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C^1-Continuous Coons-type Blending of Triangular Patches

C^1-continuous Coons-type blending of triangular patches

SAŽETAK

A Gordon–Coons-type surface construction starts from three differentiable triangular surface patches, which are defined on the same triangular parameter domain. If one boundary curve of each fits a curvilinear triangle, then the defined surface interpolates to these curves. The connection between the resulting surface and the constituents is C^1 continuous along the common boundary curves with the exception of the corner points. This surface definition is an extension of the Gordon–Coons definiton of a triangular surface patch constructed from three boundary curves.

Key words: blending surface, surface modelling, CAGD

MSC 2000: 68U05

Coonsovo povezivanje klase C^1 trokutnih dijelova

SAŽETAK

Gordon-Coonsova konstrukcija plohe kreće od tri diferencijabilna trokutna plošna dijela koji su definirani na iston trokutnom parametarskom području. Ako po jedna rubna krivulja svakog od njih odgovara krivuljnom trokutu, tada definirana ploha interpolira te krivulje. Veza između dobivene plohe i sastavnih dijelova je klase C^1 duž zajedničkih rubnih krivulja, s izuzetkom vrhova. Ova definicija plohe je proširenje Gordon–Coonsove definicije trokutnog plošnog dijela konstruiranog iz tri granične točke.

Ključne riječi: povezivanje ploha, modeliranje ploha, CAGD

1 Introduction

The presented surface definition is based on a classical interpolation method, where the constructed function of two variables has given values on the boundary of a given triangle. The original formulation of the solution of this interpolation problem is the following [1].

If the real-valued function $F(x,y)$ is continuous on the triangle T with vertices $(0,0)$, $(1,0)$ and $(0,1)$ in the xy plane, then the function given by

$$\begin{aligned} W(x,y) = \frac{1}{2}\Big\{ & \left[\frac{1-x-y}{1-y}F(0,y) + \frac{x}{1-y}F(1-y,y)\right] \\ & + \left[\frac{1-x-y}{1-x}F(x,0) + \frac{y}{1-x}F(x,1-x)\right] \\ & + \left[\frac{x}{x+y}F(x+y,0) + \frac{y}{x+y}F(0,x+y)\right] \\ & - [xF(1,0) + yF(0,1) + (1-x-y)F(0,0)] \Big\} \end{aligned}$$

is continuous over T and interpolates to the values of F on its boundary, i.e. along the curves $x=0$, $y=0$ and $1-x-y=0$ [3, §8.2].

A geometric interpretation of this interpolation problem is the construction of Gordon–Coons triangular surface patches, which is the triangular version of the well-known construction of rectangular Coons patches [2] extended in [6]. A Gordon–Coons surface patch is generated by the above formula from three continuous curve segments forming a spatial curvilinear triangle, which are the boundary curves of the generated patch. The boolean sum of convex combinations of three pairs of the given curves is corrected with a convex combination of the vertex points (Fig 1).

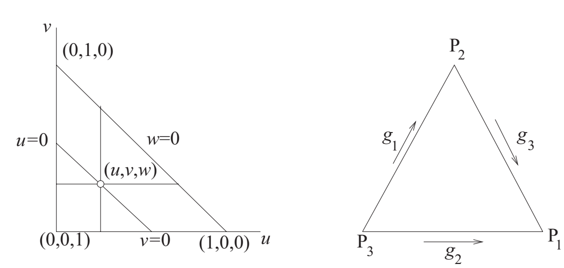

Figure 1: The parameter triangle and boundary curves.

As the convex combination is invariant with respect to affine transformations, the standard parameter triangle can

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be transformed affinely, and barycentric coordinates can be used with respect to a base triangle with the vertices $(0,0,1)$, $(1,0,0)$ and $(0,1,0)$ [4, §18]. The parameter triangle is determined by $0 \le u,v,w \le 1$ and $u+v+w=1$. The three continuous input curves are defined on the boundaries of the parameter triangle. Their representing vector functions are expressed with barycentric coordinates written in a symmetric form.

$\mathbf{g}_1(0,v,1-v)$ is defined over the edge $u=0$, $\mathbf{g}_2(u,0,1-u)$ over the edge $v=0$ and $\mathbf{g}_3(u,1-u,0)$ over the edge $w=0$, which can be written also as $\mathbf{g}_3(1-v,v,0)$ substituting $u=1-v$.

If the three curves satisfy the boundary conditions

$$\begin{aligned} \mathbf{g}_2(1,0,0) &= \mathbf{g}_3(1,0,0) = \mathbf{P}_1, \\ \mathbf{g}_1(0,1,0) &= \mathbf{g}_3(0,1,0) = \mathbf{P}_2 \quad \text{and} \\ \mathbf{g}_1(0,0,1) &= \mathbf{g}_2(0,0,1) = \mathbf{P}_3, \end{aligned}$$

then the surface patch given by the vector function

$$\begin{aligned} \mathbf{r}(u,v,w) = \frac{1}{2}\Big\{ & \left[\frac{w}{u+w}\mathbf{g}_1(0,v,1-v) + \frac{u}{u+w}\mathbf{g}_3(1-v,v,0)\right] \\ & + \left[\frac{w}{v+w}\mathbf{g}_2(u,0,1-u) + \frac{v}{v+w}\mathbf{g}_3(u,1-u,0)\right] \\ & + \left[\frac{u}{u+v}\mathbf{g}_2(u+v,0,1-u-v) \right.\\ & \qquad \left. + \frac{v}{u+v}\mathbf{g}_1(0,u+v,1-u-v)\right] \\ & - \left[u\mathbf{g}_3(1,0,0) + v\mathbf{g}_1(0,1,0) + w\mathbf{g}_2(0,0,1)\right]\Big\}, \\ & 0 \le u,v,w \le 1, \quad u+v+w=1 \end{aligned} \tag{1}$$

interpolates the input curves along the edges of the parameter triangle.

The other surface definition, which we use in our surface construction, was given for the construction of a C^1 continuous triangular interpolant in [5] as follows.

If three functions F_i, $i=1,2,3$ are C^2 differentiable on the triangle T described with the barycentric coordinates u, v and w, $0 \le u,v,w \le 1$, $u+v+w=1$, and each of them interpolates one vertex of T and a vector field along its opposite side, then the function given by

$$DF = \frac{u^2w^2F_1 + v^2w^2F_2 + u^2v^2F_3}{u^2w^2 + v^2w^2 + u^2v^2} \tag{2}$$

is differentiable, and interpolates the values and the first partial derivatives of the given "underlying" surfaces F_1, F_2 and F_3 (consequently, also the given vector fields) along the edge $u=0$, $v=0$ and $w=0$ of the triangle, respectively.

This convex combination scheme was applied and investigated for three differentiable triangular surface patches defined on the same parameter domain in [7]. However, the problem ensuring the compatibility conditions for the input surfaces at the corner points is not solved in general. Therefore, the continuity of the defined surface at the vertices is not ensured.

A generalization of the Gordon–Coons surface construction in (1) was given with three triangular surface constituents in [8] as follows.

Let $\mathbf{r}_1(u,v,w)$, $\mathbf{r}_2(u,v,w)$ and $\mathbf{r}_3(u,v,w)$ be continuous vector functions defined on the parameter triangle $0 \le u,v,w \le 1$, $u+v+w=1$, representing three triangular surface patches with common corner points (Fig 2)

$$\begin{aligned} \mathbf{r}_1(0,0,1) &= \mathbf{r}_2(0,0,1) = \mathbf{P}_3, \\ \mathbf{r}_1(0,1,0) &= \mathbf{r}_3(0,1,0) = \mathbf{P}_2, \\ \mathbf{r}_2(1,0,0) &= \mathbf{r}_3(1,0,0) = \mathbf{P}_1. \end{aligned}$$

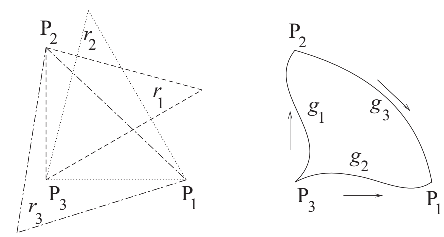

Figure 2: Three input surface patches and auxiliary curves.

The weighting ("blending") functions are

$$\begin{aligned} \mu_1 &= \frac{(1-\lambda_1)w^2}{\lambda_1 u^2 + (1-\lambda_1)w^2}, \\ \mu_2 &= \frac{(1-\lambda_2)u^2}{\lambda_2 v^2 + (1-\lambda_2)u^2}, \\ \mu_3 &= \frac{(1-\lambda_3)v^2}{\lambda_3 w^2 + (1-\lambda_3)v^2}, \end{aligned}$$

where $0 \le \lambda_1, \lambda_2, \lambda_3 \le 1$ are shape parameters of values between 0 and 1, and at the corner points

$$\mu_1(0,1,0) := 1, \quad \mu_2(0,0,1) := 1, \quad \mu_3(1,0,0) := 1$$

are required.

Definition 1. The surface patch is defined by the vector function

$$\begin{aligned} \mathbf{f}(u,v,w) = \frac{1}{2}\big[& \mu_1\mathbf{r}_1 + (1-\mu_1)\mathbf{r}_3 + \mu_2\mathbf{r}_2 + (1-\mu_2)\mathbf{r}_1 \\ & + \mu_3\mathbf{r}_3 + (1-\mu_3)\mathbf{r}_2 - \mathbf{q}(u,v,w)\big], \end{aligned} \tag{3}$$

where $\quad \mathbf{q}(u,v,w) =$

$$\begin{aligned} &\frac{v^2w^2\mathbf{g}_1(0,v,1-v) + u^2w^2\mathbf{g}_2(u,0,1-u) + u^2v^2\mathbf{g}_3(u,1-u,0)}{u^2w^2 + v^2w^2 + u^2v^2} \\ &\qquad 0 \le u,v,w \le 1, \quad u+v+w=1 \end{aligned} \tag{4}$$

is a correction term generated from the auxiliary curves $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ over the boundaries of the parameter triangle.

$$\begin{aligned}
\mathbf{g}_1(0,v,1-v) &= [\mu_3\mathbf{r}_3 + (1-\mu_3)\mathbf{r}_2]_{(0,v,1-v)}\ ,\\
\mathbf{g}_2(u,0,1-u) &= [\mu_1\mathbf{r}_1 + (1-\mu_1)\mathbf{r}_3]_{(u,0,1-u)}\ ,\\
\mathbf{g}_3(u,1-u,0) &= [\mu_2\mathbf{r}_2 + (1-\mu_2\mathbf{r}_1]_{(u,1-u,0)}
\end{aligned}$$

are blended curves over the sides $u=0$, $v=0$ and $w=0$, respectively of the triangular parameter domain. $\diamond$

The surface $\mathbf{f}(u,v,w)$ matches the boundary curves $\mathbf{r}_1(0,v,1-v)$, $\mathbf{r}_2(u,0,1-u)$ and $\mathbf{r}_3(u,1-u,0)$, $0 \le u \le 1$, $0 \le v \le 1$ [8].

The structure of this scheme is similar to that of Gordon–Coons' construction, where the boolean sum of three convex combinations of the given constituents is corrected according to the interpolation condition. Here the correction function has the structure of the scheme in (2) and fits the auxiliary curves:

$$\begin{aligned}
&\mathbf{q}(0,v,1-v) = \mathbf{g}_1(0,v,1-v),\\
&\mathbf{q}(u,0,1-u) = \mathbf{g}_2(u,0,1-u),\\
&\mathbf{q}(1-v,v,0) = \mathbf{q}(u,1-u,0) =\\
&\qquad \mathbf{g}_3(1-v,v,0) = \mathbf{g}_3(u,1-u,0).
\end{aligned}$$

The connection between the resulting surface and the input surface constituents is C^0 along the common boundary curves.

The shape parameters $(\lambda_1,\lambda_2,\lambda_3) = \underline{\lambda}$ are either specified by the user, or can be determined from a fairing condition. We have used the linearized thin plate energy function with $\overline{\mathbf{f}}(u,v) = \mathbf{f}(u,v,w)\big|_{w=1-u-v}$,

$$E(\underline{\lambda}) = \int_A (\overline{\mathbf{f}}^2_{uu} + 2\overline{\mathbf{f}}^2_{uv} + \overline{\mathbf{f}}^2_{vv})\,dA, \quad A = [0,1]\times[0,1]. \qquad (5)$$

The optimal values of λ_1, λ_2 and λ_3 are computed by minimizing $E(\underline{\lambda})$. (In the equations the indices u and v denote the partial derivatives with respect to u and v, respectively.) The integral has been approximated by an integral sum computed at 9 inner points, and the numerical minimization has been carried out by the symbolic algebraic program package Mathematica.

For drawing triangular patches with Mathematica the parameter triangle had to be transformed into a rectangle by substituting $u = t - st$, $v = st$, $s,t \in [0,1]$. Therefore, the patches appear in the figures with s and t parameter lines.

2 Examples

In Fig 3 three triangular surface patches are shown, which are defined as quadratic Bézier surfaces. One auxiliary curve and the correction term $\mathbf{q}(u,v,w)$ is shown in Fig 4 and in Fig 5, respectively. The resulting surface defined in (3) is shown in Fig 6. It joins to the input surfaces with C^0 continuity along their connection curves.

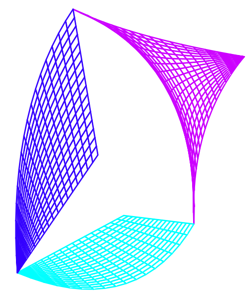

Figure 3: Three Bézier patches.

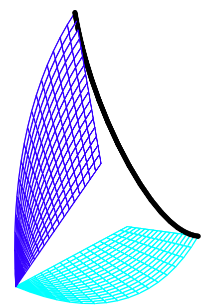

Figure 4: One auxiliary curve.

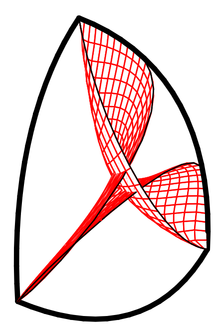

Figure 5: The correction function defined from the auxiliary curves.

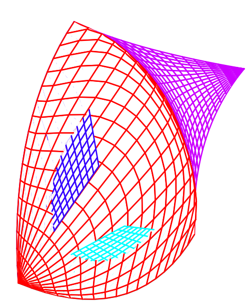

Figure 6: The resulting surface.

3 C^1 continuous blending surface constructed from differentiable patches

In this chapter a new definition of a triangular Gordon–Coons-type surface patch will be given. It is determined by three differentiable triangular patches, where three boundary curves, one of each patch, form a curvilinear triangle. The resulting patch fits these boundary curves and has a C^1-continuous connection to the given constituents along them.

Now we investigate the partial derivatives of the vector function defined in (3) along the edges of the parameter triangle T. Computing the partial derivatives with barycentric coordinates we get the following.

Along the edge $u = 0$ $\mathbf{f}_v = \mathbf{r}_{1v}$, $\mathbf{f}_w = \mathbf{r}_{1w}$ and

$$\mathbf{f}_u = \left[\mathbf{r}_{1u} + \frac{1}{2}(\mu_3\mathbf{r}_{3u} + (1-\mu_3)\mathbf{r}_{2u})\right]\Big|_{u=0}. \tag{6}$$

Along the edge $v = 0$ $\mathbf{f}_u = \mathbf{r}_{2u}$, $\mathbf{f}_w = \mathbf{r}_{2w}$ and

$$\mathbf{f}_v = \left[\mathbf{r}_{2v} + \frac{1}{2}(\mu_1\mathbf{r}_{1v} + (1-\mu_1)\mathbf{r}_{3v})\right]\Big|_{v=0}. \tag{7}$$

Along the edge $w = 0$ $\mathbf{f}_u = \mathbf{r}_{3u}$, $\mathbf{f}_v = \mathbf{r}_{3v}$ and

$$\mathbf{f}_w = \left[\mathbf{r}_{3w} + \frac{1}{2}(\mu_2\mathbf{r}_{2w} + (1-\mu_2)\mathbf{r}_{1w})\right]\Big|_{w=0}. \tag{8}$$

In order to get C^1 continuous connection between the resulting surface represented by $\mathbf{f}(u,v,w)$ and the constituents

$$\mathbf{f}_u\big|_{u=0} = \mathbf{r}_{1u}\big|_{u=0}, \quad \mathbf{f}_v\big|_{v=0} = \mathbf{r}_{2v}\big|_{v=0}, \quad \mathbf{f}_w\big|_{w=0} = \mathbf{r}_{3w}\big|_{w=0},$$

must be ensured. For this an additional correction term is needed in Definition 1. Its value has to be zero along the boundary curves, and its partial derivatives have to annulate the second terms of the partial derivatives in the expressions (6), (7) and (8). The following vector function satisfies these requirements

$$\begin{aligned}\mathbf{s}(u,v,w) = \frac{1}{2}\big[&\kappa_1(\mu_3\mathbf{r}_{3u} + (1-\mu_3)\mathbf{r}_{2u})\big|_{u=0} \\ &+ \kappa_2(\mu_1\mathbf{r}_{1v} + (1-\mu_1)\mathbf{r}_{3v})\big|_{v=0} \\ &+ \kappa_3(\mu_2\mathbf{r}_{2w} + (1-\mu_2)\mathbf{r}_{1w})\big|_{w=0}\big]\end{aligned} \tag{9}$$

with the blending functions

$$\begin{aligned}\kappa_1 = \frac{uv^2w^2}{\Sigma}, \quad \kappa_2 = \frac{vu^2w^2}{\Sigma}, \quad \kappa_3 = \frac{wu^2v^2}{\Sigma}, \\ \Sigma = u^2w^2 + v^2w^2 + u^2v^2.\end{aligned} \tag{10}$$

Obviously,

$$\begin{aligned}&\kappa_i\big|_{u=0} = 0, \quad \kappa_i\big|_{v=0} = 0, \quad \kappa_i\big|_{w=0} = 0, \quad i = 1,2,3, \\ &\kappa_{1u}\big|_{u=0} = 1, \quad \kappa_{1v}\big|_{v=0} = 0, \quad \kappa_{1w}\big|_{w=0} = 0, \\ &\kappa_{2u}\big|_{u=0} = 0, \quad \kappa_{2v}\big|_{v=0} = 1, \quad \kappa_{2w}\big|_{w=0} = 0, \\ &\kappa_{3u}\big|_{u=0} = 0, \quad \kappa_{3v}\big|_{v=0} = 0, \quad \kappa_{3w}\big|_{w=0} = 1.\end{aligned}$$

The required surface is defined by extending Definition 1 in the following way.

Definition 2.

$$\begin{aligned}\mathbf{f}(u,v,w) = \frac{1}{2}\big[&\mu_1\mathbf{r}_1 + (1-\mu_1)\mathbf{r}_3 + \mu_2\mathbf{r}_2 + (1-\mu_2)\mathbf{r}_1 \\ &+ \mu_3\mathbf{r}_3 + (1-\mu_3)\mathbf{r}_2\big] - \mathbf{q}(u,v,w) - \mathbf{s}(u,v,w), \\ &0 \le u,v,w \le 1, \quad u+v+w = 1,\end{aligned} \tag{11}$$

where $\mathbf{q}(u,v,w)$ is defined in (4), $\mathbf{s}(u,v,w)$ in (9) with the weighting functions in (10) $\diamond$

Considering the computed derivatives, we have obtained the following theorem.

Theorem 1. *Assume that three surface paches are given by the differentiable vector functions $\mathbf{r}_1(u,v,w)$, $\mathbf{r}_2(u,v,w)$ and $\mathbf{r}_3(u,v,w)$ on the parameter triangle $0 \le u,v,w \le 1$, $u+v+w = 1$ with common corner points, i.e.*

$$\begin{aligned}\mathbf{r}_1(0,0,1) &= \mathbf{r}_2(0,0,1), \\ \mathbf{r}_1(0,1,0) &= \mathbf{r}_3(0,1,0), \\ \mathbf{r}_2(1,0,0) &= \mathbf{r}_3(1,0,0).\end{aligned}$$

Then the surface represented by the vector function in Definition 2 interpolates the boundary curves $\mathbf{r}_1\big|_{u=0}$, $\mathbf{r}_2\big|_{v=0}$ and $\mathbf{r}_3\big|_{w=0}$, and joins to the corresponding surface patch C^1 continuously along the common boundary with the exception of the corner points.

Proof. The proof follows from the computations above. However, the compatibility conditions of the differentiability of the resulting surface at the vertices require further investigations. $\square$

4 Examples

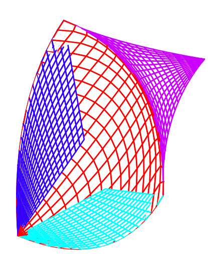

Figure 7: C^1 continuous surface defined from the constituents in Fig 3.

In Fig 7 the surface constructed by Definition 2 is shown. It is generated from the same quadratic Bézier patches as the surface in Fig 6. There is a visible difference between the C^0 and C^1 results. While the C^0 surface is rather round, and intersects the constituents, the C^1 result has common tangent planes with them along the common boundary curves. The next two figures illustrate the effect of the shape parameters λ_i included in the blending coefficients μ_i, $i = 1,2,3$. In the equation of the resulting surface in Fig 7 the shape parameters have been determined from the fairing condition by minimizing the energy function in (5). The same surface is shown from a side view in Fig 8.

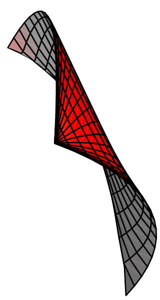

Figure 8: The surface in Fig 7 from the side.

In Fig 9 the surface is generated from the same constituents, but the shape parameters have been given as user inputs. The value of λ_3 influencing the weight of the given patch on the right hand side has been raised. Consequently, the result is less concave in the middle.

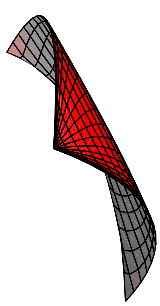

Figure 9: The surface generated with different shape parameters.

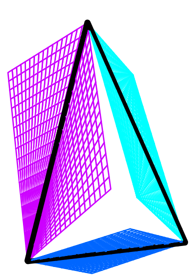

Figure 10: An open corner on a prism.

In Fig 10 an open corner on a prism is shown modelled with triangular Bézier patches. The boundary of the triangular hole is drawn with heavy lines. The constituents in the surface definition are in the inside of this triangle. The neighbouring triangles are coplanar extensions of them.

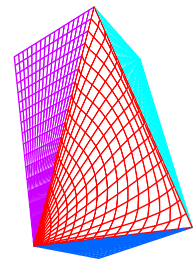

Figure 11: The C^1 resulting surface with the extensions of the constituents.

The constructed surface is shown in Fig 11. It fits the boundary and has common tangent planes with the neighbouring surfaces.

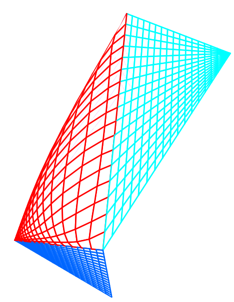

Figure 12: The same solution from a different view.

In Fig 12 the same surface is shown from a side view in order to make the comparison with the next examples easier.

The next figures show the shaping effect of the constituents. In Fig 13 different constituents with the same boundary curves and tangent planes are shown, the resulting C^1 surface is shown in Fig 14.

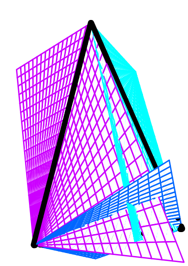

Figure 13: Constituents with the same boundaries.

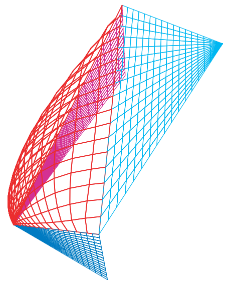

Figure 14: The result has a different shape.

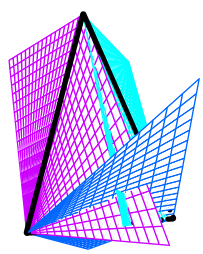

Figure 15: Changing one input patch.

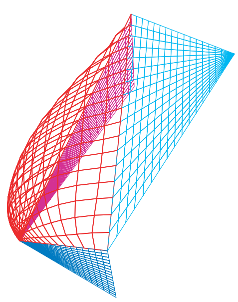

Figure 16: The effect on the inner shape of the resulting surface.

In Fig 15 the input patch on the lower side has been changed while keeping its boundary fixed. The result with these constituents is shown in Fig 16.

5 Conclusions

We have presented a new surface definition, which generates a triangular patch from three triangular surface patches. Novel in this definition is that the inner shape of the resulting surface can be modified by changing the input surface patches while keeping the boundary conditions fixed. Moreover, new is the introduction of shape parameters in the blending functions. This surface construction can be applied for filling triangular holes which occur in modelling of composite surfaces.

References

[1] Barnhill, R.E., Birkhoff, G. and Gordon, W.S.: *Smooth Interpolation in Triangles*, Journal of Approximation Theory, 8, 1973. pp. 114–128.

[2] Coons, S.A.: *Surfaces for computer-aided design of space forms*, Project MAC report, 1964.

[3] Hoschek, J., Lasser, D.: Grundlagen der geometrischen Datenverarbeitung, B. G. Teubner Stuttgart, 1992.

[4] Farin, G.: Curves and Surfaces for Computed Aided Geometric Design, Academic Press, London, 1990.

[5] Nielson, G.M.: *The Side-vertex Method for Interpolation in Triangles*, Journal of Approximation Theory, 25, 1979. pp. 318–336.

[6] Szilvási-Nagy, M., Vendel, T.P., Stachel, H.: *C^2 filling of gaps by convex combination of surfaces under boundary constrains*, KoG, 6, 2002. pp. 41–48.

[7] Szilvási-Nagy, M.: *Filling triangular holes by convex combination of surfaces*, Periodica Polytechnica Mech. Engrg., 47, 2003. pp. 81–89.

[8] Szilvási-Nagy, M., Szabó, I.: *Generalization of Coons' Construction*, manuscript, submitted in computers & graphics

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