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A NOTE ON CALCULATION OF ASYMPTOTIC ENERGY FOR GINZBURG-LANDAU FUNCTIONAL WITH ε-DEPENDENT 1-LIPSCHITZ PENALIZING TERM IN ONE DIMENSION

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Abstract. We study asymptotic behavior of the Ginzburg-Landau functional

$$I^{\varepsilon}_{g_\varepsilon}(v) = \int_\Omega \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(s)(v(s) + g_\varepsilon(s))^2\Big) ds$$

as $\varepsilon \to 0$, where (g_ε) is a given sequence of 1-Lipschitz functions. In cases where the sequence (g_ε) possesses some additional properties we calculate (rescaled) minimal macroscopic energy associated to $I^{\varepsilon}_{g_\varepsilon}$ as $\varepsilon \to 0$. Thus we obtain partial generalization of our previous results.

1. Introduction

The results presented in this note are contribution to the approach introduced by G. Alberti and S. Müller in [1]. They developed framework which allowed them to compute minimal asymptotic energy for a certain class of functionals of Ginzburg-Landau type in one dimension and to rigorously describe small-scale oscillations of the associated minimizing sequences (cf. also [8]). We consider a variant of the energy in [10] (see also [1], p. 815) which is perturbed by a sequence of 1-Lipschitz penalizing functions (g_ε). The functional I^{ε}_{g} studied in [10],

$$I^{\varepsilon}_{g}(v) := \int_\Omega \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(s)(v(s) + g(s))^2\Big) ds, \tag{1.1}$$

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is now replaced by

$$(1.2)\qquad I^{\varepsilon}_{g_\varepsilon}(v) := \int_\Omega \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(s)(v(s)+g_\varepsilon(s))^2\Big)ds,$$

where $\Omega \subset \mathbf{R}$ is open bounded interval, $v \in \mathrm{H}^2_{per}(\Omega)$, $W \in \mathrm{C}(\mathbf{R};[0,+\infty))$, $W(\xi) = 0$ if and only if $\xi \in \{-1,1\}$, W has superlinear growth in infinity, a is $\mathrm{L}^1(\Omega)$-function which is extended by periodicity to $\mathbf{R}^2$ and satisfies $a(s) \geq \alpha > 0$ (a.e. $s \in \Omega$). Typical choice for W is $W(\xi) := (\xi^2-1)^2$. We consider

$$(1.3)\qquad \mathcal{E}^\varepsilon(g_\varepsilon) := \min_{v\in \mathrm{H}^2(\Omega)} \varepsilon^{-2/3} I^{\varepsilon}_{g_\varepsilon}(v), \quad \mathcal{E}^\varepsilon_{per}(g_\varepsilon) := \min_{v\in \mathrm{H}^2_{per}(\Omega)} \varepsilon^{-2/3} I^{\varepsilon}_{g_\varepsilon}(v).$$

Typical problem in analysis of functionals like (1.2) is to determine the limit of the sequence $(\mathcal{E}^\varepsilon(g_\varepsilon))$ as $\varepsilon \to 0$. The limit is usually referred to as (rescaled) minimal asymptotic (or, equivalently, macroscopic) energy associated to (1.2) and is usually recovered by using some kind of variational convergence, like Γ-convergence, which proved to be a powerful tool in this respect (cf. [4] and references therein). In [1] Alberti and Müller calculated minimal asymptotic energy when $g = 0$ (cf. (3.2) in the case $g = 0$). There analysis also shows the following: if v_ε minimizes functional (1.1) when $g = 0$, then (v'_ε) for small $\varepsilon > 0$ exhibits two-scale behavior as shown in Figure 1. In particular, it follows that the internally created small scale of v_ε is of order $\varepsilon^{1/3}$, a result which was previously established in [8] by a quite different approach in the case when a is strictly positive constant and $g = 0$. Results concerning some similar problems can be found in [3, 5, 11]. In this note we consider the problem of stability of the minima in terms of ε-dependent perturbation coming from the sequence (g_ε). While we describe a few cases when we are able to compute minimal asymptotic energy associated to $I^\varepsilon_{g_\varepsilon}$, we point out that our conclusions are far from being complete. We expect that results can be consistently improved so as to include optimal assumptions on (g_ε) (see Conjecture 4.5).

2. Some preliminaries

We consider a compact metric space (K,d) (the space of patterns), which is defined as follows: K is the set of all measurable mappings $x : \mathbf{R} \to [-\infty,+\infty]$, endowed with the metric d defined by

$$d(x_1,x_2) := \sum_{k=1}^{\infty} \frac{1}{2^k \alpha_k} \left| \int_{\mathbf{R}} y_k \Big(\frac{2}{\pi}\arctan x_1 - \frac{2}{\pi}\arctan x_2\Big) d\lambda \right|,$$

where λ is one-dimensional Lebesgue measure, (y_k) is a sequence of bounded functions which are dense in $\mathrm{L}^1(\mathbf{R})$, such that the support of y_k is a subset of $(-k,k)$, with $\alpha_k := \|y_k\|_{\mathrm{L}^1} + \|y_k\|_{\mathrm{L}^\infty}$. As shown in [1], $\mathrm{L}^p_{loc}(\mathbf{R})$ embeds continuously in K. The Banach space $\mathrm{C}(K)$ is the space of all continuous real functions on K, whose dual is identified with the space of all finite real Borel measures on K, denoted by $\mathcal{M}(K)$ (endowed with the corresponding

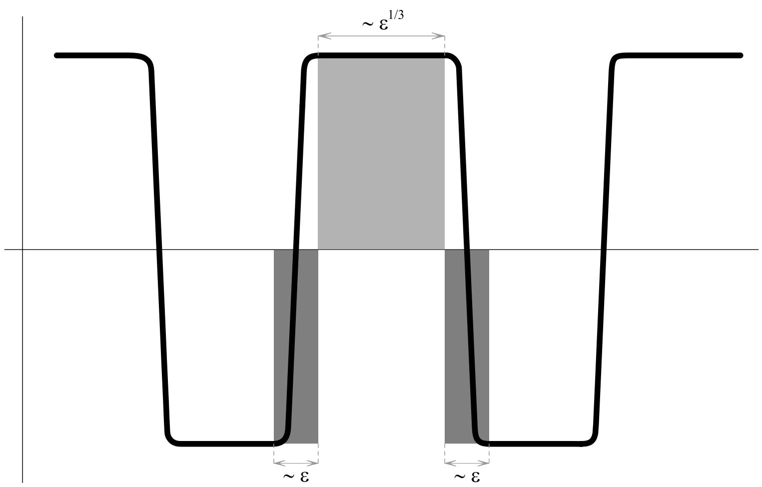

FIGURE 1. Two-scale structure of (v'_ε).

weak-star topology). A K-valued Young measure on Ω (or *Young measure on micropatterns*) is a map $\boldsymbol{\nu} \in \mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$ (where by $\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$ we denote the dual of $\mathrm{L}^1(\Omega;\mathrm{C}(K))$, cf. [2] for details) such that ν_s is a probability measure for almost every $s \in \Omega$. The set of all Young measures is denoted by $YM(\Omega;K)$ and it is always endowed with the topology of $\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$. The *elementary Young measure* associated to a measurable map $u : \Omega \to K$ is the map $\underline{\delta}_u : \Omega \to \mathcal{M}(K)$ given by $\underline{\delta}_u(s) := \delta_{u(s)}$, $s \in \Omega$. We say that a sequence of measurable maps $u^k : \Omega \to K$ *generates the Young measure* $\boldsymbol{\nu}$, if the corresponding elementary Young measures $\underline{\delta}_{u^k}$ converge to $\boldsymbol{\nu}$ in the topology of $\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$. The fundamental theorem of Young measures can be found in [2]. We say that $\mu \in \mathcal{M}(K)$ is *invariant with respect to translations* if for every $g \in \mathrm{C}(K)$ and every $\tau \in \mathbf{R}$ there holds $\langle \mu, g\rangle = \langle \mu, g \circ T_\tau\rangle$, where $T_\tau : K \to K$ is defined by $T_\tau x(t) := x(t-\tau)$, $x \in K$, $t \in \mathbf{R}$. $\mathcal{I}(K)$ denotes the class of all invariant probability measures on K. By $\mathrm{H}^2_{per}(\Omega)$ we denote a set of all real functions on Ω, extended to $\mathbf{R}$ by periodicity, which belong to $\mathrm{H}^2_{loc}(\mathbf{R})$. If $g : \Omega \to \mathbf{R}$ is a Lipschitz-continuous function, $\mathrm{Lip}(g)$ denotes the Lipschitz constant of g. Sx denotes the set of all points where function $x \in K$ is not continuous, while $|Sx|$ denotes cardinality of the set Sx.

DEFINITION 2.1 (Γ-convergence and continuous convergence). *Let X be a metric space. A sequence of functions $F^\varepsilon : X \to [0,+\infty]$ Γ-converges to F on X, and we write $F^\varepsilon \xrightarrow{\Gamma} F$, if the following is fulfilled:*

(i) *Lower-bound inequality: for every $x \in X$ and a sequence (x^ε) in X such that $x^\varepsilon \to x$ it holds $\liminf_\varepsilon F^\varepsilon(x^\varepsilon) \geq F(x)$.*

(ii) *Upper-bound inequality: For any y in X there exists a sequence (y^ε) in X such that $y^\varepsilon \to y$ and $\limsup_\varepsilon F^\varepsilon(y^\varepsilon) \le F(y)$.*

Functions F^ε continuously converge to F on X if $F^\varepsilon(x^\varepsilon) \to F(x)$ whenever $x^\varepsilon \to x$, which is written as $F^\varepsilon \xrightarrow{C} F$.

3. Formulation of the problem

Asymptotic analysis for the functional (1.1) in the case $g = 0$ is based on the following main steps (cf. [1]):

Step 1. Characterize the class of all Young measures $\boldsymbol{\nu} \in YM(\Omega; K)$ which are generated by sequences of ε-blowups $s \mapsto R_s^\varepsilon v^\varepsilon$ of functions $v^\varepsilon \in \mathrm{H}^2_{loc}(\mathbf{R})$, where $R_s^\varepsilon v(\tau) := \varepsilon^{-1/3} v(s + \varepsilon^{1/3} t)$, $t \in \mathbf{R}$.

Step 2. Rewrite the rescaled functionals $\varepsilon^{-2/3} I_0^\varepsilon(v)$ as $\int_\Omega f_s^\varepsilon(R_s^\varepsilon v) ds$ for a suitable choice of $R_s^\varepsilon v$ and f_s^ε.

Step 3. Identify the Γ-limit f_s of f_s^ε in the topology of K for almost every $s \in \Omega$.

Step 4. Identify the Γ-limit of naturally defined relaxed functionals on the space $YM(\Omega; K)$.

Step 5. Determine the minimizer for the relaxed functional in the limit and prove its uniqueness.

The steps above were subsequently successfully adjusted to the case $g_\varepsilon = g$, $\mathrm{Lip}(g) < 1$ in [10], where ε-blowup $s \mapsto R_s^{\varepsilon, g}$ defined by

$$R_s^{\varepsilon,g} v(t) := \varepsilon^{-1/3} \Big(v(s + \varepsilon^{1/3} t) + g(s) + g'(s) \varepsilon^{1/3} t \Big) \tag{3.1}$$

was used. Theorem 4.17 in [10] implies that there holds

Proposition 3.1. *For every $g \in \mathrm{W}^{1,\infty}(\Omega)$ such that* $\mathrm{Lip}(g) < 1$ *there holds*

$$\lim_{\varepsilon \to 0} \mathcal{E}^\varepsilon(g) = \lim_{\varepsilon \to 0} \mathcal{E}^\varepsilon_{per}(g) = \mathcal{E}(g), \tag{3.2}$$

where $\mathcal{E}(g)$ can be written as $\mathcal{E}(g) = \int_\Omega \psi(a(s), g'(s)) ds$ for some smooth function $\psi : (0, +\infty) \times (-1, 1) \to [0, +\infty)$ (which can be explicitly recovered).

In particular, if $g = 0$, we get minimal asymptotic energy in [1]. In the present note we discuss a more general case when (g_ε) is certain sequence of 1-Lipschitz functions. Throughout the note we always assume that there holds

$$g_\varepsilon \in \mathrm{W}^{1,\infty}(\Omega), \quad \sup\{\mathrm{Lip}(g_\varepsilon) : \varepsilon \in (0, \varepsilon_0)\} \le 1 - \delta, \quad \delta \in (0,1). \tag{3.3}$$

In effect, it is natural to assume that there exists $g \in \mathrm{W}^{1,\infty}(\Omega)$ such that $g_\varepsilon \xrightarrow{*} g$ in $W^{1,\infty}(\Omega)$. Euler-Lagrange equation associated to minimization problem for $I^\varepsilon_{g_\varepsilon}$,

$$\varepsilon^2 \frac{d^4}{ds^4} v - \frac{d}{ds}\sigma\Big(\frac{d}{ds} v\Big) + av = -g_\varepsilon, \quad \sigma := W',\ a := 1,$$

suggests that minimizers of $I^\varepsilon_{g_\varepsilon}$ are close to minimizers of I^ε_g. Then assumption (3.3) ensures that function g satisfies $\mathrm{Lip}(g) < 1$. Consequently, it is reasonable to expect that the minimization problem associated to (1.2) (according to Step 4 and Step 5) when $\varepsilon \to 0$ has the unique minimizer, which can be well-approximated by sawtooth functions with minimal period of order $O(\varepsilon^{1/3})$) (cf. Corollary 4.4). We conjecture that formula for asymptotic energy

$$\lim_{\varepsilon\to 0} \mathcal{E}^\varepsilon(g_\varepsilon) = \lim_{\varepsilon\to 0} \mathcal{E}^\varepsilon_{per}(g_\varepsilon) = \mathcal{E}(g) \tag{3.4}$$

can be obtained in this case, but we were not able to prove it. In this note we essentially require the stronger assumption so as to get (3.4), namely that the sequence (g_ε) is strongly pre-compact in $\mathrm{W}^{1,p}(\Omega)$ for some $p \in [1, +\infty]$ (say). In the case when assumption (3.3) is not satisfied, however, minimizers of $I^\varepsilon_{g_\varepsilon}$ are not easily approximated by sawtooth functions (see, for instance, Proposition 4.3 in [10]). Thus, a completely different behavior of the minimizers is expected when $\mathrm{Lip}(g) > 1$.

4. Some results

To begin with, we recall the following version of the Poincaré inequality (cf. [6], Theorem 2, p. 141):

Proposition 4.1. *Consider $p \in [1, +\infty)$. Then there exists $C_0 = C_0(p)$ such that for every $g \in \mathrm{W}^{1,p}(\Omega)$ there holds*

$$\|g - \fint_\Omega g\|_{\mathrm{L}^p(\Omega)} \le C_0 \|g'\|_{\mathrm{L}^p(\Omega)}. \tag{4.1}$$

Theorem 4.2. *Let*

$$g_\varepsilon \in \mathrm{W}^{1,\infty}(\Omega), \quad g \in \mathrm{W}^{1,\infty}(\Omega), \tag{4.2}$$

$$g'_\varepsilon(s) \longrightarrow g'(s) \ \textit{(a.e. } s \in \Omega\textit{)} \quad \textit{(or } g'_\varepsilon \xrightarrow{\mathrm{L}^p(\Omega)} g' \textit{ for some } p \in [1, +\infty]\textit{)}. \tag{4.3}$$

Then (3.4) holds.

Proof. To begin with, we note that without loss of generality we can assume that for every $\varepsilon \in (0, 1)$ there holds

$$\fint_\Omega g_\varepsilon(s) ds = \fint_\Omega g(s) ds, \tag{4.4}$$

since we can replace $s \mapsto g_\varepsilon(s)$ by $s \mapsto g_\varepsilon(s) - ⨍_\Omega g_\varepsilon + ⨍_\Omega g$ without changing the value of $\mathcal{E}^\varepsilon(g_\varepsilon)$ ($\mathcal{E}^\varepsilon_{per}(g_\varepsilon)$, resp.). This is due to the fact that $v \mapsto v + ⨍_\Omega g_\varepsilon - ⨍_\Omega g$ is one-to-one map from $\mathrm{H}^2(\Omega)$ to $\mathrm{H}^2(\Omega)$ (form $\mathrm{H}^2_{per}(\Omega)$ to $\mathrm{H}^2_{per}(\Omega)$, resp.). Idea of the proof is to consider ε-blowup (3.1). Therefore Step 1 and Step 2 can be easily completed as in [10]. In particular, every $\boldsymbol{\nu} \in YM(\Omega; K)$ which is generated by a sequence of ε-blowups (3.1) satisfies $\nu_s \in \mathcal{I}(K)$ (a.e. $s \in \Omega$). Consider the functional $f^\varepsilon_s : K \to [0, +\infty]$ defined by

$$f^\varepsilon_s(x) := ⨍_{-r}^{r} \Big(\varepsilon^{2/3} x''^2(t) + \varepsilon^{-2/3} W(x'(t) - g'(s))\Big) dt + h^\varepsilon_s(x), \quad x \in \mathrm{H}^2(-r, r)$$

(and $f^\varepsilon_s(x) := +\infty$ otherwise), where $r > 0$ is some fixed number, and $h^\varepsilon_s : K \to [0, +\infty]$ defined by

$$h^\varepsilon_s(x) := ⨍_{-r}^{r} a(s + \varepsilon^{1/3} t)\Big(x(t) - R^{\varepsilon,g}_s(-g_\varepsilon)(t)\Big)^2 dt, \quad x \in \mathrm{L}^2(-r, r)$$

(and $h^\varepsilon_s(x) := +\infty$ otherwise). Then for every $v \in \mathrm{H}^2_{per}(\Omega)$ there holds

$$\varepsilon^{-2/3} I^\varepsilon_{g_\varepsilon}(v) = \int_\Omega f^\varepsilon_s(R^{\varepsilon,g}_s v) ds. \tag{4.5}$$

We claim that the sequence (f^ε_s) Γ-converges (for almost every $s \in \Omega$) to the limit $f_s : K \to [0, +\infty]$ defined by

$$f_s(x) := \frac{A_0}{2r} |Sx' \cap [-r, r)| + a(s) ⨍_{-r}^{r} x^2(t) dt, \tag{4.6}$$

if x is a sawtooth function with slope $\{-1 + g'(s), 1 + g'(s)\}$ (and $f_s(x) := +\infty$ otherwise). To show this, it is enough to prove that the sequence (h^ε_s) continuously converges (for almost every $s \in \Omega$) to the limit $h_s : K \to [0, +\infty]$ defined by

$$h_s(x) := a(s) ⨍_{-r}^{r} x^2(t) dt,$$

if $x \in \mathrm{L}^2(r, -r)$ (and $h_s(x) := +\infty$ otherwise). Then the sequence (f^ε_s) Γ-converges by the well-known theorem of L. Modica and S. Mortola (cf. [7], see also Proposition 3.3 in [1]). Once we have proved that there holds

$$R^{\varepsilon,g}_s(-g_\varepsilon) \to 0 \quad \text{(a.e. } s \in \mathbf{R}), \tag{4.7}$$

the convergence of (h^ε_s) follows by the dominated convergence theorem. We rewrite $R^{\varepsilon,g}_s(-g_\varepsilon)$ as

$$\begin{aligned} R^{\varepsilon,g}_s(-g_\varepsilon)(t) &= \varepsilon^{-1/3}\Big(g(s + \varepsilon^{1/3} t) - g_\varepsilon(s + \varepsilon^{1/3} t)\Big) \\ &\quad - \varepsilon^{-1/3}\Big(g(s + \varepsilon^{1/3} t) - g(s) - g'(s)\varepsilon^{1/3} t\Big). \end{aligned} \tag{4.8}$$

One readily checks that the second term in (4.8) tends to zero. By (4.3) for arbitrary subsequence (ε_k) there holds $g_{\varepsilon_k} \to g$ in $\mathrm{H}^1(\Omega)$. Therefore by the

Egoroff theorem (cf. [6], p. 16) for every $m \in \mathbf{N}$ there exists a measurable set $\Omega_m \subseteq \Omega$ such that there holds

$$\lim_{k\to+\infty} \|g - g_{\varepsilon_k}\|_{\mathrm{W}^{1,\infty}(\Omega_m)} = 0, \quad \lambda(\Omega\backslash\Omega_m) \le \frac{1}{m}.$$

Set $\sigma_k := \varepsilon_k^{1/3}$, $k \in \mathbf{N}$. Consider $\varphi_k(s) := g(s) - g_{\varepsilon_k}(s)$, $s \in \Omega$ (and extend φ_k to $\mathbf{R}$ by periodicity) and $\overline{\varphi}_k(\xi) := \varphi_k(\sigma_k\xi)$, $\xi \in \mathbf{R}$. By (4.4) $\varphi_k \in \mathrm{H}^1(\Omega)$ and $\overline{\varphi}_k \in \mathrm{H}^1(\sigma_k^{-1}\Omega)$ satisfy

$$⨍_\Omega \varphi_k = 0, \quad ⨍_{\sigma_k^{-1}\Omega} \overline{\varphi}_k = 0.$$

Consequently, by Proposition 4.1 we get

$$\begin{aligned}
\sigma_k^{-1}\int_\Omega ⨍_{-r}^{r} |\varphi_k(s+\sigma_k t)|dtds &= ⨍_{-r}^{r}\int_{\sigma_k^{-1}\Omega} |\varphi_k(\sigma_k\rho)|d\rho dt \\
&\le C_1 ⨍_{-r}^{r}\int_{\sigma_k^{-1}\Omega} \left|\frac{d}{d\rho}\varphi_k(\sigma_k\rho)\right| d\rho dt \\
&= C_1\sigma_k \int_{\sigma_k^{-1}\Omega} |\varphi_k'(\sigma_k\rho)|d\rho.
\end{aligned}$$

Further on, we estimate

$$\begin{aligned}
\int_{\sigma_k^{-1}\Omega} |\varphi_k'(\sigma_k\rho)|d\rho &\le \int_{\sigma_k^{-1}\Omega_m} |\varphi_k'(\sigma_k\rho)|d\rho + \int_{\sigma_k^{-1}(\Omega\backslash\Omega_m)} |\varphi_k'(\sigma_k\rho)|d\rho \\
&\le \sigma_k^{-1}\lambda(\Omega_m)\|g' - g_{\varepsilon_k}'\|_{\mathrm{L}^\infty(\Omega_m)} + 2\mathrm{Lip}(g)\sigma_k^{-1}\lambda(\Omega\backslash\Omega_m).
\end{aligned}$$

By passing to the limit as $k \to +\infty$ and then as $m \to +\infty$, we conclude that there exists a subsequence (ε_{k_m}) with properties $\lim_{m\to+\infty}\varepsilon_{k_m} = 0$,

$$\lim_{m\to+\infty} \varepsilon_{k_m}^{-1/3} ⨍_{-r}^{r} \Big(g(s+\varepsilon_{k_m}^{1/3}t) - g_{\varepsilon_{k_m}}(s+\varepsilon_{k_m}^{1/3}t)\Big)dt = 0 \quad \text{(a.e. } s \in \Omega).$$

Thus we proved (4.7). In effect, an application of Proposition 2.11 and Proposition 3.3 in [1] yields $f_s^{\varepsilon_{k_m}} \xrightarrow{\Gamma} f_s$ on K (a.e. $s \in \Omega$) as $m \to +\infty$. Since the argument above can be carried out for arbitrary subsequence (ε_k), Step 3 can be completed. Moreover, Step 4 and Step 5 now can be restated precisely as in [10]: we define the functionals $F_{g_\varepsilon}^\varepsilon, F_g : YM(\Omega;K) \to [0,+\infty]$ by

$$(4.9)\quad F_{g_\varepsilon}^\varepsilon(\boldsymbol{\nu}) := \begin{cases} \int_\Omega \langle \nu_s, f_s^\varepsilon\rangle ds, & \text{if } \boldsymbol{\nu} = \underline{\delta}_{R^{\varepsilon,g}v} \text{ for some } v \in \mathrm{H}^2_{per}(\Omega) \\ +\infty, & \text{otherwise,} \end{cases}$$

$$(4.10)\quad F_g(\boldsymbol{\nu}) := \begin{cases} \int_\Omega \langle \nu_s, f_s\rangle ds, & \text{if } \nu_s \in \mathcal{I}(K) \text{ for a.e. } s \in \Omega \\ +\infty, & \text{otherwise.} \end{cases}$$

Then (taking into account definition of F_g, property $\mathrm{Lip}(g) < 1$ and Proposition 4.3 in [10]) property $f_s^\varepsilon \xrightarrow{\Gamma} f_s$ (a.e. $s \in \Omega$) implies that there holds

$F^{\varepsilon}_{g_\varepsilon} \xrightarrow{\Gamma} F_g$. In particular, by (4.5) there holds

$$\min_{\boldsymbol{\nu}} F^{\varepsilon}_{g_\varepsilon}(\boldsymbol{\nu}) = \min_{v} \varepsilon^{-2/3} I^{\varepsilon}_{g_\varepsilon}(v).$$

Therefore, it follows

$$\lim_{\varepsilon\to 0} \mathcal{E}^{\varepsilon}(g_\varepsilon) = \lim_{\varepsilon\to 0} \min_{\boldsymbol{\nu}} F^{\varepsilon}_{g_\varepsilon}(\boldsymbol{\nu}) = \min_{\boldsymbol{\nu}} F_g(\boldsymbol{\nu}).$$

Hence, by Corollary 5.4 in [10] we deduce (3.4). ∎

It is interesting to note that in some cases it is more natural to consider different ε-blowup rather than to use assumption (4.4) (for example, if $g_\varepsilon(s) = \alpha_\varepsilon s + \beta_\varepsilon$, where $\alpha_\varepsilon \to \alpha$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0$). Precise argument is considered in the following result.

Theorem 4.3. *Let*

$$g_\varepsilon \in \mathrm{C}^1(\Omega), \quad g \in \mathrm{C}^1(\Omega), \tag{4.11}$$

$$g'_\varepsilon \xrightarrow{C} g' \text{ on } \Omega. \tag{4.12}$$

Then (3.4) holds.

Proof. Idea is to consider ε-blowup

$$R^{\varepsilon,\varepsilon}_s v(t) := \varepsilon^{-1/3}\Big(v(s+\varepsilon^{1/3}t) + g_\varepsilon(s) + g'_\varepsilon(s)\varepsilon^{1/3}t\Big). \tag{4.13}$$

We perform the same calculations as in [10]. First, we claim that every Young measure $\boldsymbol{\nu} \in YM(\Omega; K)$ generated by a sequence of ε-blowups (4.13) associated to some sequence (v^ε) is invariant with respect to translations. To prove this, for $s \in \Omega$ we put $u^{\varepsilon,\varepsilon}_s := R^{\varepsilon,\varepsilon}_s v^\varepsilon$ and we calculate

$$\begin{aligned} T^{-1}_\tau u^{\varepsilon,\varepsilon}_s(t) - u^{\varepsilon,\varepsilon}_{s+\varepsilon^{1/3}\tau}(t) &= u^{\varepsilon,\varepsilon}_s(t+\tau) - R^{\varepsilon,\varepsilon}_{s+\varepsilon^{1/3}\tau} v^\varepsilon(t) \\ &= \varepsilon^{-1/3} g_\varepsilon(s) - g'_\varepsilon(s+\varepsilon^{1/3}\tau)t \\ &\quad -\varepsilon^{-1/3} g_\varepsilon(s+\varepsilon^{1/3}\tau) + g'_\varepsilon(s)(t+\tau). \end{aligned}$$

Invariance of $\boldsymbol{\nu}$ is easily obtained by means of Lemma 2.7 and Proposition 3.1 in [1]. Indeed, (4.12) implies that for every $t \in \mathbf{R}$, $\tau \in \mathbf{R}$ and $s \in \Omega$ it results $\lim_{\varepsilon\to 0} g'_\varepsilon(s+\varepsilon^{1/3}\tau)t = g'(s)t$. On the other hand by the Lagrange mean value theorem we can write

$$\varepsilon^{-1/3} g_\varepsilon(s+\varepsilon^{1/3}\tau) - \varepsilon^{-1/3} g_\varepsilon(s) = g'_\varepsilon(\theta_\varepsilon)\tau,$$

where $\theta_\varepsilon \to s$ as $\varepsilon \to 0$. Thus (4.12) implies that for every $\tau \in \mathbf{R}$ the functions $s \mapsto d(T^{-1}_\tau u^\varepsilon_s, u^\varepsilon_{s+\varepsilon^{1/3}\tau})$ converge in measure to 0 as $\varepsilon \to 0$. By Remark 2.6 in [1], sequences $\left(T^{-1}_\tau u^\varepsilon_s\right)$ and $\left(u^\varepsilon_{s+\varepsilon^{1/3}\tau}\right)$ generate the same Young measure $\boldsymbol{\nu}$ as $\varepsilon \to 0$. Lemma 2.7 in [1] implies that sequences $\left(u^\varepsilon_{s+\varepsilon^{1/3}\tau}\right)$ and $\left(u^\varepsilon_s\right)$ also generate the same Young measure as $\varepsilon \to 0$, which is therefore also equal to $\boldsymbol{\nu}$. Then we can check that there holds $\nu_s \in \mathcal{I}(K)$ (a.e. $s \in \Omega$) exactly as in

Proposition 3.1 in [1], which completes Step 1. In the second step we rewrite $\varepsilon^{-2/3} I^{\varepsilon}_{g_\varepsilon}(v)$ as $\int_\Omega f_s^{\varepsilon,\varepsilon}(R_s^{\varepsilon,\varepsilon} v)ds$, where the functional $f_s^{\varepsilon,\varepsilon} : K \to [0,+\infty]$ is defined by

$$f_s^{\varepsilon,\varepsilon}(x) := \fint_{-r}^{r} \Big(\varepsilon^{2/3} x''^2(t) + \varepsilon^{-2/3} W(x'(t) - g'_\varepsilon(s))\Big) dt + h_s^{\varepsilon,\varepsilon}(x), x \in \mathrm{H}^2(-r,r)$$

(and $f_s^{\varepsilon,\varepsilon}(x) := +\infty$ otherwise), where $r > 0$ is some fixed number, and $h_s^{\varepsilon,\varepsilon} : K \to [0,+\infty]$ is defined by

$$h_s^{\varepsilon,\varepsilon}(x) := \fint_{-r}^{r} a(s + \varepsilon^{1/3} t)\Big(x(t) - R_s^{\varepsilon,\varepsilon}(-g_\varepsilon)(t)\Big)^2 dt, \quad x \in \mathrm{L}^2(-r,r)$$

(and $h_s^{\varepsilon,\varepsilon}(x) := +\infty$ otherwise). We can easily check that the sequence $(f_s^{\varepsilon,\varepsilon})$ Γ-converges (for almost every $s \in \Omega$) to the limit f_s defined by (4.6). Indeed, consider $t > 0$ (or, equivalently, $t < 0$). By the Lagrange mean value theorem there exists $\theta_\varepsilon \in (s, s + \varepsilon^{1/3} t)$ such that $\theta_\varepsilon \to s$ as $\varepsilon \to 0$ and such that there holds

$$R_s^{\varepsilon,\varepsilon}(-g_\varepsilon)(t) = -g'_\varepsilon(\theta_\varepsilon)t + g'_\varepsilon(s)t.$$

(4.12) implies $R_s^{\varepsilon,\varepsilon}(-g_\varepsilon) \to 0$ (a.e. $s \in \mathbf{R}$) which (similarly as in the proof of Theorem 4.2) furnishes Step 3. We consider the functionals $F^{\varepsilon,\varepsilon}_{g_\varepsilon}, F_g : YM(\Omega; K) \to [0,+\infty]$ by

$$(4.14) \quad F^{\varepsilon,\varepsilon}_{g_\varepsilon}(\boldsymbol{\nu}) := \begin{cases} \int_\Omega \langle \nu_s, f_s^{\varepsilon,\varepsilon}\rangle ds, & \text{if } \boldsymbol{\nu} = \underline{\delta}_{R^{\varepsilon,\varepsilon} v} \text{ for some } v \in \mathrm{H}^2_{per}(\Omega) \\ +\infty, & \text{otherwise,} \end{cases}$$

$$(4.15) \quad F_g(\boldsymbol{\nu}) := \begin{cases} \int_\Omega \langle \nu_s, f_s\rangle ds, & \text{if } \nu_s \in \mathcal{I}(K) \text{ for a.e. } s \in \Omega \\ +\infty, & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 4.2, it follows $F^{\varepsilon,\varepsilon}_{g_\varepsilon} \xrightarrow{\Gamma} F_g$. Finally, we get (3.4). ∎

Convergence result established in Step 4 in the proof of Theorem 4.3 (combined with the stability of minima for Γ-convergent sequences) provides the following interpretation of asymptotic behavior of the minimizing sequences of $I^{\varepsilon}_{g_\varepsilon}$:

COROLLARY 4.4. *Suppose that (4.11) and (4.12) holds. For small $\varepsilon > 0$ ε-blowups (4.13) of minimizers of $I^{\varepsilon}_{g_\varepsilon}$ are (in the neighborhood of $s \in \Omega$) well approximated by periodic sawtooth function with slope $\pm 1 + g'_\varepsilon(s)$ and with minimal period*

$$\overline{h}_\varepsilon(s) := \Big(\frac{48 A_0}{a(s)}\Big)^{1/3} (1 - g'^2_\varepsilon(s))^{-2/3} \varepsilon^{1/3}.$$

In particular, for small $\varepsilon > 0$ minimizers of $I^{\varepsilon}_{g_\varepsilon}$ are well-approximated by sawtooth functions with slope ± 1. Besides, minimizers of $I^{\varepsilon}_{g_\varepsilon}$ are close to minimizers of I^{ε}_{g}.

Proof. Consider $s \in \Omega$. First claim is a consequence of the Theorem 4.3. Indeed, Theorem 4.3 shows that ε-blowups (4.13) of minimizers v_ε generate (in the point $s \in \Omega$ as $\varepsilon \to 0$) the unique probability measure supported on the set of all translations of $\overline{h}(s)$-periodic sawtooth function $y_s \in K$, where y_s has slope $\pm 1 + g'(s)$, period $\overline{h}(s)$ is defined by

$$\overline{h}(s) := \Big(\frac{48A_0}{a(s)}\Big)^{1/3}(1 - g'^2(s))^{-2/3},$$

and there holds $\fint_0^{\overline{h}(s)} y_s(\tau)d\tau = 0$ (the proof of this fact is essentially contained in Theorem 4.17 and Corollary 5.4 in [10]). On the other hand, if for every $\varepsilon \in (0, \varepsilon_0)$ w^ε minimizes I_g^ε, by Theorem 4.17 in [10] and the later conclusion we obtain

$$\delta_{R_s^{\varepsilon,\varepsilon} v^\varepsilon} - \delta_{R_s^{\varepsilon,g} w^\varepsilon} \xrightarrow{*} 0 \quad \text{in } \mathcal{M}(K) \quad (\text{a.e. } s \in \Omega),$$

which furnishes the second claim of the Corollary. ☐

We conclude our discussion by noting that strong convergence of (g_ε) considered in Theorem 4.2 and Theorem 4.3 was crucial. Therefore the proofs above can not be adjusted in an obvious way so as to provide the results for the general sequence (g_ε). We conjecture that there holds (cf. [9]) :

Conjecture 4.5 (Stability with respect to weak-star convergence). If $g_\varepsilon \in \mathrm{W}^{1,\infty}(\Omega)$, $g \in \mathrm{W}^{1,\infty}(\Omega)$ such that $\mathrm{Lip}(g_\varepsilon) \le 1 - \delta$,

$$g'_\varepsilon \xrightarrow{\mathrm{L}^\infty(\Omega)*} g',$$

(or in $\mathrm{L}^p(\Omega)$ (weakly)) then (3.4) holds. In particular, Corollary 4.4 holds.

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