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ON LINKING OF CANTOR SETS

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Abstract. We introduce a property $\mathcal{L}$ for a subset of a manifold which enables us to pass the geometric linking property from the manifold to this subset. We prove that cubes with handles M and N are linked if and only if subsets $X \subset \operatorname{Int} M$ and $Y \subset \operatorname{Int} N$ having property $\mathcal{L}$ are linked. We present a criterion which shows that many known Cantor sets explicitly given by defining sequences have this property. As an application of the property $\mathcal{L}$ we extend the theorem on rigid Cantor sets thus allowing slightly more complicated terms in their defining sequences.

1. Introduction

Linking of compact sets is strongly related to various problems regarding Cantor sets. Sher [7] used linking argument to prove that varying the number of components in the defining sequence of Antoine necklace yields inequivalently embedded Cantor sets. The definition of simple linking type was modelled on neighbouring tori in the defining sequence for Antoine necklace. Essentially the same linking was used by Daverman and Edwards [4] proving that there exists some class of submanifolds of codimension 2 in a given manifold which can be approximated by Cantor sets. Shilepsky [8] proved that there exists a rigid Cantor set in $\mathbb{E}^3$ using the result of Sher [7]. Wright [9] later generalized this result to arbitrary $\mathbb{E}^n$, $n \geq 3$. In that paper the notion of linking was defined by nontriviality of some homomorphism of certain fundamental groups.

Let $\mathbb{E}^n$ be the n-dimensional Euclidean space and $A, B \subset \mathbb{E}^n$ disjoint closed subsets. We say that A and B are (geometrically) *unlinked* if there exists an $(n-1)$-dimensional sphere $S \subset \mathbb{E}^n$ which separates A and B. We

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say that A and B are (geometrically) *linked* if such a sphere doesn't exist. One usually proves that two sets are geometrically unlinked by explicitly constructing a separating sphere.

Suppose now that we have manifolds M and N with subsets $X \subseteq M$ and $Y \subseteq N$. If X and Y are the cores of respective manifolds it is obvious that X and Y are geometrically linked if and only if M and N are geometrically linked. (This is the common setting for consecutive terms of defining sequence for Antoine necklace.)

We will introduce a property $\mathcal{L}$ which enables us to prove that subsets $X \subseteq M$ and $Y \subseteq N$ having this property are linked if M and N are linked (Theorem 2.4), thus extending linking lemma in [10, 2.1] or linking theorem [9, 4.4] to more complicated terms. As a corollary we will extend the theorem on rigid Cantor sets in $\mathbb{E}^3$ allowing slightly more complicated terms in their defining sequences.

Theorem 2.2 allows us to effectively check whether some finite union of cubes with handles in a given cube with handles has the property $\mathcal{L}$. Although the conditions of the theorem seem technical they can be easily checked for lot of known Cantor sets which are given by defining sequences.

If one replaces the property $\mathcal{L}$ by geometric centrality the criterion similar to Theorem 2.2 can be proved. However it is not known yet (Conjecture 3.2) whether the same replacement can be done in the linking Theorem 2.4.

2. Property $\mathcal{L}$

Let M be a compact manifold with boundary. We denote the interior of the manifold M (in the manifold sense) by $\operatorname{Int} M$ and its boundary by $\operatorname{Fr} M$.

Let $M \subset \mathbb{E}^n$ be a compact n-manifold with boundary. We say that a closed subset $A \subset \operatorname{Int} M$ has the property $\mathcal{L}$ in M, if for every n-disk $B \subset \mathbb{E}^n \setminus A$ and every open neighbourhood $U \subset M \setminus A$ of $\operatorname{Fr} M$ there exists an n-disk $B' \subset \mathbb{E}^n$ such that $B \setminus \operatorname{Int} M = B' \setminus \operatorname{Int} M$ and $B' \cap M \subset U$.

Assertion 2.1. *Let $M \subset \mathbb{E}^n$ be a compact n-manifold with boundary and $S \subset \operatorname{Int} M$ (any) core for M. Then S has property $\mathcal{L}$ in M.*

Proof. Let $B \subset \mathbb{E}^n \setminus S$ be an n-disk and $U \subset M \setminus S$ an open neighbourhood of $\operatorname{Fr} M$ in M. Since S is a core of M there exists a homeomorphism $h = (h_1, h_2)\colon M \setminus S \to \operatorname{Fr} M \times [0,1)$ satisfying $h(x) = (x, 0)$ for $x \in \operatorname{Fr} M$. Hence there exists $\tau \in (0,1)$ such that $h^{-1}(\operatorname{Fr} M \times [0,\tau]) \subset U$. The mapping $f\colon M \setminus S \to h^{-1}(\operatorname{Fr} M \times [0,\tau))$ defined by

$$f(x) = h^{-1}(\, h_1(x), \tau \cdot h_2(x)\,),$$

is a homeomorphism which is the identity on $\operatorname{Fr} M$. Finally we define $B' \cap M$ to be $f(B \cap M)$ since $f(B \cap M) \subset U$. $\square$

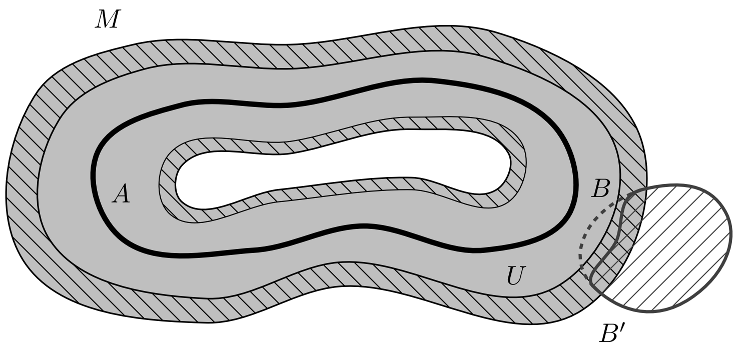

FIGURE 1. The set A has a property $\mathcal{L}$ in M.

THEOREM 2.2. *Let $M \subset \mathbb{E}^3$ be a cube with handles and $N \subset \operatorname{Int} M$ a manifold which is a finite union of cubes with handles. Suppose that there exists a finite collection $\mathfrak{D}$ of 2-disks in $\operatorname{Int} M$ with pairwise disjoint boundaries satisfying the following conditions:*

(i) disks in $\mathfrak{D}$ intersect transversally, interiors of disks in $\mathfrak{D}$ intersect $\operatorname{Fr} N$ transversally and for every disk $D \in \mathfrak{D}$ there exists a component N' of N such that $\operatorname{Fr} D = D \cap N'$;
(ii) no three disks from $\mathfrak{D}$ intersect;
(iii) for every two disks $D, E \in \mathfrak{D}$ the set $D \cap E$ is connected (it may be empty);
(iv) for every disk $D \in \mathfrak{D}$ the set $D \setminus \left(N \cup \bigcup_{E \in \mathfrak{D}\setminus\{D\}} E\right)$ is connected and simply connected;
(v) the set $M \setminus \left(N \cup \bigcup_{D \in \mathfrak{D}} D\right)$ is connected; and
(vi) there exists a set $A \subset N \cup \bigcup_{D \in \mathfrak{D}} D$ with property $\mathcal{L}$ in M.

Then N has the property $\mathcal{L}$ in M.

PROOF. Let us denote $|\mathfrak{D}'| = \bigcup_{E \in \mathfrak{D}'} E$ for any subset $\mathfrak{D}' \subset \mathfrak{D}$ and $\mathfrak{D}_D = \{E \in \mathfrak{D} \setminus \{D\} : \ E \cap D \neq \emptyset\}$ for any $D \in \mathfrak{D}$. Let $\operatorname{Fr} \mathfrak{D} = \bigcup_{D \in \mathfrak{D}} \operatorname{Fr} D$, $N^* = N \cup \bigcup_{E \in \mathfrak{D}} E$ and $N^*_D = N \cup \bigcup_{E \in \mathfrak{D}\setminus\{D\}} E$.

We shall modify the part of the disk B which lies in M so that the modified disk will not intersect A. Hence using the property $\mathcal{L}$ for A one can further modify the disk B such that $B \cap M$ lies arbitrary close to $\operatorname{Fr} M$.

Let $B \subset \mathbb{E}^3$ be a 3-disk disjoint with N. Using a small move in $\operatorname{Int} M \setminus N$ we can assume that $\operatorname{Fr} B$ intersects $|\mathfrak{D}|$ transversally. Fix an arbitrary 2-disk $D \in \mathfrak{D}$. The set $B \cap D$ is either empty or its every component is a disk with (possibly zero) holes.

From (i) it follows that every component of $\operatorname{Int} D \cap N$ is a planar 2-manifold: disk or disk with holes. From (iv) it follows that $D \setminus N$ is connected and therefore every component of $\operatorname{Int} D \cap N$ is a disk.

For every $E \in \mathfrak{D}_D$ the set $D \cap E$ is an arc and one of its boundary points lies in some disk in $\operatorname{Int} D \cap N$ and the other boundary point lies in $\operatorname{Fr} D$. (Both boundary points of the arc $D \cap E$ cannot simultaneously lie in $\operatorname{Fr} D$ since $D \setminus N^*_D$ is connected and similarly they cannot simultaneously lie in N since in this case the set $E \setminus N^*_D$ would not be connected.) By assumption of the theorem no three disks in $\mathfrak{D}$ intersect and hence arcs in $|\mathfrak{D}_D| \cap D$ are pairwise disjoint.

Let J be an arbitrary circle in $D \cap \operatorname{Fr} B$. The circle J bounds a 2-disk (say D_J) in $\operatorname{Int} D$. If $D_J \cap N \neq \emptyset$ then there exists a 2-disk $E \in \mathfrak{D}_D$ such that the arc $D \cap E$ has one boundary point on $N \cap D_J$ (otherwise $D \setminus N^*_D$ is not simply connected) and the other boundary point on $\operatorname{Fr} D$. The circle J bounds two 2-disks (say B_J and B'_J) on $\operatorname{Fr} B$ and both of them are disjoint with $\operatorname{Fr} E$. Hence the intersection number (in $\mathbb{E}^3$) of the circle $\operatorname{Fr} E$ and the 2-sphere $B_J \cup D_J$ equals to 1 which is certainly impossible. Therefore no 2-disk in $N \cap \operatorname{Int} D$ lies inside of any circle in $D \cap \operatorname{Fr} B$.

Let $E \in \mathfrak{D}_D$ be an arbitrary disk. Then $E \cap D$ is an arc having both boundary points outside D_J. If $E \cap D_J \neq \emptyset$ then using Lemma 2.3 one can find a 2-disk $E_J \subset D_J$ which boundary consists of two arcs: one of them lies in J, the other one lies in $E \cap D$. Using a small isotopy having its support in some small neighbourhood of E_J in M one can appropriately move B to reduce the number of arcs in the intersection of $E \cap D$ and D_J.

Hence after finitely many steps we end up with disjoint 1-spheres in $\operatorname{Fr} B \cap D$ and 2-disks in $E \in \mathfrak{D}_D$.

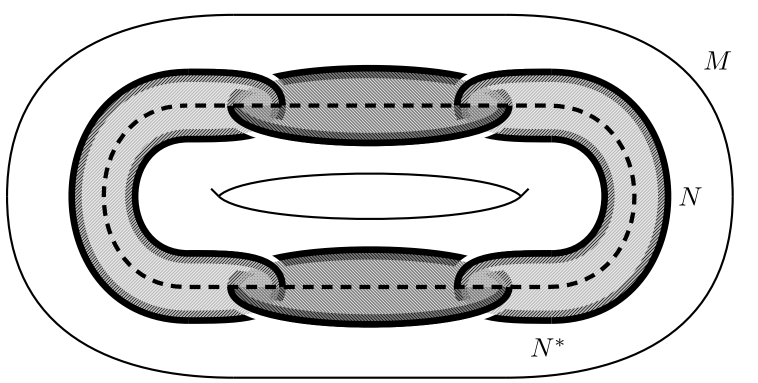

FIGURE 2.

Now choose an outermost (with respect to D) circle in $D \cap \operatorname{Fr} B$ and denote it by K. (The circle K is not necessarily unique.) The circle K bounds some 2-disk D_K in D. Using a small twosided collar of D in M we enlarge D_K to $D_K \times [-1, 1]$. Choose arbitrary $x \in \operatorname{Int} D_K$. Since $M \setminus N^*$ is connected we can connect points $(x, 1) \in \operatorname{Int} D_K \times \{1\}$ and $(x, -1) \in \operatorname{Int} D_K \times \{-1\}$ with some arc $w \subset M \setminus N^*$.

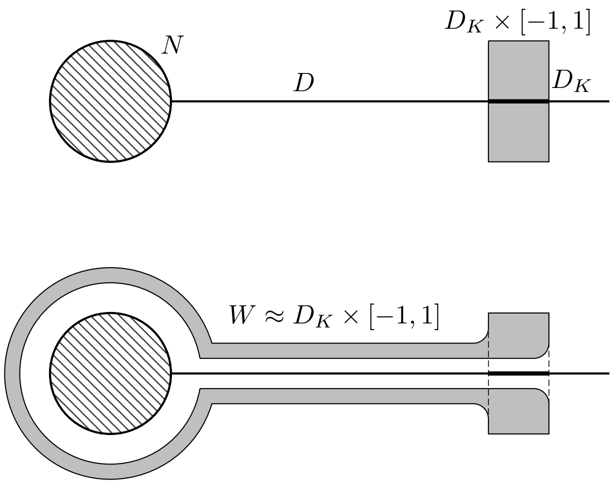

FIGURE 3. Modification of B near D_J.

Let $W = w \times B^2$ be a small tubular neighbourhood of w in $M \setminus N^*$. Obviously $W \approx D_K \times [-1, 1]$ and using an appropriate modification of W near $\operatorname{Fr} w \times B^2$ one can obtain $W \cap B \subset D_K \times [-1, 1]$ and $\operatorname{Fr} w \times B^2 = D_K \times \{-1, 1\}$.

Hence we can divert that part of B which lies in $D_K \times [-1, 1]$ to W. (The choice of outermost component of $D \cap \operatorname{Fr} B$ was necessary here. It is possible though that $B \cap (D_K \times [-1, 1])$ has more than one component.) We repeat the procedure for other circles in $D \cap \operatorname{Fr} B$ starting again with outermost ones.

We repeat the procedure for other 2-disks in $\mathfrak{D}$. We end up with 3-disk B satisfying $\operatorname{Fr} B \cap N^* = \emptyset$. Since $A \subset N^*$ has a property $\mathcal{L}$ we can modify B accordingly. $\square$

We have used the following observation:

LEMMA 2.3. *Let D be a 2-disk and T a nonempty finite collection of pairwise disjoint arcs in D properly embedded in D. The boundaries of arcs in T divide* $\operatorname{Fr} D$ *into collection L of circular arcs. Then there exists a disk $E \subset D$ bounded by an arc from T and an arc from L.*

Proof. We use induction on the number n of arcs in T. The case $n = 1$ is obvious. Now let there be $n + 1$ arcs in T and let t be one of them. Using the inductive hypothesis, there exists a 2-disk E' bounded by $t' \in T$ and $l \in L$ for some t' and l. If $t \cap E' = \emptyset$ then let $E = E'$; otherwise $\operatorname{Fr} t$ splits l in three arcs. In this case the disk E is bounded by one of them and t. $\square$

Theorem 2.4. *Let M and N be cubes with handles in $\mathbb{E}^3$ and $X \subset \operatorname{Int} M$ and $Y \subset \operatorname{Int} N$ closed subsets having property $\mathcal{L}$. Then X and Y are geometrically linked if and only if M and N are geometrically linked.*

Proof. It is obvious that a separating sphere for M and N also separates X and Y. Now assume to the contrary that there exists a separating sphere $S \subset \mathbb{E}^3$ for X and Y while M and N are geometrically linked. Let $B \subset \mathbb{E}^3$ be the 3-disk bounded by S. Due to the symmetry we may assume that $X \subset \operatorname{Int} B$ and $Y \cap B = \emptyset$. We may also assume that S intersects $\operatorname{Fr} M$ and $\operatorname{Fr} N$ transversally.

If $B \cap N \neq \emptyset$ we can use property $\mathcal{L}$ for Y in N to replace B with B' which intersects N near $\operatorname{Fr} N$. Then using a small move near $\operatorname{Fr} N$ we push B' off N to obtain B''. For simplicity we denote the 3-disk B'' with B again. Then $X \subset \operatorname{Int} B$ and $B \cap N = \emptyset$.

If $S \cap \operatorname{Fr} M \neq \emptyset$ we embed $\mathbb{E}^3$ in $S^3 \equiv \mathbb{E}^3 \cup \{\infty\}$ naturally (one-point compactification). Choose an arbitrary point $b \in \operatorname{Fr} B$. Then there exists an arc J in $S^3 \setminus N$ connecting b and ∞ which (except b) lies in $S^3 \setminus B$.

For some small regular neighbourhood $N(J)$ of the arc J in S^3 the manifold $N(J) \cup B$ is a 3-disk disjoint with N. We note that $X \subset N(J) \cup B$.

The complement of $N(J) \cup B$ in S^3 is a 3-disk B' disjoint with X. Since $\infty \in N(J) \cup B$ we use the property $\mathcal{L}$ of X to push the 3-disk B' off M. So we have obtained a 3-disk B'' whose boundary is a sphere S' separating X and Y and disjoint with $\operatorname{Fr} M \cup \operatorname{Fr} N$. Note that $Y \subset \operatorname{Int} B'$ and $X \cap B' = \emptyset$. Let us simplify the notation again and denote B'' simply by B.

Since M and N are geometrically linked we have $B \subset \operatorname{Int} M$ (i.e. $M \subset B$ is not possible). The manifold M is a cube with at least one handle because M and N are linked. Therefore there exists a properly embedded 2-disk D in M (i.e. $\operatorname{Fr} D = D \cap \operatorname{Fr} M$) such that $\operatorname{Fr} D \not\simeq 0$ in $\mathbb{E}^3 \setminus \operatorname{Int} M$. We may assume that $D \cap B = \emptyset$. A small regular neighbourhood $N(D)$ of D in $\mathbb{E}^3$ is a 3-disk which can be pushed off M using property $\mathcal{L}$. This contradicts the fact that $\operatorname{Fr} D \not\simeq 0$ in $\mathbb{E}^3 \setminus \operatorname{Int} M$. $\square$

A defining sequence for a Cantor set $X \subset \mathbb{E}^3$ is a sequence (M_i) of compact 3-manifolds M_i with boundary such that (a) each M_i consists of disjoint cubes with handles, (b) $M_{i+1} \subset \operatorname{Int} M_i$ for each i, and (c) $X = \bigcap_i M_i$. The set of all defining sequences for the Cantor set X is denoted by $\mathcal{D}(X)$. It is known (see [1]) that every Cantor set has a defining sequence, but obviously the defining sequence is not unique.

Definition 2.5. *A defining sequence* $(M_i) \in \mathcal{D}(X)$ *for a Cantor set* $X \subset \mathbb{E}^3$ *consisting of cubes with handles has the property* $\mathcal{L}$ *if for every* i *and every component* M *of* M_i *the manifold* $M \cap M_{i+1}$ *has the property* $\mathcal{L}$ *in* M.

Theorem 2.6. *Let a defining sequence* $(M_i) \in \mathcal{D}(X)$ *for a Cantor set* $X \subset \mathbb{E}^3$ *have property* $\mathcal{L}$. *Then for every* i *and for every component* M *of manifold* M_i *the Cantor set* $X \cap M$ *has property* $\mathcal{L}$ *in* M.

Proof. Let $X' = X \cap M$. Let B be a 3-disk disjoint with X'. First we prove that there exists a 3-disk D such that $D \setminus \operatorname{Int} M = B \setminus \operatorname{Int} M$ and $D \cap M \cap M_{i+1} = \emptyset$. Among all 3-disks D satisfying $D \setminus \operatorname{Int} M = B \setminus \operatorname{Int} M$ and $D \cap X' = \emptyset$ we choose such a 3-disk that the number j, $j \geq i$, such that $D \cap M \cap M_j \neq \emptyset$ and $D \cap M \cap M_{j+1} = \emptyset$ is minimal. If $j > i$ we use property $\mathcal{L}$ to push D out of every component of $M \cap M_{j+1}$ which contradicts the minimality of j. Hence $j = i$ and we use property $\mathcal{L}$ to move $D \cap M$ arbitrary close to $\operatorname{Fr} M$. □

3. Geometric centrality

Let H be a disk with (possibly zero) holes and M a manifold with nonempty boundary. According to the definition in [4] the mapping $f\colon H \to M$, $f(\operatorname{Fr} H) \subset \operatorname{Fr} M$, is *interior inessential*, if there exists some mapping $g\colon H \to \operatorname{Fr} M$, where $f|_{\operatorname{Fr} H} = g|_{\operatorname{Fr} H}$. The mapping $f\colon H \to M$, $f(\operatorname{Fr} H) \subset M$, is *interior essential* if it is not interior inessential.

Recall that $A \subseteq M$ is *geometrically central* in a manifold M if for any 2-disk with holes H and any interior essential mapping $f\colon H \to M$ we have $f(H) \cap A \neq \emptyset$. In other words: if $f(H) \cap A = \emptyset$ then $f\colon H \to M$ is interior inessential and hence there exists a map $g\colon H \to \operatorname{Fr} M$ which coincides with f on $\operatorname{Fr} H$.

It is interesting to note, that one can prove a theorem similar to Theorem 2.2 replacing property $\mathcal{L}$ with geometric centrality. Despite strong similarity between these two theorems, the proofs are completely different.

Theorem 3.1. *Let* $M \subset \mathbb{E}^3$ *be a cube with handles and* $N \subset \operatorname{Int} M$ *be manifold which is finite union of cubes with handles. Let there exists a finite collection* $\mathfrak{D}$ *of 2-disks in* $\operatorname{Int} M$ *with pairwise disjoint boundaries satisfying the following conditions:*

- *(i) disks in* $\mathfrak{D}$ *intersect transversally, interiors of disks in* $\mathfrak{D}$ *intersect* $\operatorname{Fr} N$ *transversally and for every disk* $D \in \mathfrak{D}$ *there exists a component* N' *of* N *such that* $\operatorname{Fr} D = D \cap N'$;
- *(ii) no three disks from* $\mathfrak{D}$ *intersect;*
- *(iii) for every two disks* $D, E \in \mathfrak{D}$ *the set* $D \cap E$ *is connected (it may be empty);*
- *(iv) for every disk* $D \in \mathfrak{D}$ *the set* $D \setminus \left(N \cup \bigcup_{E \in \mathfrak{D} \setminus \{D\}} E\right)$ *is simply connected;*

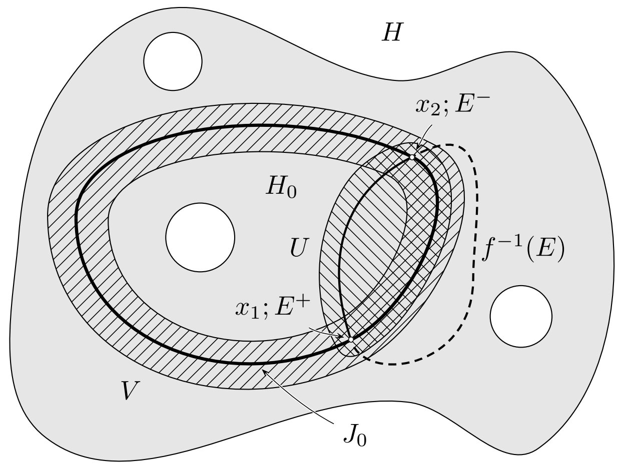

FIGURE 4. Arcs and loops in H.

(v) the set $M \setminus \left(N \cup \bigcup_{D \in \mathfrak{D}} D\right)$ *is connected; and*
(vi) there exists a set $A \subset N \cup \bigcup_{D \in \mathfrak{D}} D$ *being geometrically central in* M*.*
Then N *is geometrically central in* M*.*

PROOF. Introduce $|\mathfrak{D}'|$, $\mathfrak{D}_D$ and N^* as in the proof of Theorem 2.2.

We will make a proof by contradiction. Assume to the contrary that there exists some interior essential mapping $f\colon H \to M \setminus \operatorname{Int} N$ such that $f(\operatorname{Fr} H) \subset \operatorname{Fr} M$. As it is proven below this mapping can be modified on the set $\operatorname{Int} H$ to obtain $f(H) \cap N^* = \emptyset$. But this is impossible since there exists some $A \subset N^*$ which is geometrically central in M.

Choose arbitrary 2-disk $D \in \mathfrak{D}$ and put f transversal to $\mathfrak{D}$. Then the set $f^{-1}(D)$ consists of finitely many simply connected curves in H. The innermost of them (denote it by J_0) bounds some disk with holes H_0 such that $f^{-1}(D) \cap H_0 = J_0$. We will modify f near H_0 such that the modified f will not intersect D (on some neighbourhood of H_0).

Since 2-disks in $\mathfrak{D}$ can intersect each other, the set $f^{-1}(|\mathfrak{D}|)$ consists of transversally intersecting simple closed curves in H. Hence $f^{-1}(|\mathfrak{D}_D|) \cap H_0$ consists of arcs and loops in H_0. Both ends of all arcs lie in J_0. Because of (ii) the arcs in H_0 are pairwise disjoint.

Let us orient J_0 and all 2-disks in $\mathfrak{D}_D$. For every $x \in J_0 \cap f^{-1}(|\mathfrak{D}_D|)$ there exists a unique 2-disk $E \in \mathfrak{D}_D$ such that $x \in J_0 \cap f^{-1}(E)$. The point x will be endowed by E^+ or E^- sign whether $f(J_0)$ intersects E positively or negatively. If there exist two consecutive points (say $x_1, x_2 \in J_0 \cap f^{-1}(E)$) being endowed with different signs (say E^+ in E^-) one can modify f near J_0 and decrease the number of arcs in $f^{-1}(|\mathfrak{D}_D|) \cap H_0$.

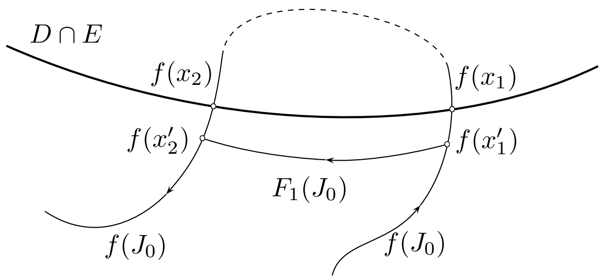

FIGURE 5. A modification of f.

More precisely: let U be some small neighbourhood in H for that 2-disk in H_0 which is bounded by an arc $\widehat{x_1x_2}$ of J_0 from x_1 to x_2 and the respective arc in $f^{-1}(E) \cap H_0$. Choose points $x'_1, x'_2 \in J_0 \setminus \widehat{x_1x_2}$ near x_1 resp. x_2 such that $f(x'_1), f(x'_2) \in U$. Since the set $D \setminus (N \cup |\mathfrak{D}_D|)$ is simply connected the set $f(\widehat{x_1x_2})$ can first be moved by a homotopy to some arc[1] between $f(x_1)$ and $f(x_2)$ which lies in $D \cap E$. Then one can push the (new) arc between $f(x'_1)$ and $f(x'_2)$ off $D \cap E$. Let us denote the join of these two homotopies by $F\colon J_0 \times I \to D \setminus \operatorname{Int} N$. Using [4, Lemma 2] for $X = U$, a manifold $M \setminus \operatorname{Int} N$, the sets $P = D \setminus \operatorname{Int} N$ and $Z = J_0 \cap U$ and appropriate small neighbourhoods for J_0 in U and for $F(J_0 \times I)$ in $M \setminus \operatorname{Int} N$ one can find a neighbourhood $W \subset U$ for J_0 and some mapping $\tilde{f}\colon H_0 \cup U \to M \setminus \operatorname{Int} N$ which coincides with f on the complement of $W \cap U$ such that $\tilde{f}(\widehat{x'_1x'_2}) \cap E = \emptyset$ and $\tilde{f}(W \setminus J_0) \cap D = \emptyset$. Finally replace f on the set $W \cap U$ by $\tilde{f}$.

After finitely many steps one can get such f that no two consecutive points in $J_0 \cap f^{-1}(|\mathfrak{D}_D|)$ have cancelling signs. If there are some arcs in $f^{-1}(|\mathfrak{D}_D|) \cap H_0$ there exits an innermost one in H_0 (say $J' = f^{-1}(E')$) which ends have equal signs. The two points in $\operatorname{Fr} J'$ bound an arc J'' in J_0 whose interior does not intersect E'. But since there exists some small neighbourhood W' of $J' \cup J''$ in H_0 such that f maps that component of $W' \setminus J'$ which intersects J'' on one side of E' the points $\operatorname{Fr} J'$ should have different signs.

Hence we can modify f such that $f^{-1}(|\mathfrak{D}_D|) \cap H_0$ consists of simple closed curves in H_0 only. The loop $f(J_0)$ does not intersect $\mathfrak{D}_D$ and so is the mapping $f|_{J_0}$ homotopic to a constant mapping into $D \setminus |\mathfrak{D}_D|$. Let the boundary of 2-disk D lie in N_D. By assumption of the theorem, the set $M \setminus N^*$ is empty hence f can be modified on some neighbourhood of J_0 such that $f(H)$ does

[1]Such arc exists since $D \cap E$ is connected.

not intersect D. (Indeed: let $[-1,1]\times J_0 \overset{c}{\approx} V \subset H$ be some small double collar of J_0 in H. Using [4, Lemma 2] again one can modify f on some set V to get a mapping g such that $g(J_0)$ is a singleton and $g(V\setminus J_0)\cap D=\emptyset$. Then for some small ε the mapping g can be modified on $V'=c([-\varepsilon,\varepsilon]\times J_0)$ such that $g|_{V'}$ is an arc which intersects $\operatorname{Int} D$ transversally and in one point. Since the set $M\setminus N^*$ is connected one can replace the arc $g(V')$ by such an arc which lies in $M\setminus N^*$.)

Now we can repeat the procedure above for the remaining 2-disks in $\mathfrak{D}$ to obtain the mapping f such that $f(H)\cap N^*=\emptyset$. But since N^* contains some geometrically central set in M we have reached the contradiction. Therefore there exists no interior essential mapping $f\colon H\to M\setminus \operatorname{Int} N$. ◻

Although geometric centrality is very similar to the property $\mathcal{L}$, it is not known yet whether the following linking theorem similar to Theorem 2.4 is valid.

Conjecture 3.2. Let M and N be cubes with handles in $\mathbb{E}^3$ and $X\subset \operatorname{Int} M$ and $Y\subset \operatorname{Int} N$ closed subsets being geometrically central in M resp. N. Then X and Y are geometrically linked if and only if M and N are geometrically linked.

4. Rigid Cantor sets

Definition 4.1. *A defining sequence $(M_i)\in\mathcal{D}(X)$ for the Cantor set $X\subset\mathbb{E}^3$ is* brittle *if for every component M of M_i and for every component M' of $M_{i+1}\cap M$ the following holds: if some loop in $\operatorname{Fr} M$ is contractible in M then this loop is contractible in $(M\setminus M_{i+1})\cup M'$ as well.*

A Cantor set which has a brittle defining sequence has some nice properties. The first and the second item of the following theorem can be proved as in [3, Lemma 5.6], the last item is a slight generalization of [10, Lemma 2.1].

Theorem 4.2. *Let $(M_i)_{i=1}^{\infty}$ be a brittle defining sequence for a Cantor set $X\subset\mathbb{E}^3$, which consists of cubes with handles. Then:*

(i) For every nonempty subset $A\subset X$ every loop $J\subset M_1$ is contractible in $(M_1\setminus X)\cup A$.
(ii) For every dense countable subset $A\subset X$ the set $(\mathbb{E}^3\setminus X)\cup A$ is 1-ULC.
(iii) For every closed proper subset $A\subset X$ there exists a 3-disk $B\subset \operatorname{Int} M_1$ such that $A\subset \operatorname{Int} B$.

Definition 4.3. *Let $A\subset\mathbb{E}^n$ be an arbitrary (closed) set. We say that the set A is* rigid*, if for every homeomorphism $f\colon\mathbb{E}^n\to\mathbb{E}^n$ it holds: if $f(A)=A$ then $f|_A=id_A$.*

There are many examples of rigid sets. Martin [6] has constructed a rigid 2-sphere in E^3, Böthe [2] has constructed a rigid simple closed curve in E^3.

Wright [10] has constructed rigid Cantor set in $\mathbb{E}^3$ using Antoine necklaces and has later [9] generalized construction to $\mathbb{E}^n$, $n \geq 3$.

The key part of the construction in [10] is Lemma 2.1. If we substitute this lemma by Theorem 2.4 we can take more general building blocks in the construction thus constructing many more rigid Cantor sets.

Let $(M_i) \in \mathcal{D}(X)$ be a defining sequence for a Cantor set X. For every component M of M_i one can define a graph Γ_i^M as follows:

- The components of $M \cap M_{i+1}$ are the vertices of Γ_i^M.
- The components M' and M'' of $M \cap M_{i+1}$ are joined by an edge in Γ_i^M if and only if M' and M'' are geometrically linked.

We say that Γ_i^M is the *linking pattern* of X in M. Let

$$\begin{aligned} \Gamma_i &= \bigsqcup_{M \text{ component of } M_i} \Gamma_i^M, \\ \Gamma(X;(M_i)) &= (\Gamma_0, \Gamma_1, \Gamma_2, \ldots). \end{aligned}$$

We say that $\Gamma(X;(M_i))$ is the *linking pattern of X with respect to the defining sequence* (M_i) or simply the *linking pattern of X*, denoted by $\Gamma(X)$.

LEMMA 4.4. *Let Cantor sets X and Y be given by defining sequences $(M_i) \in \mathcal{D}(X)$ and $(N_i) \in \mathcal{D}(Y)$ such that:*

(i) both defining sequences have property $\mathcal{L}$ (see Definition 2.5);
(ii) both defining sequences are brittle (see Definition 4.1);
(iii) for every component M of M_i the graph Γ_i^M is a cycle and for every component N of N_i the graph Γ_i^N is a cycle.

If $h(X) \subset Y$ for some homeomorphism $h\colon \mathbb{E}^3 \to \mathbb{E}^3$ then there exists $n \in \mathbb{N} \cup \{0\}$ and a component V of N_n such that $h(X) = V \cap Y$ and $\Gamma_0^{M_0} \approx \Gamma_n^V$.

PROOF. This is essentially [10, Lemma 3.1]. ∎

THEOREM 4.5. *Let Cantor set X be given by a defining sequence $(M_i) \in \mathcal{D}(X)$ such that:*

(i) defining sequence has property $\mathcal{L}$;
(ii) defining sequence is brittle;
(iii) for every component M of M_i the graph Γ_i^M is a cycle and
(iv) for every two different components M and N of M_i the sequences $\Gamma(M \cap X)$ and $\Gamma(N \cap X)$ are different.

Then X is a rigid Cantor set.

PROOF. This is essentially [10, Theorem 3.2]. ∎

As it was implicitly proven in Lemma 4.4 the Cantor set which was used as a building block in the construction [10] of a rigid Cantor set was unsplittable (i.e. no two of its points can by separated by a 2-sphere in its complement).

Hence by Theorem 4.5 we get a rigid Cantor set whose complement has a nontrivial fundamental group.

However, using a different approach Garity, Repovš and Željko proved [5] that there exists a rigid Cantor set in $\mathbb{E}^3$ whose complement has a trivial fundamental group.

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