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FINITE 2-GROUPS G WITH $\Omega_2^*(G)$ METACYCLIC

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Abstract. In this paper we classify finite non-metacyclic 2-groups G such that $\Omega_2^*(G)$ (the subgroup generated by all elements of order 4) is metacyclic. However, if G is a finite 2-group such that $\Omega_2(G)$ (the subgroup generated by all elements of order ≤ 4) is metacyclic, then G is metacyclic.

1. Introduction

A famous result of N. Blackburn (see Proposition 1.4) states that if G is a finite 2-group such that the subgroup $\Omega_2(G)$ (the subgroup generated by all elements of order ≤ 4) is metacyclic, then G is metacyclic. What can we say in the case, where G is a finite 2-group and we know that only the subgroup $\Omega_2^*(G)$ (the subgroup generated by all elements of order 4) is metacyclic? The purpose of this paper is to classify finite non-metacyclic 2-groups G such that $\Omega_2^*(G)$ is metacyclic. We have seen in Janko [3] that such a subgroup $\Omega_2^*(G)$ has the strong influence on the structure of the whole group G so that the structure of the 2-group G is almost uniquely determined, when $\Omega_2^*(G)$ is known.

All groups considered here are finite and our notation is standard. In particular,

$$M_{2^n} = \langle a, t \mid a^{2^{n-1}} = t^2 = 1,\ a^t = a^{1+2^{n-2}},\ n \geq 4\rangle,$$

and 2-groups of maximal class are dihedral groups D_{2^n} (of order 2^n, $n \geq 3$), generalized quaternion groups Q_{2^n} (of order 2^n, $n \geq 3$), and semi-dihedral groups SD_{2^n} (of order 2^n, $n \geq 4$).

For convenience, we state here some known results which are used in this paper.

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Proposition 1.1 ([5, Proposition 1.4]). *Let G be a 2-group of order $\geq 2^4$ satisfying $|\Omega_2(G)| \leq 2^3$. If $Z(G)$ is noncyclic, then G is abelian of type $(2, 2^n)$, $n \geq 3$.*

Proposition 1.2 ([5, Theorem 2.1]). *Let G be a metacyclic 2-group which is neither cyclic nor of maximal class. Then G has exactly three involutions.*

Proposition 1.3 ([2, Theorem 4.1]). *Let G be a 2-group of order $> 2^4$ all of whose elements of order 4 generate the subgroup $H = \Omega_2^*(G)$ of order 2^4. Assume in addition that G has exactly 6 cyclic subgroups of order 4 and $|\Omega_2(G)| > 2^4$. Then we have the following possibilities:*

(a) *$H \cong Q_8 \times C_2$ and $G \cong SD_{2^4} \times C_2$.*
(b) *$H \cong \langle a, b \mid a^4 = b^4 = 1,\ a^b = a^{-1}\rangle$ and*

$$G = \langle b, t \mid b^4 = t^2 = 1,\ b^t = ab,\ a^4 = 1,\ a^b = a^{-1},\ a^t = a^{-1}\rangle.$$

Here $|G| = 2^5$, $H = \langle a, b\rangle$, $\Phi(G) = \langle a, b^2\rangle \cong C_4 \times C_2$, $\Omega_2(G) = G$, and $Z(G) = \langle a^2, b^2\rangle \cong E_4$.

(c) *$H \cong C_4 \times C_4$ and G has a metacyclic maximal subgroup M such that $\Omega_2(M) = H$, $G = M\langle t\rangle$, where t is an involution with $C_M(t) = \Omega_1(M) \cong E_4$ and t inverts each element of $C_M(H)$ so that $C_M(H)$ is abelian. (The last statement actually follows from the proof of Theorem 4.1 in [2].)*

Proposition 1.4 (N. Blackburn, [2, Proposition 1.8]). *If G is a 2-group such that $\Omega_2(G)$ is metacyclic, then G is metacyclic, too.*

Proposition 1.5 ([4, Theorem 5.1]). *Let G be a 2-group containing exactly one abelian subgroup of type $(4, 2)$. Then one of the following holds:*

(a) *$|\Omega_2(G)| = 8$.*
(b) *$G \cong C_2 \times D_{2^{n+1}}$, $n \geq 2$.*
(c) *$G = \langle b, t \mid b^{2^{n+1}} = t^2 = 1, b^t = b^{-1+2^{n-1}}u, u^2 = [u, t] = 1, b^u = b^{1+2^n}, n \geq 2\rangle$. Here $|G| = 2^{n+3}$, $Z(G) = \langle b^{2^n}\rangle$ is of order 2, $\Phi(G) = \langle b^2, u\rangle$, $E = \langle b^{2^n}, u, t\rangle \cong E_8$ is self-centralizing in G, $\Omega_2(G) = \langle u\rangle \times \langle b^2, t\rangle \cong C_2 \times D_{2^{n+1}}$, $G' = \langle b^{2^n}, u\rangle \cong E_4$ in case $n = 2$, and $G' = \langle b^2u\rangle \cong C_{2^n}$ for $n \geq 3$.*

Proposition 1.6 ([4, Proposition 1.12]). *Let G be a p-group with a nonabelian subgroup P of order p^3. If $C_G(P) \leq P$, then G is of maximal class.*

Proposition 1.7 (Janko, [1, Proposition 1.10]). *Let τ be an involutory automorphism acting on an abelian 2-group B so that $C_B(\tau) = W_0$ is contained in $\Omega_1(B)$. Then τ acts invertingly on $\mho_1(B)$ and on B/W_0.*

Proposition 1.8 ([2, Introduction]). *Suppose that a 2-group G is neither cyclic nor of maximal class. Then the number $c_n(G)$ ($n > 1$, n fixed) of cyclic* *subgroups of order 2^n is even. (This result is also due to G. A. Miller and appears in section 51 of the 1915 book on "Finite groups" by Miller-Blichfeldt-Dickson.)*

We prove here the following new result.

Theorem 1.9. *Let G be a non-metacyclic 2-group of exponent > 2 such that $H = \Omega_2^*(G)$ is metacyclic. Then one of the following holds:*

(i) *$H \cong C_4 \times C_2$ is the unique abelian subgroup of G of type $(4, 2)$ and G is isomorphic to one of the groups given in (b) and (c) of Proposition 1.5.*

(ii) *$H \cong C_4 \times C_4$ and G is isomorphic to one of the groups given in (c) of Proposition 1.3.*

(iii) *$G = \langle t, c \mid t^2 = c^{2^{n+1}} = 1,\ n \geq 2,\ tc = b,\ b^4 = [b^2, c] = 1\rangle$, where $|G| = 2^{n+3}$, $n \geq 2$, $H = \Omega_2^*(G) = \langle c^2, b\rangle$ with $(c^2)^b = c^{-2}$ and H is a splitting metacyclic maximal subgroup , $\langle b^2\rangle \times \langle c\rangle$ is the unique abelian maximal subgroup (of type $(2, 2^{n+1})$), $Z(G) = \langle b^2, c^{2^n}\rangle \cong E_4$, $G' = \langle c^2 b^2\rangle \cong C_{2^n}$, and $\langle t, b^2, c^{2^n}\rangle \cong E_8$ (so that G is non-metacyclic).*

2. Proof of Theorem 1.9

Let G be a non-metacyclic 2-group of exponent > 2 such that the subgroup $H = \Omega_2^*(G)$ is metacyclic. If $H = \Omega_2(G)$, then a result of N. Blackburn (Proposition 1.4) implies that G is metacyclic, a contradiction. Hence $\Omega_2(G) > H$ and so there exist involutions in $G - H$.

Suppose that H is cyclic. Then $H \cong C_4$ and so G has exactly one cyclic subgroup of order 4. But then Proposition 1.8 implies that G is metacyclic, a contradiction. Hence H is noncyclic.

Assume that H is abelian (of rank 2). Since $H = \Omega_2^*(H)$, we have either $H \cong C_4 \times C_2$ or $H \cong C_4 \times C_4$.

Suppose $H \cong C_4 \times C_2$. In that case H is the unique abelian subgroup of type $(4, 2)$ in G. Suppose that this is not the case. Then there is an involution $i \in G - H$ which centralizes an element $v \in H$ of order 4, where $\langle i\rangle \times \langle v\rangle \cong C_2 \times C_4$. But then $o(iv) = 4$ and $iv \in G - H$, a contradiction. (We need this uniqueness proof so that we are able to use Proposition 1.5.) By Proposition 1.5, G is isomorphic to a group given in parts (b) and (c) of that proposition.

Suppose $H \cong C_4 \times C_4$. In that case G has exactly 6 cyclic subgroups of order 4 and $\Omega_2(G) > H$ and so G is isomorphic to a group given in the part (c) of Proposition 1.3.

From now on we assume that H is nonabelian. Suppose in addition that H has a cyclic subgroup of index 2. Since $\Omega_2^*(H) = H$, we get $H \cong Q_{2^n}$, $n \geq 3$. Let $H_0 \cong Q_8$ be a quaternion subgroup of H so that $C_H(H_0) = Z(H_0) = Z(H) \cong C_2$. If $C_G(H_0) \leq H_0$, then G is of maximal class (Proposition 1.6) and so G is metacyclic, a contradiction. Hence $D = C_G(H_0) \not\leq H_0$ so that $D \cap H = Z(H_0)$, $D > Z(H_0)$, and D must be elementary abelian. Let $d \in D - Z(H_0)$ and $s \in H_0$ with $o(s) = 4$. Then $o(ds) = 4$ and $ds \notin H$, a contradiction.

Our subgroup $H = \Omega_2^*(G)$ is metacyclic nonabelian and H has no cyclic subgroup of index 2 and so, by Proposition 1.2, H has exactly three involutions and $\Omega_1(H) \cong E_4$. Let $Z = \langle a \rangle$ be a cyclic normal subgroup of H such that H/Z is cyclic and we have $|H/Z| \geq 4$. Let K/Z be the subgroup of index 2 in H/Z. Since $\Omega_2^*(H) = H$, there is an element b of order 4 in $H - K$. This implies $|H/Z| = 4$, $H = \langle a \rangle \langle b \rangle$ with $\langle a \rangle \cap \langle b \rangle = \{1\}$ and so H is splitting over Z. We set $o(a) = 2^n$ with $n \geq 2$ since H is nonabelian. Since $K = \langle a \rangle \langle b^2 \rangle$ contains exactly three involutions, K is either abelian of type $(2, 2^n)$, $n \geq 2$ or $K \cong M_{2^{n+1}}$, $n \geq 3$. In the last case, $\langle b \rangle \cong C_4$ acts faithfully on $\langle a \rangle$ and so in that case $n \geq 4$.

First assume $K \cong M_{2^{n+1}}$, $n \geq 4$, where $\langle b \rangle$ acts faithfully on $Z = \langle a \rangle$. We have $a^b = av$ or $a^b = a^{-1}v$, where v is an element of order 4 in $\langle a \rangle$. Set $v^2 = z$, where $z \in Z(H)$.

Suppose $a^b = av$ so that $H' = \langle v \rangle$ and $(a^4)^b = (av)^4 = a^4$. Since $\langle v \rangle \leq \langle a^4 \rangle$, we have $H' \leq Z(H)$. If $x, y \in H$ with $o(x) \leq 8$ and $o(y) \leq 8$, then $(xy)^8 = x^8 y^8 [y, x]^{28} = 1$ and so $\Omega_3(H) < H$ because $o(a) \geq 2^4$. This is a contradiction since we must have $\Omega_2^*(H) = H$ but $\Omega_2^*(H) \leq \Omega_3(H)$.

Assume $a^b = a^{-1}v$ so that $(a^2)^b = (a^{-1}v)^2 = a^{-2}z$ and $(a^4)^b = (a^{-1}v)^4 = a^{-4}$. Therefore b inverts $\langle a^4 \rangle$ and so $v^b = v^{-1}$. Also,

$$a^{b^2} = (a^{-1}v)^b = (a^{-1}v)^{-1}v^{-1} = av^{-2} = az, \quad \text{and} \quad a = (a^{b^{-1}})^{-1}v^{b^{-1}},$$

which gives $a^{b^{-1}} = a^{-1}v^{-1}$ and $a^{b^\eta} = a^{-1}v^\eta$, where $\eta = \pm 1$. We compute:

$$(ba^2)^2 = ba^2ba^2 = b^2(a^2)^b a^2 = b^2 a^{-2} z a^2 = b^2 z,$$

and so $o(ba^2) = 4$. This implies $\Omega_2^*(H) \geq \langle b, a^2 \rangle$, where $L = \langle b, a^2 \rangle$ is a maximal subgroup of H. We claim that the set $H - L$ contains no elements of order 4 and this gives us a contradiction. Indeed, each element in $H - L$ has the form $(b^j a^{2i})a = b^j a^{2i+1}$ (i, j are integers). If $j = 2$, then

$$(b^2a^{2i+1})^2 = b^2a^{2i+1}b^2a^{2i+1} = b^4(a^{b^2})^{2i+1}a^{2i+1} = (az)^{2i+1}a^{2i+1} = (a^{4i}z)a^2,$$

which is an element of order ≥ 8. If $j = \eta = \pm 1$, then

$$\begin{aligned} (b^\eta a^{2i+1})^2 &= b^\eta a^{2i+1} b^\eta a^{2i+1} = b^{2\eta}(a^{b^\eta})^{2i+1}a^{2i+1} \\ &= b^2(a^{-1}v^\eta)^{2i+1}a^{2i+1} = b^2(v^\eta)^{2i}v^\eta = b^2 z^i v^\eta, \end{aligned}$$

which is an element of order 4 since $[b^2, v] = 1$.

We have proved that $K = \langle b^2, a \rangle$ must be abelian of type $(2, 2^n)$, $n \geq 2$, $E_4 \cong \Omega_1(H) = \langle b^2, z \rangle \leq Z(H)$, where we have set $z = a^{2^{n-1}}$. The element b induces on $\langle a \rangle$ an involutory automorphism and so we have either $a^b = az$, $n \geq 3$ or $a^b = a^{-1}z^\epsilon$, $\epsilon = 0, 1$, $n \geq 2$ (and if $\epsilon = 1$, then $n \geq 3$).

First assume $a^b = az$, $n \geq 3$, where $H' = \langle z \rangle$ and so H is of class 2. In that case, if $x, y \in H$ with $o(x) \leq 4$ and $o(y) \leq 4$, then $(xy)^4 = x^4 y^4 [y, x]^6 = 1$ and so $exp(\Omega_2(H)) = 4$. But $o(a) = 2^n \geq 8$ and so $\Omega_2^*(H) \leq \Omega_2(H) < H$, a contradiction.

We have proved that $a^b = a^{-1} z^{\epsilon}$, $\epsilon = 0, 1$, $n \geq 2$, and if $\epsilon = 1$, then $n \geq 3$. Assume $n = 2$ so that $H = \langle a, b \,|\, a^4 = b^4 = 1,\, a^b = a^{-1} \rangle$. By Proposition 1.3(b), G is isomorphic to the following (uniquely determined) group of order 2^5:

$$G = \langle b, t \mid b^4 = t^2 = 1,\ b^t = ab,\ a^4 = 1,\ a^b = a^{-1},\ a^t = a^{-1} \rangle, \tag{2.1}$$

where $\Omega_2^*(G) = \langle a, b \rangle$, $\Phi(G) = \langle a, b^2 \rangle \cong C_4 \times C_2$, and $\Omega_2(G) = G$.

It remains to study the case $n \geq 3$, where

$$H = \langle a, b \,|\, a^{2^n} = b^4 = 1,\ n \geq 3,\ a^b = a^{-1} z^{\epsilon},\ \epsilon = 0, 1,\ z = a^{2^{n-1}} \rangle,$$

$H' = \langle a^2 \rangle \cong C_{2^{n-1}}$, $\Omega_1(H) = Z(H) = \langle b^2, z \rangle \cong E_4$, and $K = \langle b^2, a \rangle$ is the unique abelian maximal subgroup (of type $(2, 2^n)$) of H.

Let t be an involution in $G - H$ and set $L = H\langle t \rangle$. Since $\langle z \rangle = \Omega_1(H')$, $z \in Z(G)$ and let $\langle v \rangle$ be the cyclic subgroup of order 4 in H' so that $\langle v \rangle$ is normal in G. Note that $v^b = v^{-1}$ and $C_H(v) = K$ so that $C = C_G(v)$ covers G/H. Set $C_0 = C_L(v)$ and we see that

$$|G : C| = |L : C_0| = 2,\ L = C_0\langle b \rangle,\ G = C\langle b \rangle,\ C \cap H = K.$$

If t does not centralize $Z(H) = \langle b^2, z \rangle$, then $\langle t, Z(H) \rangle \cong D_8$ and tb^2 is an element of order 4 in $L - H$, a contradiction. Thus t centralizes $Z(H)$ and so $Z(H) \leq Z(L)$. Also, t does not centralize any element of order 4 in H and so $C_H(t) = \langle b^2, z \rangle = \Omega_1(H)$.

Since $\langle v \rangle$ is central in C, there are no involutions in $C - K$. But there are no elements of order 4 in $C - K$ and so $\Omega_2(C) = \Omega_2(K) = \langle b^2 \rangle \times \langle v \rangle \cong C_2 \times C_4$. The fact that $C_K(t) = Z(H)$ also implies $C_C(t) = Z(H) = \Omega_1(C)$. Note that $Z(H) \leq Z(L)$ implies that $Z(C_0) \geq \langle b^2, z \rangle$ and so $Z(C_0)$ is noncyclic. By Proposition 1.1, C_0 is abelian of type $(2, 2^{n+1})$.

We act with the involution t on the abelian group C_0 and apply Proposition 1.7. It follows that t acts invertingly on $C_0/\langle b^2, z \rangle$. We get $a^t = a^{-1}s$, where $s \in \langle b^2, z \rangle$. Then $(ta)^2 = tata = a^t a = a^{-1} s a = s$ and so $s = 1$ since $ta \notin H$ and ta cannot be an element of order 4. We get $a^t = a^{-1}$ and so t acts invertingly on K. On the other hand, $b = tc_0$ with $c_0 \in K$ and so $a^b = a^{tc_0} = (a^{-1})^{c_0} = a^{-1}$ because C_0 is abelian. We have proved that $\epsilon = 0$ and so b also acts invertingly on K.

We show that the involution $b^2 z$ is not a square in H. Indeed, for any $x \in K$, we get

$$(bx)^2 = bxbx = b^2 x^b x = b^2 x^{-1} x = b^2.$$

On the other hand, b^2 and z are squares in H and so $\langle b^2 z\rangle$ is a characteristic subgroup of H and therefore $b^2 z \in Z(G)$. It follows that $Z(H) = \langle b^2, z\rangle \leq Z(G)$.

We use again Proposition 1.1 and get that C is also abelian (of type $(2, 2^k)$, $k \geq n+1$). If $C \neq C_0$, then there is an element $d \in C_0 - K$ such that $d \in \mho_1(C)$. By Proposition 1.7, t acts invertingly on $\mho_1(C)$ and so $d^t = t^{-1}$. But then t inverts each element in C_0 which implies that all elements in $tC_0 = L - C_0$ are involutions. This is a contradiction since $b \in L - C_0$ and $o(b) = 4$.

We have proved that $C = C_0$ and so $G = L$. Since t acts invertingly on K, all elements in tK are involutions. But b is not an involution and so $b = tc$ with a suitable element $c \in C_0 - K$ so that $o(c) = 2^{n+1}$. Since C_0 is abelian, we have $[b^2, c] = 1$. We have obtained the following group of order 2^{n+3}:

$$G = \langle c, t \mid c^{2^{n+1}} = t^2 = 1,\ n \geq 3,\ tc = b,\ b^4 = [b^2, c] = 1\rangle, \tag{2.2}$$

where $\Omega_2^*(G) = \langle c^2, b\rangle$ with $(c^2)^b = c^{-2}$.

If we set $n = 2$ in (2.2), we get a group G of order 2^5 with

$$\Omega_2^*(G) = \langle c^2, b \mid (c^2)^4 = b^4 = 1,\ (c^2)^b = c^{-2}\rangle$$

and $\Omega_2(G) = G$ and so this group G (because Proposition 1.3(b) implies the uniqueness of such a group) must be isomorphic to the group given in (2.1). We have obtained the groups given in part (iii) of our theorem for all $n \geq 2$.

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