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COMMUTATIVITY PRESERVING MAPS ON $\mathbf{M_n}(\mathbb{R})$

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Abstract. We obtain the general form of continuous injective maps on $M_n(\mathbb{R})$, $n > 3$, that preserve commutativity.

1. Introduction and the statement of the main result

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem which concerns characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant (see [8, 11]). Such problems arise in most parts of mathematics. In fact, it turns out that in many cases the corresponding results provide important information on the automorphisms of the underlying structures. In the last few decades a lot of results on linear preservers on matrix algebras as well as on more general rings and operator algebras have been obtained (see [9]). Besides linear preservers also a more general problem of characterizing additive preservers and related problem of characterizing multiplicative preservers on matrix algebras were studied a lot. It is surprising that in some cases we can get nice structural results for preservers without any algebraic assumption like linearity, additivity or multiplicativity. Probably the first fundamental attempt to attack non-linear preserver problems was made by Baribeau and Ransford in [1]. They studied spectrum preserving non-linear maps of matrix algebras under some mild differentiability condition.

Linear preserver problems concerning commutativity are one of the most extensively studied preserver problems both on matrix algebras and on operator algebras (see, for example, [2, 3, 4, 10, 15] and the references therein). The reason is that the assumption of preserving commutativity can be considered

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as the assumption of preserving zero Lie products. Because of applications in quantum mechanics it is of interest to study also a more difficult problem of characterizing non-linear commutativity preserving maps.

In this paper we will study non-linear commutativity preserving maps on $M_n(\mathbb{R})$, the algebra of all $n \times n$ real matrices. A map $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ *preserves commutativity* if $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$, $A, B \in M_n(\mathbb{R})$. If ϕ is bijective and both ϕ and ϕ^{-1} preserve commutativity, then we say that ϕ *preserves commutativity in both directions*. In [13] Šemrl characterized bijective continuous maps on $M_n(\mathbb{C})$, the algebra of all $n \times n$ complex matrices, that preserve commutativity in both directions. The continuity assumption and the assumption $n \geq 3$ are indispensable in this theorem (see [13] for counterexamples). The analogous result holds true also for $M_n(\mathbb{R})$ (see [5]). Recently Šemrl [14] considered injective commutativity preserving maps on $M_n(\mathbb{C})$, $n > 3$, that are continuous and are not assumed to be linear. He proved that every continuous injective map on $M_n(\mathbb{C})$, $n > 3$, that preserves commutativity is of the form $\phi(A) = Tp_A(A)T^{-1}$ for all $A \in M_n(\mathbb{C})$, or $\phi(A) = Tp_A(A^t)T^{-1}$ for all $A \in M_n(\mathbb{C})$, or $\phi(A) = Tp_A(\overline{A})T^{-1}$ for all $A \in M_n(\mathbb{C})$, or $\phi(A) = Tp_A(A^*)T^{-1}$ for all $A \in M_n(\mathbb{C})$, where $T \in M_n(\mathbb{C})$ is an invertible matrix and $A \mapsto p_A(A)$ is a locally polynomial map. The natural question here is whether an analogue holds true for real matrices. Theorem 1.1 answers this question in the affirmative.

What are examples of (not necessarily linear) commutativity preserving maps acting on $M_n(\mathbb{R})$? We start our list of examples with the standard examples. Every similarity transformation $A \mapsto TAT^{-1}$, where T is an invertible $n \times n$ real matrix, is a bijective linear map on $M_n(\mathbb{R})$ that preserves commutativity in both directions. The same holds for the transposition map $A \mapsto A^t$. But there are also many nonadditive maps $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ that preserve commutativity. To see this observe that if A, B is any pair of commuting matrices and p and q are any real polynomials, then $p(A)$ and $q(B)$ commute as well. So, if we associate to each $A \in M_n(\mathbb{R})$ a real polynomial p_A, then the map $A \mapsto p_A(A)$ preserves commutativity. Such maps will be called *locally polynomial maps*. Our main result states that every continuous injective map on $M_n(\mathbb{R})$, $n > 3$, preserving commutativity is a composition of maps described above.

THEOREM 1.1. *Let $n > 3$ and let $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be an injective commutativity preserving continuous map. Then there exist an invertible matrix $T \in M_n(\mathbb{R})$ and a locally polynomial map $A \mapsto p_A(A)$ such that either*

$$\phi(A) = Tp_A(A)T^{-1}$$

for all $A \in M_n(\mathbb{R})$, or

$$\phi(A) = Tp_A(A^t)T^{-1}$$

for all $A \in M_n(\mathbb{R})$.

2. Preliminary results

We will first introduce the *real Jordan canonical form.* Let $A \in M_n(\mathbb{R})$. Then all the nonreal eigenvalues of A must occur in conjugate pairs. Moreover, if A has only real entries, then $\operatorname{rank}(A-\lambda I)^k = \operatorname{rank}\overline{(A-\lambda I)}^k = \operatorname{rank}(A-\overline{\lambda} I)^k$ for all $\lambda \in \mathbb{C}$ and all $k = 1, 2, \ldots$, and hence the structure of the Jordan blocks corresponding to any eigenvalue λ is the same as the structure of the Jordan blocks corresponding to the conjugate eigenvalue $\overline{\lambda}$. Thus, all the Jordan blocks of all sizes (not just 1×1 blocks) corresponding to nonreal eigenvalues occur in conjugate pairs of equal size. For example, if λ is a nonreal eigenvalue of the real matrix A, and if $J_2(\lambda)$ appears in the Jordan canonical form of A with a certain multiplicity (here $J_2(\lambda)$ denotes the 2×2 Jordan block corresponding to the eigenvalue λ), $J_2(\overline{\lambda})$ must also appear with the same multiplicity. The block matrix

$$\begin{bmatrix} J_2(\lambda) & 0 \\ 0 & J_2(\overline{\lambda}) \end{bmatrix} \tag{2.1}$$

is permutation-similar (interchange rows and columns 2 and 3) to the block matrix

$$\begin{bmatrix} D(\lambda) & I \\ 0 & D(\lambda) \end{bmatrix},$$

where

$$D(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \in M_2(\mathbb{C}).$$

Each 2×2 diagonal block $D(\lambda)$ is similar to a real 2×2 matrix

$$SD(\lambda)S^{-1} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = C(a,b),$$

where $\lambda = a + ib$, $a, b \in \mathbb{R}$, and $S = \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix}$. Thus, every block pair of conjugate 2×2 Jordan blocks (2.1) with nonreal λ is similar via $\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$ to a real 4×4 block of the form

$$C_2(a,b) = \begin{bmatrix} C(a,b) & I \\ 0 & C(a,b) \end{bmatrix}.$$

This observations lead us to the *real Jordan canonical form* (see also [7]).

THEOREM 2.1. *Each real matrix $A \in M_n(\mathbb{R})$ is similar to a block diagonal real matrix of the form*

$$\begin{bmatrix} C_{n_1}(a_1,b_1) & & & & & & \\ & C_{n_2}(a_2,b_2) & & & & 0 & \\ & & \ddots & & & & \\ & & & C_{n_k}(a_k,b_k) & & & \\ & & & & J_{m_1}(c_1) & & \\ & 0 & & & & \ddots & \\ & & & & & & J_{m_h}(c_h) \end{bmatrix},$$

where a_j and b_j are real numbers, $\lambda_j = a_j + ib_j$ is a nonreal eigenvalue of A for $j = 1, 2, \ldots, k$, $c_1, \ldots, c_h$ are real eigenvalues of A, and $J_{m_1}(c_1), \ldots, J_{m_h}(c_h)$ are Jordan blocks. Each real block triangular matrix $C_{n_j}(a_j, b_j) \in M_{2n_j}(\mathbb{R})$ is of the form

$$C_{n_j}(a_j,b_j) = \begin{bmatrix} C(a_j,b_j) & I & \ldots & 0 \\ 0 & C(a_j,b_j) & \ddots & \vdots \\ \vdots & \vdots & \ddots & I \\ 0 & 0 & \ldots & C(a_j,b_j) \end{bmatrix}$$

and corresponds to a pair of conjugate Jordan blocks $J_{n_j}(\lambda_j), J_{n_j}(\overline{\lambda_j}) \in M_{n_j}(\mathbb{C})$ with nonreal $\lambda_j = a_j + ib_j$ in the Jordan canonical form of A. Moreover, there is always a real nonsingular matrix S such that SAS^{-1} is in the real Jordan canonical form described above.

We will say that a matrix $A \in M_n(\mathbb{R})$ is *diagonalizable* if every eigenvalue of A has algebraic multiplicity one. So, a matrix $A \in M_n(\mathbb{R})$ is diagonalizable if and only if $n_1 = n_2 = \ldots = n_k = m_1 = m_2 = \ldots = m_h = 1$, where $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_h$ are positive integers from the above theorem.

Let $\mathcal{S}$ be a subset of $M_n(\mathbb{R})$. Recall that its commutant $\mathcal{S}'$ is the space of all matrices from $M_n(\mathbb{R})$ that commute with all matrices from $\mathcal{S}$. When $\mathcal{S} = \{A\}$ we write shortly $A' = \{A\}'$. Clearly, for $A \in M_n(\mathbb{R})$ we have $A' = M_n(\mathbb{R})$ if and only if A is a scalar matrix. A matrix A is said to be *nonderogatory* if $\lambda_j = a_j + ib_j$, $j = 1, 2, \ldots, k$, from Theorem 2.1 are distinct nonreal eigenvalues of A and $c_1, \ldots, c_h$ from Theorem 2.1 are distinct real eigenvalues of A.

Now, assume that A is a nonderogatory matrix. Then there exists an invertible $S \in M_n(\mathbb{R})$ such that

$$A = S \operatorname{diag}\left(C_{n_1}(a_1,b_1), \ldots, C_{n_k}(a_k,b_k), J_{m_1}(c_1), \ldots, J_{m_h}(c_h)\right) S^{-1},$$

where $a_j + ib_j$, $j = 1, \ldots, k$, are distinct nonreal eigenvalues of A and $c_1, \ldots, c_h$ are distinct real eigenvalues of A. Let $B \in M_n(\mathbb{R})$ be a matrix that commutes with A. If we write $S^{-1}BS$ in the partitioned form $S^{-1}BS = [B_{ij}]$, which conforms with the above decomposition of $S^{-1}AS$, then it is easy to see that the corresponding off-diagonal blocks have to be zero since all real and nonreal eigenvalues of A are different. The matrix $S^{-1}BS$ must therefore be a block diagonal matrix

$$S^{-1}BS = \operatorname{diag}(B_1, \ldots, B_k, \overline{B}_1, \ldots, \overline{B}_h)$$

with each $B_i \in M_{2n_i}(\mathbb{R})$, $i = 1, \ldots, k$, and $\overline{B}_j \in M_{m_j}(\mathbb{R})$, $j = 1, \ldots, h$. The commutativity assumption says that $B_i C_{n_i}(a_i, b_i) = C_{n_i}(a_i, b_i) B_i$ for all $i = 1, \ldots, k$. An explicit calculation shows that each B_i must be of the form

$$B_i = \begin{bmatrix} C_1^{(i)} & C_2^{(i)} & \ldots & C_{n_i}^{(i)} \\ 0 & C_1^{(i)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & C_2^{(i)} \\ 0 & 0 & \ldots & C_1^{(i)} \end{bmatrix},$$

where $C_j^{(i)} = C(\alpha_j^{(i)}, \beta_j^{(i)})$, $j = 1, \ldots, n_i$. Since B commutes with A we also have $\overline{B}_i J_i(c_i) = J_i(c_i) \overline{B}_i$ for all $i = 1, \ldots, h$. Therefore $\overline{B}_i$ must be an upper triangular matrix of Toeplitz type, that is,

$$\overline{B}_i = \begin{bmatrix} \gamma_1^{(i)} & \gamma_2^{(i)} & \ldots & \gamma_{m_i}^{(i)} \\ 0 & \gamma_1^{(i)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \gamma_2^{(i)} \\ 0 & 0 & \ldots & \gamma_1^{(i)} \end{bmatrix},$$

where the entries are constant down the diagonals. This yields (see [5, Lemma 2.6]) that $S^{-1}BS$ is the matrix of the form

$$\operatorname{diag}(p_1(C_{n_1}(a_1, b_1)), \ldots, p_k(C_{n_k}(a_k, b_k)), q_1(J_{m_1}(c_1)), \ldots, q_h(J_{m_h}(c_h))),$$

where $p_1, \ldots, p_k, q_1, \ldots, q_h$ are polynomials with real coefficients. Hence, there exists a real polynomial p such that $B = p(A)$ (see also [7, Theorem 3.2.4.2.]). Moreover, a real matrix B commutes with a nonderogatory matrix $A \in M_n(\mathbb{R})$ if and only if $B = p(A)$ for some polynomial p with real coefficients.

Let $A \in M_n(\mathbb{R})$. Then, of course, $(rA + sI)' = A'$ for every pair $r, s \in \mathbb{R}$ with $r \neq 0$. Let E_{ij} be the matrix with all entries equal to zero except the (i, j)-entry that is equal to one. A straightforward computation shows that the commutant of E_{11} is the set of all real matrices of the form

$$\begin{bmatrix} * & 0 & 0 & \ldots & 0 \\ 0 & * & * & \ldots & * \\ 0 & * & * & \ldots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \ldots & * \end{bmatrix},$$

while the commutant of E_{12} is the set of all real matrices of the form

$$\begin{bmatrix} a & * & * & \dots & * \\ 0 & a & 0 & \dots & 0 \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \dots & * \end{bmatrix}.$$

Every rank one matrix is similar either to a nonzero multiple of E_{11}, or to E_{12}. Thus, the commutant of every rank one matrix has dimension $n^2 - 2(n-1)$. It is also not difficult to see that $\dim A' \geq n^2 - 2(n-1)$ if and only if A is a real linear combination of a scalar matrix and a rank one matrix.

We will represent vectors $x \in \mathbb{R}^n$ as $n \times 1$ real matrices. Note that the standard basis $e_1, e_2, \dots, e_n$ of $\mathbb{R}^n$ is the set of all $n \times 1$ matrices having all entries equal to zero but one that is equal to one. If $x, y \in \mathbb{R}^n$ are two nonzero vectors, then xy^t is a rank one matrix. Every rank one matrix can be written in this form. In particular $E_{ij} = e_i e_j^t$, $1 \leq i, j \leq n$. If xy^t and uv^t are two rank one operators, then we will write

$$xy^t \sim uv^t$$

if and only if x and u are linearly dependent or y and v are linearly dependent. For nonzero vectors $x, y \in \mathbb{R}^n$ we denote

$$L_x = \{xv^t \ : \ v \in \mathbb{R}^n \setminus \{0\}\}$$

and

$$R_y = \{uy^t \ : \ u \in \mathbb{R}^n \setminus \{0\}\}.$$

Clearly, if $A, B \in M_n(\mathbb{R})$ are rank one matrices, then $A \sim B$ if and only if $A, B \in L_x$ for some nonzero vector x, or $A, B \in R_y$ for some nonzero vector y.

In the proof of our main result we will also use the next lemma which was proved in [14, Theorem 3.1] for the field of complex numbers. Since the same idea works for the field of real numbers we will omit the proof.

Lemma 2.2. *Let $n > 3$ and let $A, B \in M_n(\mathbb{R})$ be two linearly independent rank one matrices. Then the following statements are equivalent*

(i) $A \sim B$,
(ii) $\dim(A' \cap B') = n^2 - 3n + 3$,
(iii) $\dim(A' \cap B') \geq n^2 - 3n + 3$.

The main idea of the proof of Theorem 1.1 is similar to those in [14, Theorem 3.1]. Under the continuity assumption we can again apply the dimension arguments as in the linear case. This follows from the invariance of domain theorem [6, p. 344] stating that if U is an open subset of $\mathbb{R}^m$ and $F : U \to \mathbb{R}^m$ a continuous injective map, then $F(U)$ is open. In particular, there is no injective continuous map from $\mathbb{R}^k$ into $\mathbb{R}^m$ whenever $m < k$. Now, let $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be an injective continuous commutativity preserving map. Then, obviously, for every matrix $A \in M_n(\mathbb{R})$ we have $\phi(A') \subseteq \phi(A)'$. Therefore, if A has a commutant of a large dimension, then the commutant of $\phi(A)$ must be of a large dimension as well.

3. Proof of the main result

The goal of this section is to prove Theorem 1.1. So, let us assume that $n > 3$ and let $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be an injective commutativity preserving continuous map. Then $\phi(A') \subseteq \phi(A)'$ for every matrix $A \in M_n(\mathbb{R})$. In particular, for every real number t we have

$$\phi(M_n(\mathbb{R})) = \phi((tI)') \subseteq \phi(tI)' \subseteq M_n(\mathbb{R}).$$

By the invariance of domain theorem the subspace $\phi(tI)'$ cannot be a proper subspace of the full matrix algebra $M_n(\mathbb{R})$. This yields that $\phi(tI)' = M_n(\mathbb{R})$, and consequently, ϕ maps scalar matrices into scalar matrices.

We already know that $\dim A' \geq n^2-2(n-1)$ if and only if A is a real linear combination of a scalar matrix and a rank one matrix. As a consequence, every $A \in M_n(\mathbb{R})$ which is a real linear combination of a scalar matrix and a rank one matrix is mapped into a matrix of the same type. Otherwise ϕ would map A', which is of dimension at least $n^2 - 2(n-1)$, continuously and injectively into $\phi(A)'$, whose dimension would be strictly smaller than $n^2 - 2(n-1)$, contradicting the invariance of domain theorem.

In the next step we will show that for every matrix $A \in M_n(\mathbb{R})$ of rank one there exists a nonzero real number t such that $\phi(tA)$ is not a scalar matrix. Assume on the contrary that $\phi(tA) = f(t)I$, $t \in \mathbb{R}$, for some continuous injective function $f : \mathbb{R} \to \mathbb{R}$. We already know that $\phi(tI) = g(t)I$, $t \in \mathbb{R}$, for some continuous injective function $g : \mathbb{R} \to \mathbb{R}$. Note that the range of g is an open subset of $\mathbb{R}$. Since f is continuous $\lim_{t\to 0} f(t) = f(0) = g(0)$. Thus, there exist nonzero real numbers t_1 and t_2 such that $f(t_1) = g(t_2)$, contradicting the injectivity of ϕ. So, for every rank one matrix $A \in M_n(\mathbb{R})$ there exist a nonzero scalar t, a rank one real matrix B, and scalars $r, s \in \mathbb{R}$, $r \neq 0$, such that $\phi(tA) = rB+sI$. Moreover, $\phi(\mathbb{R}A+\mathbb{R}I) \subseteq \mathbb{R}B+\mathbb{R}I$. Indeed, if there were scalars $r \neq 0$ and s such that $\phi(rA+sI) \notin \mathbb{R}B+\mathbb{R}I$, then ϕ would map $A' = (rA+sI)'$ injectively and continuously into $B' \cap \phi(rA+sI)'$, which would be a proper subset of B'. But this is impossible by the invariance of domain theorem. Note also that if $\phi(\mathbb{R}A+\mathbb{R}I) \subseteq \mathbb{R}B_1+\mathbb{R}I$ and $\phi(\mathbb{R}A+\mathbb{R}I) \subseteq \mathbb{R}B_2 + \mathbb{R}I$ for two real matrices B_1 and B_2 of rank one, then $\mathbb{R}B_1 + \mathbb{R}I = \mathbb{R}B_2 + \mathbb{R}I$, which further yields that B_1 and B_2 are linearly dependent. In other words, a matrix B with the property $\phi(\mathbb{R}A+\mathbb{R}I) \subseteq \mathbb{R}B+\mathbb{R}I$ is uniquely determined up to a linear dependence.

Suppose that A_1 and A_2 are two linearly independent rank one real matrices and $\phi(\mathbb{R}A_k+\mathbb{R}I) \subseteq \mathbb{R}B+\mathbb{R}I$, $k = 1, 2$, for some rank one matrix B. By the invariance of domain theorem, $\phi(A_1')$ is an open subset of B' containing

$\phi(0) = t_0 I$. Let C be a real matrix that commutes with A_2 and does not commute with A_1. In other words $C \in A_2' \setminus A_1'$. As $\lim_{t\to 0} \phi(tC) = \phi(0) = t_0 I$ we can find a nonzero scalar t such that $\phi(tC) \in \phi(A_1')$, contradicting the injectivity of ϕ. So, we proved that if A_1 and A_2 are two linearly independent rank one matrices and $\phi(\mathbb{R}A_k + \mathbb{R}I) \subseteq \mathbb{R}B_k + \mathbb{R}I$, $k = 1, 2$, for some rank one matrices B_1, B_2, then B_1 and B_2 have to be linearly independent as well.

Denote by $M_n^1(\mathbb{R}) \subset M_n(\mathbb{R})$ the set of all rank one real matrices. Note that this is not a vector space. But we can define the corresponding projective space as the set

$$\mathbb{P}M_n^1(\mathbb{R}) = \{[A] \ : \ A \in M_n^1(\mathbb{R})\},$$

where $[A] = \{tA \ : \ t \in \mathbb{R} \setminus \{0\}\}$. For an arbitrary subset $\mathcal{S} \subseteq M_n^1(\mathbb{R})$ we will write $\mathbb{P}\mathcal{S} = \{[A] \ : \ A \in \mathcal{S}\}$. By what we have proved, the map ϕ induces an injective map $\varphi : \mathbb{P}M_n^1(\mathbb{R}) \to \mathbb{P}M_n^1(\mathbb{R})$ defined by $\varphi([A]) = [B]$, where A and B are rank one matrices such that $\phi(\mathbb{R}A + \mathbb{R}I) \subseteq \mathbb{R}B + \mathbb{R}I$.

Suppose that $A, B \in M_n^1(\mathbb{R})$ are two linearly independent rank one matrices satisfying $A \sim B$ and $\varphi([A]) = [A_1]$, $\varphi([B]) = [B_1]$. Then from $\phi(A' \cap B') \subseteq A_1' \cap B_1'$, Lemma (2.2), and the invariance of domain theorem we get

$$n^2 - 3n + 3 = \dim(A' \cap B') \leq \dim(A_1' \cap B_1').$$

This yields that $A_1 \sim B_1$ and consequently, for every nonzero $x \in \mathbb{R}^n$ we have either $\varphi(\mathbb{P}L_x) \subseteq \mathbb{P}L_z$ for some nonzero $z \in \mathbb{R}^n$, or $\varphi(\mathbb{P}L_x) \subseteq \mathbb{P}R_y$ for some nonzero $y \in \mathbb{R}^n$. After composing ϕ with the transposition map, if necessary, we may assume that there exists a nonzero $x \in \mathbb{R}^n$ such that $\varphi(\mathbb{P}L_x) \subseteq \mathbb{P}L_z$ for some nonzero $z \in \mathbb{R}^n$.

In the next step we will prove that $\varphi(\mathbb{P}L_x) = \mathbb{P}L_z$. We know that ϕ maps the $(n+1)$-dimensional space $\{rI + xv^t \ : \ r \in \mathbb{R}, \ v \in \mathbb{R}^n\}$ injectively and continuously into the $(n+1)$-dimensional space $\{rI + zv^t \ : \ r \in \mathbb{R}, \ v \in \mathbb{R}^n\}$. Since the ϕ-image of the first space is an open subset of the second one containing at least one scalar matrix it follows that $\varphi(\mathbb{P}L_x) = \mathbb{P}L_z$, as desired.

Let $y \in \mathbb{R}^n \setminus \{0\}$. The subspaces

$$V_1 = \{rI + xv^t \ : \ r \in \mathbb{R}, \ v \in \mathbb{R}^n\}$$

and

$$V_2 = \{rI + uy^t \ : \ r \in \mathbb{R}, \ u \in \mathbb{R}^n\}$$

have a two-dimensional intersection

$$V_1 \cap V_2 = \{rI + sxy^t \ : \ r, s \in \mathbb{R}\}.$$

We have either $\varphi(\mathbb{P}R_y) = \mathbb{P}L_u$ for some nonzero $u \in \mathbb{R}^n$, or $\varphi(\mathbb{P}R_y) = \mathbb{P}R_v$ for some nonzero $v \in \mathbb{R}^n$. We will prove that the first possibility cannot occur.

Assume that $\varphi(\mathbb{P}R_y) = \mathbb{P}L_u$ for some nonzero $u \in \mathbb{R}^n$. Then either $\mathbb{P}L_z \cap \mathbb{P}L_u = \emptyset$, or $\mathbb{P}L_z = \mathbb{P}L_u$. In the first case the two-dimensional subspace $V_1 \cap V_2$ is mapped injectively and continuously into the one-dimensional space of scalar matrices, a contradiction. In the second case V_1 is mapped onto some open subset W_1 of the subspace $\{rI + zv^t \ : \ r \in \mathbb{R},\ v \in \mathbb{R}^n\}$. Similarly, V_2 is mapped onto some open subset W_2 of the same subspace. If $\phi(V_1 \cap V_2)$ is a proper subset of $W_1 \cap W_2$, then we can find matrices $A_1 \in V_1 \setminus (V_1 \cap V_2)$ and $A_2 \in V_2 \setminus (V_1 \cap V_2)$ such that $\phi(A_1) = \phi(A_2) \in W_1 \cap W_2$, contradicting the injectivity of ϕ. Thus, $\phi(V_1 \cap V_2) = W_1 \cap W_2$, which is again impossible as $V_1 \cap V_2$ is a two-dimensional subspace, while $W_1 \cap W_2$ is a nonempty open subset of an $(n+1)$-dimensional subspace.

So, we have proved that for every nonzero $y \in \mathbb{R}^n$ there exists a nonzero $v \in \mathbb{R}^n$ such that $\varphi(\mathbb{P}R_y) = \mathbb{P}R_v$. Using exactly the same arguments we can prove that for every nonzero $x \in \mathbb{R}^n$ there exists a nonzero $z \in \mathbb{R}^n$ such that $\varphi(\mathbb{P}L_x) = \mathbb{P}L_z$.

In the next step we will prove that the map φ is bijective. We already know that it is injective. Choose vectors $x, y \in \mathbb{R}^n$ and let $\varphi(\mathbb{P}L_x) = \mathbb{P}L_z$ and $\varphi(\mathbb{P}R_y) = \mathbb{P}R_v$. Further, let $A = pq^t$ be any rank one real matrix. If p and z are linearly dependent, then $[A]$ is contained in the range of φ. The same is true if q and v are linearly dependent. Hence, suppose that p and z are linearly independent and q and v are linearly independent. Since $pv^t \in R_v$ it follows from $\varphi(\mathbb{P}R_y) = \mathbb{P}R_v$ that $\varphi([p_1 y^t]) = [pv^t]$ for some nonzero vector $p_1 \in \mathbb{R}^n$. Similarly, $\varphi([xq_1^t]) = [zq^t]$ for some nonzero vector $q_1 \in \mathbb{R}^n$. Set $\varphi([p_1 q_1^t]) = [ab^t]$. Then $ab^t \sim pv^t$ and $ab^t \sim zq^t$. If a and p are linearly dependent, then a and z are linearly independent. It follows from $ab^t \sim zq^t$ that b and q are linearly dependent. Thus, $[ab^t] = [pq^t] = [A]$ is contained in the range of φ and we are done. If a and p are linearly independent, then $ab^t \sim pv^t$ yields that b and v are linearly dependent. Then we get from $ab^t \sim zq^t$ that a and z are linearly dependent. From here we further conclude that

$$\varphi([p_1 q_1^t]) = [ab^t] = [zv^t] = \varphi([xy^t]).$$

By the injectivity of φ the vectors p_1 and x are linearly dependent. Hence, from $\varphi([p_1 y^t]) = [pv^t]$ we conclude that $\varphi(\mathbb{P}L_x) = \mathbb{P}L_p$ which together with $\varphi(\mathbb{P}L_x) = \mathbb{P}L_z$ yield that p and z are linearly dependent, a contradiction.

Now, let $A = xy^t$ and $B = uv^t$ be two rank one matrices with $AB = 0$ and $\varphi([A]) = [A_1]$, $\varphi([B]) = [B_1]$. Note that $y^t u = 0$. We will prove that also $A_1 B_1 = 0$. We know that $\varphi(\mathbb{P}L_u) = \mathbb{P}L_z$ for some nonzero $z \in \mathbb{R}^n$ and $\varphi(\mathbb{P}R_y) = \mathbb{P}R_w$ for some nonzero $w \in \mathbb{R}^n$. Of course, $A_1 \in R_w$ and $B_1 \in L_z$. Choose $C \in L_u$ such that A and C are linearly independent commutative matrices. Note that $\varphi([C]) = [C_1]$ and $C_1 \in L_z$. Then A_1 and C_1 are also linearly independent commutative matrices of rank one. This yields that $A_1 C_1 = 0$. Hence, $w^t z = 0$ and consequently $A_1 B_1 = 0$, as desired.

We can apply [12, Lemma 2.2] to conclude that there exists an invertible matrix $T \in M_n(\mathbb{R})$ such that

$$\varphi([A]) = [TAT^{-1}], \quad [A] \in \mathbb{P}M_n^1(\mathbb{R}).$$

After composing ϕ with the similarity transformation, $A \mapsto T^{-1}AT$, we may assume that T is the identity. Thus, for every rank one matrix $A \in M_n(\mathbb{R})$ there exist scalars $r, s \in \mathbb{R}$ such that $\phi(A) = rA + sI$.

Let $P \in M_n(\mathbb{R})$ be any non-trivial idempotent matrix. Then we can write

$$P = S \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$$

for some invertible matrix $S \in M_n(\mathbb{R})$. The matrix P commutes with every scalar multiple of a rank one idempotent Q of the form

$$Q = S \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} S^{-1},$$

where R is a rank one idempotent matrix of the appropriate size. Similarly, P commutes with every scalar multiple of a rank one idempotent Q of the form

$$Q = S \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} S^{-1},$$

where R is a rank one idempotent matrix of the appropriate size. Since $\phi(tP)$, $t \in \mathbb{R}$, commutes with every $\phi(rQ)$, where r is any scalar and Q is as above, it follows that $\phi(tP)$ is a linear combination of P and I for every real number t. In the same way as in the case of rank one matrices we can apply the continuity assumption to conclude that there is at least one nonzero t such that $\phi(tP)$ is not a scalar matrix. Let $A \in M_n(\mathbb{R})$ be a matrix that commutes with P. Then $\phi(A)$ commutes with $\phi(tP)$ which yields that $\phi(A)$ commutes with P as well.

Suppose that $k > 1$ and $h = n - 2k$. Let a, b, c be real numbers, $b \neq 0$, $S \in M_n(\mathbb{R})$ an invertible matrix, and let B be a matrix of the form

$$B = S\mathrm{diag}\,(\underbrace{C(a,b), C(a,b), \ldots, C(a,b)}_{k-\text{times}}, \underbrace{c, c, \ldots, c}_{h-\text{times}})S^{-1}. \tag{3.2}$$

Because B commutes with idempotents

$$\begin{gathered} S\mathrm{diag}\,(I, \ldots, 0, 0, \ldots, 0)S^{-1}, \\ \vdots \\ S\mathrm{diag}\,(0, \ldots, I, 0, \ldots, 0)S^{-1}, \\ S\mathrm{diag}\,(0, \ldots, 0, 1, \ldots, 0)S^{-1}, \\ \vdots \\ S\mathrm{diag}\,(0, \ldots, 0, 0, \ldots, 1)S^{-1}, \end{gathered}$$

the matrix $\phi(B)$ commutes with these idempotents as well, and therefore,

$$\phi(B) = S\mathrm{diag}\,(B_1, B_2, \ldots, B_k, \gamma_1, \gamma_2, \ldots, \gamma_h)S^{-1},$$

where $B_1, B_2, \ldots, B_k$ are 2×2 matrices and $\gamma_1, \gamma_2, \ldots, \gamma_h \in \mathbb{R}$. Define

$$D = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

Since B commutes with the idempotent matrix $P = S(D \oplus 0)S^{-1}$, where 0 denotes the $(n-4) \times (n-4)$ zero matrix, it follows that also the matrix $\phi(B)$ commutes with this matrix. This yields that $B_1 = B_2 = C(\alpha, \beta)$ for some real numbers α and β. In the same way we prove that $B_i = C(\alpha, \beta)$, $i = 3, \ldots, k$. As before we can apply the continuity assumption to conclude that there exists at least one $B \in M_n(\mathbb{R})$ of the form (3.2) such that $\phi(B)$ has a nonreal eigenvalue $\alpha + i\beta$, $\beta \neq 0$. Note also that if $k = 1$, then a matrix

$$B = S\mathrm{diag}\,(C(a,b), \underbrace{c, c, \ldots, c}_{(n-2)-\text{times}})S^{-1}$$

commutes with the matrix

$$S\mathrm{diag}\,(C(a,b), C(a,b), \underbrace{c, c, \ldots, c}_{(n-4)-\text{times}})S^{-1},$$

which yields that

$$\phi(B) = S\mathrm{diag}\,(C(\alpha, \beta), \gamma_1, \gamma_2, \ldots, \gamma_{n-2})S^{-1}.$$

Now, let $A \in M_n(\mathbb{R})$ be a diagonalizable matrix. Then we can write

$$A = S\mathrm{diag}\,(C(a_1, b_1), C(a_2, b_2), \ldots, C(a_k, b_k), c_1, c_2, \ldots, c_h)S^{-1}$$

for some invertible matrix $S \in M_n(\mathbb{R})$ and some real numbers $a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_h$. Here, $k \geq 0$ and $h = n - 2k$. As above we can show that

$$\phi(A) = S\mathrm{diag}\,(A_1, A_2, \ldots, A_k, \gamma_1, \gamma_2, \ldots, \gamma_h)S^{-1},$$

where $A_1, A_2, \ldots, A_k$ are 2×2 matrices and $\gamma_1, \gamma_2, \ldots, \gamma_h \in \mathbb{R}$. Further, matrices A and B commute which implies that this is also true for the matrices $\phi(A)$ and $\phi(B)$. It follows that $A_i = C(\alpha_i, \beta_i)$, $i = 1, 2, \ldots, k$, where α_i and β_i are real numbers. In particular, $A\phi(A) = \phi(A)A$ for every diagonalizable matrix A. The set of all diagonalizable matrices is dense in $M_n(\mathbb{R})$, and thus, $\phi(A)$ commutes with A for every $A \in M_n(\mathbb{R})$.

Now, let $A \in M_n(\mathbb{R})$ be a nonderogatory matrix. We already know that a nonderogatory matrix A commutes with B if and only if B is a polynomial of A. Since A commutes with $\phi(A)$ it follows that $\phi(A) = p_A(A)$ for some real polynomial p_A.

It remains to prove that for every matrix $A \in M_n(\mathbb{R})$ there exists a polynomial p_A such that $\phi(A) = p_A(A)$. First we will prove this for matrices with only real eigenvalues. Here we will consider just matrices with two Jordan cells in the Jordan canonical form, since the same idea works in the general case as well. So, let

$$A = S\ \mathrm{diag}(J_k(a), J_{n-k}(b))\ S^{-1},$$

where a and b are real eigenvalues of A and $k \geq n-k$. If $a \neq b$, then A is a nonderogatory matrix and the proof is complete. It remains to consider the case when $a = b$. Set

$$A_m = S\ \mathrm{diag}(J_k(a), J_{n-k}(a + \frac{1}{m}))\ S^{-1}$$

for every $m \in \mathbb{N}$. We already know that $\phi(A_m) = p_{A_m}(A_m)$ for some real polynomials p_{A_m}. This means that each A_m is mapped into a matrix of the form

$$S \left[\begin{array}{cccc|cccc} c_1 & c_2 & \dots & c_k & & & & \\ 0 & c_1 & \ddots & \vdots & & & & \\ \vdots & \vdots & \ddots & c_2 & & 0 & & \\ 0 & 0 & \dots & c_1 & & & & \\ & & & & d_1 & d_2 & \dots & d_{n-k} \\ & & 0 & & 0 & d_1 & \ddots & \vdots \\ & & & & \vdots & \vdots & \ddots & d_2 \\ & & & & 0 & 0 & \dots & d_1 \end{array} \right] S^{-1}.$$

Since $A = \lim_{m\to\infty} A_m$ it follows that $\phi(A) = \lim_{m\to\infty} \phi(A_m)$. This yields that $\phi(A)$ is also of the form described above. We only have to show that $c_1 = d_1, c_2 = d_2, \dots, c_{n-k} = d_{n-k}$. The matrix A commutes with the idempotent matrix

$$P = S \begin{bmatrix} I & Q \\ 0 & 0 \end{bmatrix} S^{-1},$$

where $Q = \begin{bmatrix} I \\ 0 \end{bmatrix}$. Of course, in both cases I denotes the identity matrix of the appropriate size. Hence, $\phi(A)$ commutes with P as well. This gives the desired equalities $c_1 = d_1, c_2 = d_2, \dots, c_{n-k} = d_{n-k}$.

In order to complete the proof we have to show that $\phi(A) = p_A(A)$ for matrices of the form

$$A = S\ \mathrm{diag}(C_k(a,b), C_h(a,b))\ S^{-1},$$

where $S \in M_n(\mathbb{R})$ is an invertible matrix, $a, b \in \mathbb{R}$, $b \neq 0$, $n = 2k + 2h$, and $k \geq h$. We will treat only the matrices of the above form since we already know that $\phi(A) = p_A(A)$ for nonderogatory matrices and matrices with only real eigenvalues. Note also that if n is odd, then we take $A = S\ \mathrm{diag}(C_k(a,b), C_h(a,b), c)\ S^{-1}$, where c is a real number and $n = 2k+2h+1$.

Let

$$A_m = S\ \mathrm{diag}(C_k(a,b), C_h(a+\frac{1}{m}, b))\ S^{-1}$$

for every $m \in \mathbb{N}$. We already know that $\phi(A_m) = p_{A_m}(A_m)$ for some real polynomials p_{A_m}. This means that each A_m is mapped into a matrix of the form

$$S \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix} S^{-1},$$

where

$$B = \begin{bmatrix} C(c_1,d_1) & C(c_2,d_2) & \dots & C(c_k,d_k) \\ 0 & C(c_1,d_1) & \ddots & \vdots \\ \vdots & \vdots & \ddots & C(c_2,d_2) \\ 0 & 0 & \dots & C(c_1,d_1) \end{bmatrix}$$

and

$$D = \begin{bmatrix} C(e_1,f_1) & C(e_2,f_2) & \dots & C(e_h,f_h) \\ 0 & C(e_1,f_1) & \ddots & \vdots \\ \vdots & \vdots & \ddots & C(e_2,f_2) \\ 0 & 0 & \dots & C(e_1,f_1) \end{bmatrix}.$$

Of course, $A = \lim_{m\to\infty} A_m$. Thus, $\phi(A) = \lim_{m\to\infty} \phi(A_m)$. This yields that $\phi(A)$ is also of the form described above. We have to show that $C(c_1,d_1) = C(e_1,f_1), \dots, C(c_h,d_h) = C(e_h,f_h)$. The matrix A commutes with the idempotent matrix

$$P = S \begin{bmatrix} I & Q \\ 0 & 0 \end{bmatrix} S^{-1},$$

where $Q = \begin{bmatrix} I \\ 0 \end{bmatrix}$. The first I denotes the $2k \times 2k$ identity matrix and the second I denotes the $2h \times 2h$ identity matrix. We already know that the matrix $\phi(A)$ commutes with P as well which gives the desired equalities $C(c_1,d_1) = C(e_1,f_1), \dots, C(c_h,d_h) = C(e_h,f_h)$. The proof is completed.

References

[1] L. Baribeau and T. Ransford, *Non-linear spectrum-preserving maps*, Bull. London Math. Soc. **32** (2000), 8–14.

[2] M. Brešar, *Commuting traces of biadditive mappings, commutativity preserving mappings, and Lie mappings*, Trans. Amer. Math. Soc. **335** (1993), 525–546.

[3] M. Brešar and C. R. Miers, *Commutativity preserving mappings of von Neumann algebras*, Canad. J. Math. **45** (1993), 695–708.

[4] M. D. Choi, A. A. Jafarian and H. Radjavi, *Linear maps preserving commutativity*, Linear Algebra Appl. **87** (1987), 227–241.

[5] A. Fošner, *Non-linear commutativity preserving maps on $M_n(\mathbb{R})$*, Linear and Multilinear Algebra **53** (2005), 323–344.

[6] W. Fulton, Algebraic topology. A first course, Springer, Graduate Texts in Mathematics **153**, Springer-Verlag, New York, 1995.

[7] R. A. Horn and C. R. Johnson, Matrix analysis, Cambridge University Press, Cambridge, 1985.

[8] C. K. Li and N. K. Tsing, *Linear preserver problems: A brief introduction and some special techniques*, Linear Algebra Appl. **162/164** (1992), 217–235.

[9] L. Molnár, Selected preserver problems on algebraic structures of linear operators and on function spaces, Springer-Verlag, Berlin, 2007.

[10] M. Omladič, H. Radjavi and P. Šemrl, *Preserving commutativity*, J. Pure Appl. Algebra **156** (2001), 309–328.

[11] S. Pierce et.al., *A survey of linear preserver problems*, Linear and Multilinear Algebra **33** (1992), 1–129.

[12] L. Rodman and P. Šemrl, *Orthogonality preserving bijective maps on real and complex projective spaces*, Linear and Multilinear Algebra. **54** (2006), 335–367.

[13] P. Šemrl, *Non-linear commutativity preserving maps*, Acta Sci. Math. (Szeged) **71** (2005), 781–819.

[14] P. Šemrl, *Commutativity preserving maps*, Linear Algebra Appl. **429** (2008), 1051–1070.

[15] W. Watkins, *Polynomial functions that preserve commuting pairs of matrices*, Linear and Multilinear Algebra **5** (1977/78), 87–90.

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