \Omega_1(A) \cong E_8$. Thus there are exactly 3 involutions in T and there is some involution $c \in \Omega_1(A) - \Omega_1(T) \subseteq A - T$. Now, obviously, $A = \langle T, c\rangle = T \times \langle c\rangle$. $\square$

(iii) *Denote $N = \langle a\rangle \lhd M$. Then either*

1) $N \leq T$ and $T = \langle a, b^2\rangle,\ \nu \geq 2$, or

2) $N \not\leq T$ and, without loss, $T = \langle a^2, b\rangle$.

PROOF. If $N = \langle a\rangle \leq T$, then $b \notin T$, but $b^2 \in T$, as $M/T \cong Z_2$. Thus $\langle a, b^2\rangle \leq T$ and $|M : \langle a, b^2\rangle| = 2 = |M : T|$. It follows $T = \langle a, b^2\rangle$. If $N \not\leq T$, then $M = NT$ and $NT/N = M/N \cong T/N \cap T \cong Z_{2^\nu}$. Henceforth $|N : N \cap T| = |M : T| = 2,\ N \cap T = \langle a^2\rangle$ and there exists $b' \in T - (N \cap T)$ such that ${b'}^{2^\nu} \in N \cap T$ and $b'^{2^{\nu-1}} \notin N \cap T$. Now $T = \langle N \cap T, b'\rangle = \langle a^2, b'\rangle$ and $M = \langle a, b'\rangle$. Replacing b by b' we get $M = \langle a, b\rangle,\ T = \langle a^2, b\rangle$. $\square$

(iv) *$\Phi(M) = \mho_1(M) = \langle a^2, b^2\rangle$ is abelian.*

PROOF. We know that $\Phi(M) = \mho_1(M)$ for 2-groups and $M/\Phi(M) \cong E_4$ since M is metacyclic. As $\langle a^2, b^2\rangle \leq \mho_1(M)$, and $|M : \langle a^2, b^2\rangle| = 4$, it follows $\Phi(M) = \langle a^2, b^2\rangle$. Also $\Phi(M) \leq T$, because T is maximal in M, and so $\Phi(M)$ is abelian. $\square$

In the following, we consider the involutions in T. In $N = \langle a\rangle$ there is only one involution $\tau = a^{2^{\mu-1}}$. If σ is another involution in T, then $\Omega_1(T) = \langle \sigma, \tau\rangle$.

(v) *If $\nu \geq 2,\ \rho \geq 2$, then $\Omega_1(T) = \langle \sigma, \tau\rangle$, where $\sigma = a^{-2^{\rho-1}} b^{2^{\nu-1}}$ and $\tau = a^{2^{\mu-1}}$, and thus $\Omega_1(T) \leq \langle a^2, b^2\rangle = \Phi(M)$. Besides, $\Omega_1(A) = \langle \sigma, \tau, c\rangle \cong E_8$.*

PROOF. Here $\sigma \in T - \langle a\rangle$, and $\sigma^2 = (a^{-2^{\rho-1}} b^{2^{\nu-1}})^2 = a^{-2^\rho} b^{2^\nu} = a^{-2^\rho} a^{2^\rho} = 1$. So σ and $\sigma\tau$ are both involutions in $T - \langle a\rangle$. Since $A = T \times \langle c\rangle$, obviously $\Omega_1(A) = \Omega_1(T) \times \langle c\rangle = \langle \sigma, \tau, c\rangle$. $\square$

(vi) *If $\nu \geq 2,\ \rho \geq 2$, then $G = M \times \langle c\rangle$ and M is minimal nonabelian.*

PROOF. By (ii) and (iii) we have $[T, c] = 1$ and either 1) $T = \langle a, b^2\rangle$ or 2) $T = \langle a^2, b\rangle$. Thus $a^c = a$ in case 1) and $b^c = b$ in case 2). Among the generators a, b of M denote the one belonging to T by x, and the one outside of T by y. Thus $x^c = x$. It is $y \notin T$, but $y^2 \in T$ and we have $(y^2)^c = y^2,\ y^{c^2} = y^1 = y$. Since $G/T \cong E_4$ and $G/T = \langle Ty, Tc\rangle$, it is $y^c = ty$, for some $t \in T$. Hence $y^{c^2} = (y^c)^c = (ty)^c = t^c \cdot ty = t \cdot ty = y,\ (y^2)^c = y^c \cdot y^c = ty \cdot ty = ty^2 t^y = tt^y y^2 = y^2$. It follows that $t^2 = 1$ and $t^y = t$, thus t is some involution in T and $[y, t] = 1$.

We assert that $\Phi(G)=\Phi(M)=\langle a^2,b^2\rangle$. As $G=\langle M,c\rangle$, the elements of G are of the form $g=x^\alpha y^\beta$ or $g=x^\alpha y^\beta c$. If $g=x^\alpha y^\beta\in\langle a,b\rangle=M$, then $g^2\in\Phi(M)=\langle a^2,b^2\rangle$. If $g=x^\alpha y^\beta c$, then $g^2=(x^\alpha y^\beta c)^2=x^\alpha y^\beta c^2(x^c)^\alpha(y^c)^\beta=x^\alpha y^\beta\cdot 1\cdot x^\alpha(ty)^\beta=x^\alpha y^\beta x^\alpha t^\beta y^\beta=(x^\alpha y^\beta)^2\cdot t^\beta$, because of $[x,t]=[y,t]=1$. Since $(x^\alpha y^\beta)^2\in\Phi(M)$ and $t\in\langle\sigma,\tau\rangle\subseteq\Phi(M)$, it follows $g^2\in\Phi(M)$ in any case. Therefore $\mho_1(G)=\Phi(G)\le\Phi(M)\le\Phi(G)$, and so $\Phi(G)=\Phi(M)=\langle a^2,b^2\rangle$.

Now $G=\langle\Phi(G),a,b,c\rangle=\langle x,y,c\rangle$. The subgroup $M_1=\langle\Phi(G),y,c\rangle=\langle\Phi(M),y,c\rangle$ is a maximal subgroup of G containing $\langle\sigma,\tau,c\rangle\cong E_8$. Thus M_1 is not metacyclic. So it must be abelian, and $y^c=y$. Since also $x^c=x$, we have $[a,c]=[b,c]=1$, and so $G=M\times\langle c\rangle$.

For each maximal subgroup T_1 of M, we have $T_1\ge\Phi(M)\ge\langle\sigma,\tau\rangle$. The group $T_1\times\langle c\rangle$ is maximal in G and contains $\langle\sigma,\tau,c\rangle\cong E_8$. By the above argument $T_1\times\langle c\rangle$ is also abelian, and so is T_1. It follows that all proper subgroups of M are abelian and so M is minimal nonabelian metacyclic group. ∎

Now we consider the remaining cases, when $\nu=1$ or $\rho=1$.

(vii) *Both cases $\nu=1$, or $\rho=1$ reduce to the case $\nu=1$, that is*

$$M=\langle a,b\mid a^{2^\mu}=1,\ b^2=a^{2^\rho},\ a^b=a^s\rangle,\ 2\nmid s.$$

Proof. If $\nu=1$, then M is as stated above and it is a metacyclic group with a cyclic maximal subgroup $\langle a\rangle$.

If $\rho=1$, then $M=\langle a,b\mid a^{2^\mu}=1,b^{2^\nu}=a^2,a^b=a^s\rangle$. Now, $|b^{2^\nu}|=|a^2|=2^{\mu-1}$, and so $|b|=2^{\nu+\mu-1}$. As $|M|=2^{\mu+\nu}$, it now follows that$\langle b\rangle$ is a cyclic maximal subgroup of M. Interchanging the notation for a and b, we get again the same relations for M as in the assertion. ∎

(viii) *In any case, $G=\langle c\rangle\times M$ and M is minimal nonabelian.*

Proof. We continue considering the remaining case $\nu=1$. Since $T=A\cap M$, $d(T)=2$, therefore $T\ne\langle a\rangle$. From $|M:T|=2$ and $|M:\langle a\rangle|=2$, it follows $T\cap\langle a\rangle=\langle a^2\rangle$. Since $\langle\sigma,\tau\rangle\le T$ and $\sigma\notin\langle a\rangle$, it is $M=\langle a,\sigma\rangle$, and we can replace b by σ. Now, $b=\sigma\in T$, $b^2=\sigma^2=1$ and $T=\langle a^2,b\rangle$, $M=\langle a,b\mid a^{2^\mu}=b^2=1,a^b=a^s\rangle$.

It is $(a^2)^b=a^2=(a^b)^2=a^{2s}$and thus $2^\mu|2(s-1)$. It follows that $s=1+2^{\mu-1}$ and so $a^b=a^{2^{\mu-1}}\cdot a=\tau a$. Similarly as in (vi), we have $a^c=ta$, for some $t\in T$, and from $a^{c^2}=a^1=a$ and $(a^2)^c=a^2$ we conclude again that $t^2=1$ and $t^a=t$. Therefore $t\in\langle\sigma,\tau\rangle$. Because of $\tau^a=\tau$, $\sigma^a=b^a\ne b=\sigma$, it must be $t\in\langle\tau\rangle$, that is

$$G=\langle a,b,c\mid a^{2^\mu}=b^2=c^2=1,a^b=\tau a,a^c=\tau^\eta a,b^c=b\rangle,\ \eta\in\{0,1\}.$$

If $\eta = 0$, then obviously $G = M \times \langle c \rangle$, where

$$M = \langle a, b \mid a^{2^\mu} = b^2 = 1,\ a^b = a^{1+2^{\mu-1}} \rangle.$$

Otherwise, if $\eta = 1$, replacing c by $c' = bc = \sigma c$, we have $c'^2 = (\sigma c)^2 = 1,\ a^{c'} = a^{bc} = (\tau a)^c = \tau \cdot \tau a = a$, and again $G = M \times \langle c \rangle$. The maximal subgroups of M are $\langle \Phi(M), a \rangle = \langle a \rangle,\ \langle \Phi(M), \sigma \rangle = \langle a^2, \sigma \rangle$ and $\langle \Phi(M), a\sigma \rangle = \langle a^2, a\sigma \rangle$, all of them being abelian. Thus M is minimal nonabelian group. ∎

(ix) *G is isomorphic to some of the groups:*

$$G = \langle a, b, c \mid a^{2^\mu} = b^{2^\nu} = c^2 = 1, a^b = a^{1+2^{\mu-1}}, a^c = a, b^c = b \rangle,$$

$\mu \geq 2,\ \nu \geq 1$.

Proof. This follows immediately by (vi),(viii) and Theorem 1, as the groups from Theorem 1(b) are not metacyclic, and $Q_8 \times Z_2$ is minimal nonmetacyclic. ∎

(x) *All groups listed in the Theorem have the stated property.*

Proof. It remains to show that every maximal subgroup of such a group G is abelian or metacyclic. We know that M is minimal nonabelian and $M/\Phi(M) \cong E_4$. Thus $\Phi(M)$ is intersection of abelian maximal subgroups and so lies in $Z(M)$ and $Z(M) = \Phi(M)$. Since $G = M \times \langle c \rangle$, obviously $\Phi(M) = \Omega_1(M) = \Omega_1(G) = \Phi(G)$ and $Z(G) = Z(M) \times \langle c \rangle = \Phi(G) \times \langle c \rangle$.

The Frattini factor group $G/\Phi(G) = \langle \overline{a}, \overline{b}, \overline{c} \rangle \cong E_8$ has 7 maximal subgroups: $\overline{H}_1 = \langle \overline{a}, \overline{b} \rangle$, $\overline{H}_2 = \langle \overline{a}, \overline{c} \rangle$, $\overline{H}_3 = \langle \overline{b}, \overline{c} \rangle$, $\overline{H}_4 = \langle \overline{ab}, \overline{c} \rangle$, $\overline{H}_5 = \langle \overline{a}, \overline{bc} \rangle$, $\overline{H}_6 = \langle \overline{ac}, \overline{b} \rangle$, and $\overline{H}_7 = \langle \overline{ac}, \overline{bc} \rangle$. They are in the one to one correspondence with maximal subgroups of G, according the correspondence law:

$$\overline{H_i} = \langle \overline{x}, \overline{y} \rangle \leftrightarrow H_i = \langle x, y, \Phi(G) \rangle.$$

We see that: $H_1 = \langle a, b, \Phi(G) \rangle = \langle a, b, \Phi(M) \rangle = M$ is metacyclic, nonabelian, $H_2 = \langle a, c, \Phi(G) \rangle,\ H_3 = \langle b, c, \Phi(G) \rangle$ and $H_4 = \langle ab, c, \Phi(G) \rangle$ are all abelian, because they are cyclic extensions of $Z(G) = \Phi(G) \times \langle c \rangle$.

The groups H_3 and H_4 are moreover nonmetacyclic in both cases $\nu \geq 2$ and $\nu = 1$, while H_2 is metacyclic in the latter case, as for $\nu = 1$ the group $\Phi(G) = \langle a^2 \rangle$.

Since $c \in Z(G)$ and $c^2 = 1$, it is $(ac)^2 = a^2,\ (bc)^2 = b^2,\ |ac| = |a|,\ |bc| = |b|$ and $[a, bc] = [ac, b] = [ac, bc] = [a, b] = a^{2^{\mu-1}} = (ac)^{2^{\mu-1}}$. Therefore:

$$H_5 = \langle a, bc, \Phi(G) \rangle = \langle a, bc \mid a^{2^\mu} = (bc)^{2^\nu} = 1,\ [a, bc] = a^{2^{\mu-1}} \rangle \cong H_1$$

$$H_6 = \langle ac, b, \Phi(G) \rangle = \langle ac, b \mid (ac)^{2^\mu} = b^{2^\nu} = 1,\ [ac, b] = (ac)^{2^{\mu-1}} \rangle \cong H_1$$

$$H_7 = \langle ac, bc, \Phi(G) \rangle = \langle ac, bc \mid (ac)^{2^\mu} = (bc)^{2^\nu} = 1,\ [ac, bc] = (ac)^{2^{\mu-1}} \rangle \cong H_1,$$

and H_5, H_6, H_7 are all metacyclic nonabelian. ∎

Our Theorem is proved.

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References

[1] Z. Janko, *Finite 2-groups with exactly four cyclic subgroups of order* 2^n, J.Reine Angew. Math. **566** (2004), 135-181.

[2] G. A. Miller and H. Moreno, *Nonabelian groups in which every subgroup is abelian*, Trans. Am. Math. Soc. **4** (1903), 398-404.

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