GLASNIK MATEMATIČKI
Vol. 41(61)(2006), 177 – 186

LARGE TIME BEHAVIOR OF DIRICHLET HEAT KERNELS ON UNBOUNDED DOMAINS ABOVE THE GRAPH OF A BOUNDED LIPSCHITZ FUNCTION

Kittipat Wong
Chulalongkorn University, Thailand

Abstract. Let $D \subseteq \mathbb{R}^d, d \geq 2$ be the unbounded domain above the graph of a bounded Lipschitz function. We study the asymptotic behavior of the transition density $p^D(t, x, y)$ of killed Brownian motions in D and show that $\lim_{t\to\infty} t^{\frac{d+2}{2}} p^D(t, x, y) = C_1 u(x)u(y)$, where u is a minimal harmonic function corresponding to the Martin point at infinity and C_1 is a positive constant.

1. Introduction

This article is concerned with the large time asymptotic behavior of the Dirichlet heat kernel $p^D(t, x, y)$ on the unbounded domain D above the graph of a bounded Lipschitz function f. Here and in the sequel, by a domain in $\mathbb{R}^d, d \geq 2$, we mean an open connected subset of $\mathbb{R}^d$. The Dirichlet heat kernel is the transition density of the Brownian motion killed upon leaving the domain D. This work was inspired by the results of Pierre Collet, Servet Martínez and Jaime San Martín [2]. Those authors obtained the asymptotic behavior of the Dirichlet heat kernel on an exterior domain, i.e., an unbounded domain which is the complement of a compact nonpolar subset of $\mathbb{R}^d$. More specifically, they proved that for x, y in the plannar exterior domain D,

$$\lim_{t\uparrow\infty} t(\log t)^2 p^D(t, x, y) = \frac{2}{\pi} u_1(x)u_1(y), \tag{1.1}$$

where $u_1(x) = \pi \lim_{|y|\to\infty} G^D(x, y)$ and $G^D(x, y) = \int_0^\infty p^D(t, x, y)\, dt$. Those authors also obtained the result in higher dimensions, namely, for $d \geq 3$, and

2000 *Mathematics Subject Classification.* 60J65, 35J05.
Key words and phrases. Dirichlet heat kernels, asymptotic behavior, Brownian motions.

$x, y \in D$,

$$\lim_{t\uparrow\infty} t^{\frac{d}{2}} p^D(t,x,y) = (2\pi)^{-\frac{d}{2}} u_2(x) u_2(y), \tag{1.2}$$

where $u_2(x) = \lim_{t\uparrow\infty} P_x(\tau_D > t)$ and τ_D is the first exit time of the exterior domain D. We adapt the techniques used by Collet, Martínez and San Martín to obtain the asymptotic behavior of the Dirichlet heat kernel in the case of the unbounded domain D above the graph of a bounded Lipschitz function. Our main result on the asymptotic behavior of the Dirichlet heat kernel $p^D(t,x,y)$ on the unbounded domain above the graph of a bounded Lipschitz function is the following (see Theorem 2.5 for more details). Let $D \subseteq \mathbb{R}^d$ be the domain above the graph of a bounded Lipschitz function f. Then, for any $x, y \in D$, we have

$$\lim_{t\uparrow\infty} t^{\frac{d+2}{2}} p^D(t,x,y) = Hu(x)u(y),$$

for some positive constant H and positive harmonic function u vanishing on the boundary ∂D of D.

2. Main results

In this section, we are going to establish our main results. We fix a bounded real-valued Lipschitz function $f\colon \mathbb{R}^{d-1} \to \mathbb{R}$. Recall that a Lipschitz function f on $\mathbb{R}^{d-1}$ means that there exists a positive constant C such that

$$|f(\tilde{x}_1) - f(\tilde{x}_2)| \le C|\tilde{x}_1 - \tilde{x}_2|, \tag{2.1}$$

for all $\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^{d-1}$. We also fix a domain $D \subseteq \mathbb{R}^d$ as follows:

$$D = \{x = (\tilde{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} = \mathbb{R}^d \colon x_d > f(\tilde{x})\}. \tag{2.2}$$

The unbounded domain D above is called the *domain above the graph of a bounded Lipschitz function*. Since f is bounded, there exist constants a and b such that

$$a \le f(\tilde{x}) \le b,$$

for each $\tilde{x} \in \mathbb{R}^{d-1}$. Define

$$\mathcal{H} = \Big\{h\colon h \text{ is nonnegative and harmonic in } D \text{ and } \lim_{D\ni x\to y} h(x) = 0, \text{ for all } y \in \partial D\Big\}.$$

Notice that $\partial D = \left\{x = (\tilde{x}, x_d) \in \mathbb{R}^d \colon x_d = f(\tilde{x})\right\}$.

Theorem 2.1. *There is only one point corresponding to infinity on the Martin boundary of D and this Martin point is minimal. In particular, this means that $\mathcal{H}$ defined above is one dimensional.*

PROOF. Fix a point $z_0 = (\tilde{z}_0, a-2) \in \mathbb{R}^d$ and consider the inversion with respect to the sphere $S_1(z_0) := \{x \in \mathbb{R}^d \colon |x - z_0| = 1\}$. The image D^* of D under this inversion is a bounded Lipschitz domain with z_0 being the image of infinity. Notice that z_0 is on the boundary of D^*. By Theorem 1.5 on page 337 in [5], the Martin boundary, the minimal Martin boundary and the Euclidean boundary of D^* coincide. This means that each boundary point z of D^* corresponds to a minimal harmonic function v_z. Define a function $u_z(x) = |x^* - z_0|^{d-2} v_z(x^*)$, for $x \in D$ and $x^* = z_0 + \frac{1}{|x-z_0|^2}(x - z_0)$. Then, the Laplacian of u_z vanishes on D (see [1]) and therefore, $\mathcal{H}$ is one dimensional. Moreover, there is only one point corresponding to infinity on the Martin boundary of D and this Martin point is minimal. ∎

REMARK 2.2. From now on we will use u to denote a positive harmonic function in D corresponding to the Martin point at infinity.

We will obtain some property of the function u after the following observation.

Let us recall the Green function G_H for the half space $H := \{x = (\tilde{x}, x_d) \in \mathbb{R}^d \colon x_d > 0\}$. For $x, y \in H$ with $x \neq y$,

$$G_H(x,y) = k(x,y) - k(x,y'),$$

where

$$k(x,y) = \begin{cases} \frac{1}{\pi} \ln\left(\frac{1}{|x-y|}\right) & \text{if } d = 2; \\ |x-y|^{2-d} & \text{if } d \geq 3, \end{cases}$$

and $y' = (y_1, y_2, \ldots, y_{d-1}, -y_d)$ (see page 113 in [6]). Then, for $d = 2$,

$$G_H(x,y) = \frac{1}{\pi} \ln\left(\frac{|x-y'|}{|x-y|}\right).$$

Notice that

$$\begin{aligned} |x-y'|^2 &= |x-y|^2 + 4x_d y_d \\ &= |x-y|^2(1+\beta), \end{aligned}$$

where $\beta = \frac{4x_d y_d}{|x-y|^2} > 0$. Therefore,

$$\begin{aligned} G_H(x,y) &= \frac{1}{\pi} \ln\left(\frac{|x-y|\sqrt{1+\beta}}{|x-y|}\right) \\ &= \frac{1}{2\pi} \ln\left(1 + \frac{4x_d y_d}{|x-y|^2}\right). \end{aligned} \tag{2.3}$$

The above equation will be used in the proof of the next theorem. The notation $f_1 \approx f_2$ means that there exists a positive constant C such that $\frac{1}{C} f_2 \leq f_1 \leq C f_2$.

THEOREM 2.3. *For sufficiently large x_d, we have*

$$u\left(\tilde{x}, x_d\right) \approx x_d.$$

PROOF. Let $x_0 = \left(\tilde{0}, b+100\right) \in D$ and $y_n = \left(\tilde{0}, b+n\right) \in D$. Then, Theorem 1.5 on page 337 in [5] implies that for any $x \in D$,

$$\lim_{n\to\infty} \frac{G_D(x,y_n)}{G_D(x_0,y_n)} = u(x).$$

Define

$$D_a = \{x = (\tilde{x}, x_d) : x_d > a\}, \qquad D_b = \{x = (\tilde{x}, x_d) : x_d > b\}.$$

Then, for any $x \in D_b$, we have

$$G_{D_b}(x,y_n) \le G_D(x,y_n) \le G_{D_a}(x,y_n).$$

Thus, for any $x \in D_b$,

$$\liminf_{n\to\infty} \frac{G_{D_b}(x,y_n)}{G_{D_a}(x_0,y_n)} \le u(x) \le \limsup_{n\to\infty} \frac{G_{D_a}(x,y_n)}{G_{D_b}(x_0,y_n)}. \tag{2.4}$$

From the explicit formulae for G_{D_a} and G_{D_b}, it can be shown that for $d \ge 3$,

$$\begin{aligned} G_{D_a}(x,y) &\approx |x-y|^{2-d} \min\left\{1, \frac{(x_d-a)(y_d-a)}{|x-y|^2}\right\}, \\ G_{D_b}(x,y) &\approx |x-y|^{2-d} \min\left\{1, \frac{(x_d-b)(y_d-b)}{|x-y|^2}\right\}. \end{aligned}$$

Thus, for any $x \in D_b$,

$$u(x) \le C \limsup_{n\to\infty} \frac{\frac{(x_d-a)(y_d^{(n)}-a)}{|x-y_n|^2} \wedge 1}{\frac{(x_d^{(0)}-b)(y_d^{(n)}-b)}{|x_0-y_n|^2} \wedge 1} = C\ \frac{x_d-a}{100},$$

where $x_d^{(0)}$ and $y_d^{(n)}$ are the $d-$th component of x_0 and y_n, respectively. Also,

$$\begin{aligned} u(x) &\ge C \liminf_{n\to\infty} \frac{\frac{(x_d-b)(y_d^{(n)}-b)}{|x-y_n|^2} \wedge 1}{\frac{(x_d^{(0)}-a)(y_d^{(n)}-a)}{|x_0-y_n|^2} \wedge 1} \\ &= C \frac{x_d-b}{x_d^{(0)}-a} = C\frac{x_d-b}{b+100-a}. \end{aligned}$$

For $d = 2$, it follows from (2.3) that the Green functions G_{D_a} and G_{D_b} for D_a and D_b respectively are given by

$$\begin{aligned} G_{D_a}(x,y) &= \frac{1}{2\pi} \ln\left(1 + \frac{4(x_d-a)(y_d-a)}{|x-y|^2}\right) \text{ and} \\ G_{D_b}(x,y) &= \frac{1}{2\pi} \ln\left(1 + \frac{4(x_d-b)(y_d-b)}{|x-y|^2}\right). \end{aligned}$$

Recall the fact that $\ln(1+x) \approx x$ as $x \to 0$. By (2.4),

$$\begin{aligned} u(x) &\leq \limsup_{n\to\infty} \frac{\ln\left(1+\frac{4(x_d-a)(y_d^{(n)}-a)}{|x-y_n|^2}\right)}{\ln\left(1+\frac{4(x_d^{(0)}-b)(y_d^{(n)}-b)}{|x_0-y_n|^2}\right)} \\ &= \limsup_{n\to\infty} \frac{\ln\left(1+\frac{4(x_d-a)(b+n-a)}{|x-y_n|^2}\right)}{\ln\left(1+\frac{4(100)(b+n-b)}{|x_0-y_n|^2}\right)} \leq Cx_d, \end{aligned}$$

for some positive constant C, since both $\frac{4(x_d-a)(b+n-a)}{|x-y_n|^2}$ and $\frac{4(100)(b+n-b)}{|x_0-y_n|^2}$ converge to 0 as $n \to \infty$.

Similarly,

$$\begin{aligned} u(x) &\geq \liminf_{n\to\infty} \frac{\ln\left(1+\frac{4(x_d-b)(y_d^{(n)}-b)}{|x-y_n|^2}\right)}{\ln\left(1+\frac{4(x_d^{(0)}-a)(y_d^{(0)}-a)}{|x_0-y_n|^2}\right)} \\ &= \liminf_{n\to\infty} \frac{\ln\left(1+\frac{4(x_d-b)(n)}{|x-y_n|^2}\right)}{\ln\left(1+\frac{4(b+100-a)(b+n-a)}{|x_0-y_n|^2}\right)} = Cx_d, \end{aligned}$$

for some positive constant C. ∎

THEOREM 2.4. *The function u defined above is invariant, i.e.,*

$$u(x) = \int_D p^D(t,x,y)u(y)\,dy$$

for all $t > 0$ and $x \in D$.

PROOF. To show that u is invariant, it suffices to show (see pages 728 and 669 in [3]) that for some $x \in D$,

$$\lim_{t\uparrow\infty} \int_D p^D(t,x,y)u(y)\,dy > 0.$$

In other words, the condition above means that the function u is not purely excessive. By Theorem 2.3, take $M > b$ so large that

$$u(y) \approx y_d \quad \text{for } \ y_d > M$$

and put $x = (\tilde{0}, 1+M) \in D$. Then, we obtain

$$\begin{aligned}\int_D p^D(t,x,y)u(y)\,dy &\geq \int_{D_b} p^D(t,x,y)u(y)\,dy \\ &\geq \int_{D_b} p^{D_b}(t,x,y)u(y)\,dy \\ &\geq C\int_{\{y:\, y_d>M\}} p^{D_M}(t,x,y)y_d\,dy.\end{aligned}$$

By the explicit formula of

$$p^{D_M}(t,x,y) = C\left(\frac{x_d - M}{\sqrt{t}} \wedge 1\right)\left(\frac{y_d - M}{\sqrt{t}} \wedge 1\right) t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{2t}\right)$$

and for t so large that $1 - \frac{1}{\sqrt{t}} > \frac{1}{2}$ and $\frac{1}{\sqrt{t}} \wedge 1 = \frac{1}{\sqrt{t}}$, we have

$$\begin{aligned}&\int_{\{y:\, y_d>M\}} p^{D_M}(t,x,y)y_d\,dy \\ &\geq C\int_{\{y:\, y_d>M\}}\left(\frac{1}{\sqrt{t}} \wedge 1\right)\left(\frac{y_d - M}{\sqrt{t}} \wedge 1\right) t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{2t}\right)(y_d - M)\,dy \\ &= C\frac{1}{\sqrt{t}}\int_{\{y:\, y_d>M\}}\left(\frac{y_d - M}{\sqrt{t}} \wedge 1\right)(y_d - M)t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{2t}\right)\,dy \\ &= C\frac{1}{\sqrt{t}}\int_M^\infty \left(\frac{y_d - M}{\sqrt{t}} \wedge 1\right)(y_d - M)t^{-\frac{1}{2}} \exp\left(-\frac{|y_d - M - 1|^2}{2t}\right)\,dy_d \\ &\geq C\frac{1}{t}\int_M^{M+\sqrt{t}} (y_d - M)^2 t^{-\frac{1}{2}} \exp\left(-\frac{|y_d - M - 1|^2}{2t}\right)\,dy_d \\ &\geq C\frac{1}{t}\int_M^{M+\sqrt{t}} (y_d - M - 1)^2 t^{-\frac{1}{2}} \exp\left(-\frac{|y_d - M - 1|^2}{2t}\right)\,dy_d \\ &= C\frac{1}{t}\int_{-1}^{\sqrt{t}-1} v^2 t^{-\frac{1}{2}} \exp\left(-\frac{v^2}{2t}\right)\,dv \\ &= C\int_{-\frac{1}{\sqrt{t}}}^{1-\frac{1}{\sqrt{t}}} u^2 \exp\left(-\frac{u^2}{2}\right)\,du \\ &> C\int_0^{\frac{1}{2}} u^2 \exp\left(-\frac{u^2}{2}\right)\,du > 0.\end{aligned}$$

□

Let $d(x)$ be the Euclidean distance from x to the boundary ∂D of D and recall that the function u is the Martin kernel corresponding to the point at infinity and $x_0 = (\tilde{0}, b+100) \in D$. Now, we are ready to establish the main result.

THEOREM 2.5. *Let $D \subseteq \mathbb{R}^d, d \geq 2$, be the domain above the graph of f. For any $x, y \in D$, we have*

$$\lim_{t \uparrow \infty} t^{\frac{d+2}{2}} p^D(t,x,y) = Cu(x)u(y),$$

where u is the function introduced in Remark 2.2 and C is a positive constant. The convergence is uniform on compact subsets of $D \times D$.

PROOF. Let $x, y \in D$ and fix $x_0 = (\tilde{0}, b+100) \in D \cap D_b = D_b$. It is easy to see from the explicit formula of the Dirichlet heat kernel on the half space D_a that $p^{D_a}(t,x_0,x_0) \geq p^{D_a}(t,x,y)$. Therefore,

$$\frac{p^D(t,x,y)}{p^{D_a}(t,x_0,x_0)} \leq \frac{p^D(t,x,y)}{p^{D_a}(t,x,y)} \leq 1. \tag{2.5}$$

Let K be a compact subset of D. Then, there exists $k \in \mathbb{N}$ such that $K \subseteq D_k$, where

$$D_k = \left\{ x \in D \colon d(x) > \frac{1}{k}, |\tilde{x}| \leq k \ \text{ and } \ x_d \leq b + 100 + k \right\}.$$

Therefore, the parabolic Harnack inequality (see Theorem 1 in [4]) implies the existence of a positive constant C such that for $x, y \in K$ and $t > t_0$, for some positive t_0,

$$\begin{aligned} p^D(t,x,y) &\geq Cp^D(t-\varepsilon, x_0, y) \\ &\geq Cp^D(t-\varepsilon, x_0, x_0) \\ &\geq Cp^{D_b}(t-\varepsilon, x_0, x_0), \end{aligned}$$

for some $\varepsilon > 0$. Notice that we can assume that $t_0 > \varepsilon$.

So, we have as $t \uparrow \infty$,

$$\begin{aligned} \frac{p^D(t,x,y)}{p^{D_a}(t,x_0,x_0)} &\geq C\frac{p^{D_b}(t-\varepsilon,x_0,x_0)}{p^{D_a}(t,x_0,x_0)} \\ &\longrightarrow C\frac{(x_d^{(0)}-b)^2}{(x_d^{(0)}-a)^2} > 0, \end{aligned}$$

where $x_d^{(0)}$ is the d-th component of x_0. Thus, the family of functions $\left\{ \frac{p^D(t,x,y)}{p^{D_a}(t,x_0,x_0)} \colon t > t_0 \right\}$ is bounded on compact subsets of $D \times D$. Next, we claim that

$$\sup_{t \geq t_0, |s| \leq 2} \frac{p^{D_a}(t,x_0,x_0)}{p^{D_a}(t+s,x_0,x_0)} < \infty.$$

To see this, for sufficiently large $t > t_0$ and $|s| \le 2$,

$$\begin{aligned}\frac{p^{D_a}(t,x_0,x_0)}{p^{D_a}(t+s,x_0,x_0)} &= \frac{\frac{2}{(2\pi)^{d/2}}(b+100-a)^2 t^{-\frac{d+2}{2}}}{\frac{2}{(2\pi)^{d/2}}(b+100-a)^2 (t+s)^{-\frac{d+2}{2}}} \\ &= \left(\frac{t+s}{t}\right)^{\frac{d+2}{2}} = \left(1+\frac{s}{t}\right)^{\frac{d+2}{2}} < \infty.\end{aligned}$$

Therefore, the family of functions $\left\{\frac{p^D(t,x,y)}{p^{D_a}(t,x_0,x_0)} : t > t_0\right\}$ is equicontinuous on compact subsets of $D \times D$ by Lemma 2.1 in [2]. Therefore, Arzela-Ascoli theorem implies that any sequence converging to infinity contains a subsequence $t_n \uparrow \infty$ such that

$$\lim_{t_n \uparrow \infty} \frac{p^D(t_n,x,y)}{p^{D_a}(t_n,x_0,x_0)} = V(x,y)$$

for some continuous function $V(\cdot,y)$, where the convergence is uniform on compact subsets of D. Note that $V(x,y) \ge C\frac{10^4}{(b+100-a)^2} > 0$ and therefore $V(x,y)$ is nontrivial.

From the semigroup property, we get that for any $s > 0$,

$$\frac{p^D(t_n+s,x,y)}{p^{D_a}(t_n,x_0,x_0)} = \int_D \frac{p^D(t_n,x,\xi)}{p^{D_a}(t_n,x_0,x_0)} \cdot p^D(s,\xi,y)\,d\xi.$$

Recall from (2.5) that $\frac{p^D(t,x,y)}{p^{D_a}(t,x_0,x_0)} \le 1$ for all $t > 0$; $x, y \in D$ and $x_0 \in D_b$. Using this inequality, the Gaussian bound for p^D and the dominated convergence theorem, we obtain

$$V(x,y) = \int_D V(x,\xi)p^D(s,\xi,y)\,d\xi.$$

Therefore, $V(\cdot,y)$ is a nontrivial positive harmonic function vanishing at the boundary ∂D of D. Since $\mathcal{H}$ is one dimensional by Theorem 2.1, $V(\cdot,y) = a(y)u(\cdot)$ for some function $a = a(y)$. By the symmetry of the problem, we have

$$V(x,y) = C_1 u(x)u(y)$$

for some positive constant C_1, which may depend on the subsequence. So,

$$\lim_{t_n \uparrow \infty} \frac{p^D(t_n,x,y)}{p^{D_a}(t_n,x_0,x_0)} = C_1 u(x)u(y).$$

On the other hand, by the explicit formula of p^{D_a}, we obtain

$$\lim_{t_n \uparrow \infty} \frac{p^D(t_n,x,y)}{p^{D_a}(t_n,x_0,x_0)} = \lim_{t_n \uparrow \infty} \frac{t_n^{\frac{d+2}{2}} p^D(t_n,x,y)}{\frac{2}{(2\pi)^{d/2}}(b+100-a)^2}.$$

So,

$$\lim_{t_n \uparrow \infty} \frac{t_n^{\frac{d+2}{2}} p^D(t_n, x, y)}{\frac{2}{(2\pi)^{d/2}}(b + 100 - a)^2} = C_1 u(x)u(y).$$

Hence,

$$\lim_{t_n \uparrow \infty} t_n^{\frac{d+2}{2}} p^D(t_n, x, y) = Cu(x)u(y),$$

where $C = \frac{2C_1}{(2\pi)^{d/2}}(b + 100 - a)^2$. Notice that

$$t_n^{\frac{d+2}{2}} p^{D_b}(t_n, y, y) \leq t_n^{\frac{d+2}{2}} p^D(t_n, y, y) \leq t_n^{\frac{d+2}{2}} p^{D_a}(t_n, y, y).$$

Taking $t_n \uparrow \infty$ and y_d being sufficiently large give us

$$\frac{2}{(2\pi)^{d/2}}(y_d - b)^2 \leq C\,(u(y))^2 \leq \frac{2}{(2\pi)^{d/2}}(y_d - a)^2.$$

Since $u(y)$ behaves asymptotically as y_d by Theorem 2.3, we see that the limit of $t_n^{\frac{d+2}{2}} p^D(t_n, y, y)$ as $t \uparrow \infty$ does not depend on the subsequence t_n, and we conclude that

$$\lim_{t \uparrow \infty} t^{\frac{d+2}{2}} p^D(t, x, y) = Cu(x)u(y).$$

The convergence is uniform on compact subsets of $D \times D$. □

Acknowledgements.
I would like to thank Professor Renming Song for the guidance and discussion of this paper. Also, many thanks go to the referees on the comments of this paper.

References

[1] D. H. Armitage and S. J. Gardiner, Classical Potential Theory, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2001.
[2] P. Collet, S. Martínez and J. San Martín, *Asymptotic behavior of a Brownian motion on exterior domains*, Probab. Theory Related Fields **116** (2000), 303-316.
[3] J. L. Doob, Classical Potential Theory and its Probabilistic Counterpart, Springer-Verlag, Berlin, 2001.
[4] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101-134.
[5] R. G. Pinsky, Positive Harmonic Functions and Diffusion, Cambridge Studies in Advanced Mathematics 45, Cambridge University Press, Cambridge, 1995.
[6] S. C. Port and C. J. Stone, Brownian Motion and Classical Potential Theory, Academic Press, New York, 1978.
[7] N. Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math. **21** (1968), 205-226.

K. Wong
Department of Mathematics
Faculty of Science
Chulalongkorn University
Bangkok, 10330
Thailand
E-mail: kittipat.w@chula.ac.th

Received: 19.1.2005.
Revised: 19.7.2005.