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AN IDEAL-BASED ZERO-DIVISOR GRAPH OF A COMMUTATIVE SEMIRING

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Abstract. There is a natural graph associated to the zero-divisors of a commutative semiring with non-zero identity. In this article we essentially study zero-divisor graphs with respect to primal and non-primal ideals of a commutative semiring R and investigate the interplay between the semiring-theoretic properties of R and the graph-theoretic properties of $\Gamma_I(R)$ for some ideal I of R. We also show that the zero-divisor graph with respect to primal ideals commutes by the semiring of fractions of R.

1. Introduction

Throughout all semirings are assumed to be commutative semirings with non-zero identity. The zero-divisor graph of a semiring is the (simple) graph whose vertex set is the set of non-zero zero-divisors, and an edge is drawn between two distinct vertices if their product is zero. This definition is the same as that introduced by D. F. Anderson and P. S. Livingston in [1]. Let I be an ideal of a commutative semiring R. We define an undirected graph $\Gamma_I(R)$ with vertices $V(\Gamma_{\rm I}(R)) = \{x \in R - I : xy \in I \text{ for some } y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$ [6]. This definition is the same as that introduced by S. P. Redmond in [13]. The zero-divisor graph with respect to ideals of a commutative semiring has been studied in [6]. So, if $I = 0$ then $\Gamma_I(R) = \Gamma(R)$, and I is a non-zero prime ideal if and only if $\Gamma_I(R) = \emptyset$ [6, Lemma 2.1]. If I is a Q-deal of R, then the graphs $\Gamma_I(R)$ and $\Gamma(R/I)$ are different graphs [6, Theorem 2.4]. Hence the graph $\Gamma_I(R)$ is worthy of study.

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring, we mean a commutative semigroup

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$(R, \cdot)$ and a commutative monoid $(R, +, 0)$ in which 0 is the additive identity and $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$, both are connected by ring-like distributivity. In this paper, all semirings considered will be assumed to be commutative semirings with non-zero identity. A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A subtractive ideal (= k-ideal) K is an ideal such that if $x, x + y \in K$ then $y \in K$ (so $\{0\}$ is a k-ideal of R). The k-closure $\mathrm{cl}(K)$ of K is defined by $\mathrm{cl}(K) = \{a \in R : a + c = d$ for some $c, d \in K\}$ is an ideal of R satisfying $K \subseteq \mathrm{cl}(K)$ and $\mathrm{cl}(\mathrm{cl}(K)) = \mathrm{cl}(K)$. So an ideal K of R is a k-ideal if and only if $K = \mathrm{cl}(K)$. A prime ideal of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. If I is an ideal of R, then the radical of I, denoted by $\sqrt{I}$, is the set of all $x \in R$ for which $x^n \in I$ for some positive integer n. This is an ideal of R, contains I, and is the intersection of all the prime ideals of R that contain I ([2]). An ideal I of R is called a radical ideal if $I = \sqrt{I}$. A semiring R is called reduced if it contains no non-zero nilpotent elements.

Let R be a commutative semiring. We recall from [4] (also see [8]), that an element $a \in R$ is called prime to an ideal I of R if $ra \in I$ (where $r \in R$) implies that $r \in I$. Denote by $p(I)$ the set of elements of R that are not prime to I. A proper ideal I of R is said to be primal if $p(I)$ forms an ideal (so 0 is not necessarily primal); this ideal is always a prime ideal, called the adjoint ideal P of I. In this case we also say that I is a P-primal ideal of R.

Let R be a commutative semiring with non-zero identity. We use the notation A^* to refer to the non-zero elements of A. For two distinct vertices a and b in a graph Γ, the distance between a and b, denoted $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $\mathrm{d}(a, b) = \infty$. The diameter of a graph Γ is $\mathrm{diam}(\Gamma) = sup\{d(a, b) :$ a and b are distinct vertices of $\Gamma\}$. We will use use the notation $\mathrm{diam}(\Gamma(R))$ to denote the diameter of the graph of $Z^*(R)$. A graph is said to be connected if there exists a path between any two distinct vertices, and a graph is complete if it is connected with diameter at most one. The grith of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle, otherwise, $\mathrm{gr}(\Gamma) = \infty$. We will use the notation $\mathrm{gr}(\Gamma(R))$ to denote the grith of the graph of $Z^*(R)$.

Let R be a given semiring, and let S be the set of all multiplicatively cancelable elements of R (so $1 \in S$). Clearly, the set S is multiplicatively closed. Define a relation $\sim$ on $R \times S$ as follows: for $(a, s), (b, t) \in R \times S$, we write $(a, s) \sim (b, t)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation on $R \times S$. For $(a, s) \in R \times S$, denote the equivalence class of $\sim$ which contains (a, s) by a/s, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S can be given the structure of a commutative semiring under operations for which $a/s + b/t = (ta + sb)/st$, $(a/s)(b/t) = (ab)/st$ for all $a, b \in R$ and $s, t \in S$. This new semiring R_S is called the semiring of fractions of R with respect to S; its zero element is $0/1$, its multiplicative identity element is $1/1$ and each element of S has a multiplicative inverse in R_S (see [12]). Throughout this paper we shall assume unless otherwise stated, that **S** is the set of all multiplicatively cancelable elements of a semiring R. Now suppose that I is an ideal of a semiring R. The ideal generated by I in R_S, that is, the set of all finite sums $s_1a_1 + ..., s_na_n$ where $a_i \in R_S$ and $s_i \in I$, is called the extention of I to R_S, and it is denoted by IR_S. Again, if J is an ideal of R_S then by the contraction of J in R we mean $J \cap R = \{r \in R : r/1 \in J\}$, which is clearly an ideal of R.

Let R be a semiring, X a non-empty subset of R and I an ideal of R. Set $XR_S = \{a/s : a \in X, s \in S\} \subseteq R_S$. We say that a zero-divisor graph with respect to I commutes with semiring of fractions of R with respect to S if $\Gamma_I(R)R_S = \Gamma_{IR_S}(R_S)$.

The main goal in this paper is to generalize some of the results in the paper listed as [7], from commutative ring theory to commutative semiring theory. We shortly summarize the content of the paper. Let R be a semiring. In section 2, it is shown that (Theorem 2.5) if I and J are primal ideals of R, then $I = J$ if and only if $\Gamma_I(R) = \Gamma_J(R)$. It is proved that (Theorem 2.8) if I is a primal Q-ideal of a Noetherian semiring R, then $\text{diam}(\Gamma(R/I)) \leq 2$. It is shown that (Theorem 2.12) if I is a radical Q-ideal of a semiring R, then $\text{diam}(\Gamma(R/I)) = \text{diam}(\Gamma(R))$. In commutative ring theory, one of the essential questions is whether a zero-divisor graph with respect to an ideal commutes with localization, and this case, what are the relations between the diameters (resp. girths) of such graphs. In [7], they give a condition giving an affirmative answer to these questions. Here, we show that the zero-divisor graph with respect to primal ideals commutes by the semiring of fractions of R with respect to S, $\text{diam}(\Gamma_I(R)) = \text{diam}(\Gamma_{IR_S}(R_S))$ and $\text{gr}(\Gamma_I(R)) = \text{gr}(\Gamma_{IR_S}(R_S))$ (see Proposition 2.15, Theorem 2.16 and Theorem 2.17). In section 3, It is proved that if I is a radical Q-ideal of a semiring R, then $V(\Gamma_I(R)) \cup I = \bigcup_{P \in \text{Min}_k(I)} P$ (Theorem 3.6). Also, it is shown that (Theorem 3.13) if I is a Q-ideal of a semiring R which is not a primal ideal of R with $I \neq \sqrt{I}$, then $\text{diam}(\Gamma_I(R)) = 3$.

2. Primal ideals

In this section, we will investigate the ideal-based zero-divisor graph with respect to primal ideals of a semiring. The class of primal ideals is a large class. For example, all primary ideals and irreducible ideals are primal [4]. Therefore, the structure of zero-divisor graphs with respect to primal ideals is worthy of study. We begin with the key lemma of this article.

Lemma 2.1. *Let I be a proper ideal of a semiring R. Then $I \subseteq p(I)$ and $V(\Gamma_I(R)) = p(I) - I$. In particular, $V(\Gamma_I(R)) \cup I = p(I)$.*

Proof. Let $x \in I$. Since $x1_R \in R$ with $1_R \notin I$, we must have x is not prime to I. Therefore, $I \subseteq p(I)$. Let $a \in \Gamma_I(R)$. Then $a \notin I$ and $ab \in I$ for some $b \notin I$, so a is not prime to I; hence $\Gamma_I(R) \subseteq p(I) - I$. For the reverse inclusion, assume that $c \in p(I) - I$. Then there is an element $d \in R - I$ such that $cd \in I$; hence $c \in \Gamma_I(R)$, so we have equality. □

LEMMA 2.2. *Let I and P be ideals of a semiring R with $I \subseteq P$. Then I is a P-primal ideal of R if and only if $V(\Gamma_I(R)) = P - I$.*

PROOF. By Lemma 2.1, it suffices to show that P is exactly the set of elements of R that are not prime to I. First, suppose that $p \in P$. We may assume that $p \in P - I$. Then there is an element $q \in R - I$ such that $pq \in I$, so p is not prime to I. Next, suppose that r is not prime to I. We may assume that $r \notin I$. Then there exists $s \notin I$ such that $rs \in I$; hence $r \in \Gamma_I(R) \subseteq P$. Thus I is a P-primal ideal of R. □

THEOREM 2.3. *Let I be an ideal of a semiring R. Then I is a primal ideal of R if and only if $V(\Gamma_I(R)) \cup I$ is an (prime) ideal of R.*

PROOF. This follows from Lemma 2.2. □

THEOREM 2.4. *Let R be a semiring. Then 0 is a primal ideal of R if and only if $Z(R)$ is an (prime) ideal of R.*

PROOF. It is easy to see that $p(0) = Z(R)$. Now the assertion follows from the definition. □

THEOREM 2.5. *Let I and J be P-primal ideals of a semiring R. Then $\Gamma_I(R) = \Gamma_J(R)$ if and only if $I = J$.*

PROOF. By Lemma 2.1, we must have $I \subseteq P$ and $J \subseteq P$. Now the result follows from Lemma 2.2. □

LEMMA 2.6. *If I is a P-primal k-ideal of a semiring R, then P is a k-ideal of R.*

PROOF. Let $a, a + b \in P$; we show that $b \in P$. There are elements $r, s \in R - I$ such that $ra \in I$ and $s(a + b) = sa + sb \in I$, so $rsb \in I$ since I is a k-ideal. If $rs \in I$, then a is prime to I, which is a contradiction. Therefore, $rsb \in I$ with $rs \notin I$ gives that $b \in P$. □

An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) if there exists a subset Q of R such that $R = \cup\{q + I : q \in Q\}$ and if $q_1, q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$. Let I be a Q-ideal of a semiring R and let $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$, and $(q_1 + I) \odot (q_2 + I) = q_4 + I$, where $q_4 \in Q$ is the unique element such that $q_1 q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I.

By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0+I \subseteq q_0+I$. Then q_0+I is the zero element of R/I. Clearly, if R is commutative, then so is R/I (see [9]).

THEOREM 2.7. *Assume that I is a P-primal Q-ideal of a semiring R and let q_0 be the unique element in Q such that q_0+I is the zero in R/I. Then $V(\Gamma(R/I)) \cup \{q_0+I\}$ is a prime k-ideal of R/I.*

PROOF. Suppose that I is a P-primal ideal of R. It follows from Lemma 2.6 and [5, Proposition 2.2 and Theorem 2.5] that P/I is a prime k-ideal of R/I. It is enough to show that $\Gamma(R/I) \cup \{q_0+I\} = P/I$. Let $q+I \in \Gamma(R/I) \cup \{q_0+I\}$, where $q \in Q$. If $q+I = q_0+I$, then we are done. So we may assume that $q+I \neq q_0+I$. Then there is an element $q_0+I \neq q_1+I \in R/I$ such that $(q+I) \odot (q_1+I) = q_0+I$, where $q_1 \in Q$ and $q_1q+I \subseteq q_0+I = I$, so $q_1q \in I$ with $q_1 \notin I$ by [6, Lemma 2.3] and the fact that I is a k-ideal. Then $q \in P \cap Q$ since I is a P-primal ideal; hence $q+I \in P/I$ by [5, Proposition 2.2]. Therefore, $\Gamma(R/I) \cup \{q_0+I\} \subseteq P/I$. For the other containment, suppose that $q+I \in P/I$, where $q \in Q \cap P$. We may assume that $q_0+I \neq q+I$. Then $q \notin I$ and there is a an element $s \in R-I$ such that $qs \in I$ since I is primal. There exist $q_1 \in Q$ and $a \in I$ such that $s = q_1+a$; so $qs = qq_1+aq$. Hence $qq_1 \in I$ since I is a k-ideal. There exists the unique element q_2 of Q with $(q+I) \odot (q_1+I) = q_2+I$ and $qq_1+I \subseteq q_2+I$. Then $qq_1+c = q_2+d$ for some $c, d \in I$; so $q_2 \in I$. Hence $q_2 = q_0$ by [6, Lemma 2.3]. Thus $q+I \in \Gamma(R/I)$, and so we have equality. □

THEOREM 2.8. *Let I be a P-primal Q-ideal of a Noetherian semiring R. Then* $\operatorname{diam}(\Gamma(R/I)) \leq 2$.

PROOF. Let q_0 be the unique element in Q such that q_0+I is the zero in R/I. By Theorem 2.7, $\Gamma(R/I) \cup \{q_0+I\} = P/I$ is a prime ideal of R/I. It follows from [6, Lemma 3.6 (ii)] that P/I is the union of all the associated primes of R/I; hence $P/I \in \operatorname{Ass}(R/I)$. It follows from [6, Lemma 3.5 (ii) and Remark 4.6 (7)] that $P/I = (0 : \bar{p})$ for some $\bar{p} \in \Gamma(R/I)$; hence $\operatorname{diam}(\Gamma(R/I)) \leq 2$. □

Recall some definition in [3]. For a graph G, the set of vertices adjacent to vertex x in G is denoted by $N(x)$. Let a, b be vertices of G, we define $a \sim b$ if $N(a) = N(b)$. Clearly, $\sim$ is an equivalence relation on the vertex set of G, and $G/\sim$ is also a graph in a natural way with $[x]$ and $[y]$ adjacent in $G/\sim$ if and only if x and y are adjacent in G. Then by [3, Lemma 2.1], we have the following lemma:

LEMMA 2.9. *Let R be a semiring. Then* $\operatorname{diam}(\Gamma_I(R)) = \operatorname{diam}(\Gamma_I(R)/\sim)$. *In particular, $\Gamma_I(R)$ is connected if and only if $\Gamma_I(R))/\sim$ is connected. Moreover,* $\operatorname{gr}(\Gamma_I(R) = \operatorname{gr}(\Gamma_I(R)/\sim)$.

Lemma 2.10. *Let I be a proper radical Q-ideal of a semiring R and let $x = q_1 + a$, $y = q_2 + b \in R - I$, where $q_1, q_2 \in Q$ and $a, b \in I$. Then x is adjacent to y in $\Gamma_I(R)$ if and only if $q_1 + I$ is adjacent to $q_2 + I$ in $\Gamma(R/I)$.*

Proof. This follows from [6, Theorem 2.4]. □

Proposition 2.11. *Let I be a radical Q-ideal of a semiring R. Then the graph $\Gamma_I(R)/\sim$ is isomorphic to $\Gamma(R/I)/\sim$.*

Proof. Define the map $\psi : \Gamma_I(R)/\sim \to \Gamma(R/I)/\sim$ by $\psi([x]) = [q + I]$, where $x = q + a$ ($q \in Q$, $a \in I$) is a vertex of $\Gamma_I(R)$. By Lemma 2.10, it is routine to check that ψ is a graph isomorphism. □

Theorem 2.12. *If I is a proper radical Q-ideal of a semiring R, then* $\mathrm{diam}(\Gamma_I(R)) = \mathrm{diam}(\Gamma(R/I))$

Proof. This follows from Lemma 2.9 and Proposition 2.11. □

Theorem 2.13. *Let I be a primal radical Q-ideal of a Noetherian semiring R. Then* $\mathrm{diam}(\Gamma_I(R)) \leq 2$.

Proof. This follows from Theorem 2.8 and Theorem 2.12. □

Lemma 2.14. *Let I be a P-primal ideal of a semiring R with $P \cap S = \emptyset$. Then the following hold:*
(i) If $a/s \in IR_S$, then $a \in I$.
(ii) IR_S is a PR_S-primal ideal of R_S.

Proof. (i) Suppose that $a/s \in IR_S$, but $a \notin I$. Then there are elements $a' \in I$ and $t \in S$ such that $a/s = a'/t$ by [4, Lemma 2.3 (i)], so $ta = sa' \in I$. It follows that t is not prime to I; hence $t \in P \cap S$, which is a contradiction. Therefore, $a \in I$.

(ii) By [4, Lemma 2.3 (iv)], we must have PR_S is a prime ideal of R_S. It is enough to show that PR_S is exactly the set of elements of R_S that are not prime to IR_S. Let $r/s \in PR_S$. Then r is not prime to I, so there exists $c \in R - I$ with $rc \in I$. Since $P \cap S = \emptyset$, we get $sc \notin I$; hence $(sc)/1 \notin IR_S$ by (i). Since $(r/s)(sc)/1 \in IR_S$, we must have r/s is not prime to IR_S. Next, suppose that r/s is not prime to IR_S. Then there exists $d/t \notin IR_S$ with $(r/s)(d/t) \in IR_S$; hence $rd \in I$ by (i). Since $d \notin I$, it follows that r is not prime to I. Thus $r \in P$, and hence $r/s \in PR_S$, as required. □

Proposition 2.15. *Let I be a P-primal ideal of a semiring R with $P \cap S = \emptyset$. Then $\Gamma_I(R)R_S = \Gamma_{IR_S}(R_S)$.*

Proof. By Lemma 2.2 and Lemma 2.14, we must have $\Gamma_{IR_S}(R_S) = PR_S - IR_S$. It suffices to show that $(P - I)R_S = PR_S - IR_S$. First, suppose that $a/s \in PR_S - IR_S$. Then $a/s \notin IR_S$ (so $a \notin I$) and $(a/s)(b/t) = ab/st \in IR_S$ for some $b/t \notin IR_S$ (so $b \notin I$), so $ab \in I$ by Lemma 2.14

(i); hence a is not prime to I. It follows that $a/s \in (P - I)R_S$. Thus $PR_S - IR_S \subseteq (P - I)R_S$. Next, assume that $a/s \in (P - I)R_S$. Then $a \in P - I$ implies that $ab \in I$ for some $b \notin I$ by Lemma 2.2, so by Lemma 2.14 (i), $b/1 \notin IR_S$. Now $(a/s)(b/1) = ab/s \in IR_S$ gives that a/s is not prime to IR_S, so $a/s \in PR_S - IR_S$; hence we have equality. □

THEOREM 2.16. *Let I be a P-primal ideal of a semiring R with $P \cap S = \emptyset$. Then* $\mathrm{diam}(\Gamma_I(R)) = \mathrm{diam}(\Gamma_{IR_S}(R_S))$.

PROOF. Suppose that $\mathrm{diam}(\Gamma_I(R)) = 1$. For every distinct vertices $a/s, b/t$ of $\Gamma_{IR_S}(R_S))$, Proposition 2.15 gives that a and b are distinct elements of $\Gamma_I(R)$, so $ab \in I$; hence $(a/s)(b/t) \in IR_S$. Thus $\mathrm{diam}(\Gamma_{IR_S}(R_S)) = 1$. If $\mathrm{diam}(\Gamma_{IR_S}(R_S)) = 1$, then for every distinct vertices a, b of $\Gamma_I(R) = P - I$, we must have the distinct vertices $a/1, b/1 \in (P - I)R_S = \Gamma_{IR_S}(R_S)$ by Proposition 2.15 (since if $a/1 = b/1$, then $a = b$, which is a contradiction), so $(a/1)(b/1) \in IR_S$. It follows from Lemma 3.14 (i) that $ab \in I$. Thus $\mathrm{diam}(\Gamma_I(R)) = 1$.

Now assume that $\mathrm{diam}(\Gamma_I(R)) = 2$. Let $a/s, b/t \in \Gamma_{IR_S}(R_S)$. If $(a/s)(b/t) \notin IR_S$, then $ab \notin I$, so there exists $c \in \Gamma_I(R)$ such that $ac \in I$ and $bc \in I$, so $c/1 \in \Gamma_{IR_S}(R_S)$ by Proposition 2.15. As $(a/s)(c/1) \in IR_S$ and $(c/1)(b/t) \in IR_S$, we must have $\mathrm{diam}(\Gamma_{IR_S}R_S)) = 2$. Conversely, assume that $\mathrm{diam}(\Gamma_{IR_S}(R_S)) = 2$. Let $a, b \in \Gamma_I(R)$ with $a \neq b$. If $ab \notin I$, then $ab/1 \notin IR_S$ by Lemma 2.14 (i), so there is an element c/s of $\Gamma_{IR_S}(R_S)$ with $(a/1)(c/s) \in IR_S$ and $(c/s)(b/1) \in IR_S$. In this case, by Proposition 2.15, we must have $c \in \Gamma_I(R)$. Moreover, Lemma 2.14 (i) gives $ac \in I$ and $cb \in I$; hence $\mathrm{diam}(\Gamma_I(R)) = 2$. Since, in general, the diameter of every zero-divisor graph with respect to an ideal of a commutative semiring is at most 3 (see [6, Lemma 2.1]), we have proved the result. □

THEOREM 2.17. *Let I be a P-primal Q-ideal of a semiring R with $P \cap S = \emptyset$. Then* $\mathrm{gr}(\Gamma_I(R)) = \mathrm{gr}(\Gamma_{IR_S}(R_S))$.

PROOF. First, assume that $\mathrm{gr}(\Gamma_I(R)) = \infty$. If $\mathrm{gr}(\Gamma_{IR_S}(R_S)) = n$, then there is a cycle $a_1/s_1 - a_2/s_2 - ... - a_n/s_n$ in $\Gamma_{IR_S}(R_S)$. In this case, $a_1 - a_2 - ... - a_n$ forms a cycle in $\Gamma_I(R)$ by Lemma 2.14 (i), which is a contradiction. So $\mathrm{gr}(\Gamma_{IR_S}(R_S)) = \infty$. If $\mathrm{gr}(\Gamma_{IR_S}(R_S)) = \infty$, then since $\Gamma_I(R)$ is a subgraph of $\Gamma_{IR_S}(R_S)$, we must have $\mathrm{gr}(\Gamma_I(R)) = \infty$. By [6, Theorem 3.3], the girth of every ideal-based zero-divisor graph of a commutative semiring, when finite, is either 3 or 4. Assume that $\mathrm{gr}(\Gamma_{IR_S}(R_S)) = 3$. So there exist distinct vertices $a/s, b/t, c/u$ in $\Gamma_{IR_S}(R_S)$ such that $(a/s)(b/t)$, $(b/t)(c/u)$ and $(c/u)(a/s)$ are elements of IR_S, so Proposition 2.15 gives that a, b and c are distinct vertices of $\Gamma_I(R)$; hence $ab, bc, ca \in I$ by Lemma 2.14 (i). It follows that $\mathrm{gr}(\Gamma_I(R)) = 3$. Conversely, assume that $\mathrm{gr}(\Gamma_I(R)) = 3$. Since the canonical homomorphism $R \to R_S$ is injective, we can assume that R is a sub-semiring of R_S, so in this case, $\Gamma_I(R)$ is a subgraph of $\Gamma_{IR_S}(R_S)$ (for if $a \in \Gamma_I(R)$, then $a/1 \in \Gamma_I(R))R_S = \Gamma_{IR_S}(R_S)$ by Proposition 2.15; hence $\mathrm{gr}(\Gamma_{IR_S}(R_S)) \leq \mathrm{gr}(\Gamma_I(R)) = 3$. Since the girth of a graph is at least 3, we must have $\mathrm{gr}(\Gamma_{IR_S}(R_S)) = 3$. Now it is clear that $\mathrm{gr}(\Gamma_{IR_S}(R_S)) = 4$ if and only if $\mathrm{gr}(\Gamma_I(R)) = 4$. ∎

3. Non-primal ideals

Let I be a k-ideal of a semiring R. A prime k-ideal P of R is said to be minimal prime k-ideal of I if J is a prime k-ideal in R such that $I \subsetneqq J \subseteq P$, then $J = P$. Denote by $\mathrm{Min}_k(I)$ the set of minimal prime k-ideals of R containing I (note that $\mathrm{Min}_k(0) = \mathrm{Min}_k(R)$). Now we study the diameter of $\Gamma_I(R)$, where I is not a primal ideal.

The proof of the following remark is completely straightforward.

Remark 3.1. Let R be a commutative semiring with non-zero identity. We denote by τ_R the set of all k-ideals of R and denote by $\mathrm{Spec}_k(R)$ the set of all prime k-ideals of R. Then:

(i) Let I be a k-ideal of R, T a multiplicatively closed subset of R such that $I \cap T = \emptyset$ and set $\Delta = \{J \in \tau_R : I \subseteq J \text{ and } J \cap T = \emptyset\}$. Then $I \in \Delta$ and the set Δ of k-ideals of R (partially ordered by inclusion) has at least one maximal element, and any such maximal element of Δ is a prime k-ideal of R.

(ii) Assume that I is a k-ideal of R and let $\mathrm{Var}_k(I) = \{P \in \mathrm{Spec}_k(R) : I \subseteq P\}$ (since each proper k-deal of R is contained in a maximal k-ideal of R, we must have $\mathrm{Var}_k(I) \neq \emptyset$). Then by using (i), $\sqrt{I} = \bigcap_{P \in \mathrm{Var}_k(I)} P$.

(iii) Let I be a proper k-ideal of R. Since an intersection of a family of k-ideals of R is k-ideal by [5, Lemma 2.12]), then $\mathrm{Var}_k(I)$ has at least one minimal member with respect to inclusion (by partially ordering $\mathrm{Var}_k(I)$ by reverse inclusion and using Zorn's Lemma). Therefore, $\mathrm{Min}_k(I) \neq \emptyset$.

(iv) Let P, I be k-ideals of R with P prime and $I \subseteq P$. Then the non-empty set $\Delta = \{P' \in \mathrm{Spec}_k(R) : I \subseteq P' \subseteq P\}$ has a minimal element P_1 with respect to inclusion (by partially ordering Δ by reverse inclusion and using Zorn's Lemma), where $P_1 \in \mathrm{Min}_k(I)$.

(v) If I is a proper k-ideal of R, then $\sqrt{I} = \bigcap_{P \in \mathrm{Min}_k(I)} P$ (using (i) and (iv)). In particular, if R is reduced, then $\mathrm{Min}_k(0) = 0$.

Compare the first two results with [10, Theorem 2.1 and Corollary 2.3].

Proposition 3.2. *Let $I \subseteq P$ be k-ideals of a semiring R, where P is a prime k-ideal. Then the following conditions are equivalent:*

(i) $P \in \mathrm{Min}_k(I)$.

(ii) $R - P$ is a multiplicatively closed set which is maximal with respect to the property $(R - P) \cap I = \emptyset$.

(iii) For each $x \in P$, there is a $y \notin P$ and non-negative integer n such that $yx^n \in I$.

Proof. $(i) \to (ii)$: Let T be a multiplicatively closed set which is maximal with respect to the property $T \cap I = \emptyset$ and $R - P \subseteq T$. If J is a k-ideal containing I that is maximal with respect to the property $J \cap T = \emptyset$, then J is prime k-ideal of R by Remark 3.1 (i). As $J \cap (R - P) = \emptyset$, we must have $P = J$. Thus $T = R - P$.

$(ii) \to (iii)$: Assume that $0 \neq x \in P$ and let $T = \{yx^i : y \in R - P, i = 0, 1, 2, ...\}$. Then T is a multiplicatively closed set with $R - P \subsetneqq T$, so $T \cap I \neq \emptyset$. So there is an element $y \in R - P$ and a positive integer n such that $yx^n \in I$.

$(iii) \to (i)$: Suppose that $I \subsetneqq J \subset P$, where J is a prime k-ideal; we show that $P = J$. Otherwise, there is an element $x \in P$ such that $x \notin J$. So there exist $y \notin P$ and a positive integer n such that $yx^n \in I \subseteq J$, which is a contradiction. ◻

Proposition 3.3. *Let I be a finitely generated k-ideal of a reduced semiring R. Then I is contained in a minimal prime k-ideal of R if and only if $\mathrm{Ann}(I) \neq 0$. In particular, $Z(R) = \bigcup_{P \in \mathrm{Min}_k(0)} P$.*

Proof. Let $I = \langle a_1, a_2, ..., a_n \rangle$. If $I \subseteq P$ for some minimal prime k-ideal P of R, then by Proposition 3.2, there exist $b_i \notin P$ and non-negative integers s_i such that $b_i a_i^{s_i} = (b_i a_i)^{s_i} = b_i a_i = 0$; hence $b_i a_i = 0$ for $i = 1, ..., n$ since R is reduced. Then $0 \neq b_1 b_2 ... b_n \in \mathrm{Ann}(I)$. Conversely, if $\mathrm{Ann}(I) \neq 0$, then $\mathrm{Ann}(I) \nsubseteq P$ for some minimal prime k-ideal of R (otherwise, $\mathrm{Ann}(I) = 0$ by Remark 3.1 (v)). Let $x \in I$. By assumption, there is an element $r \in R - P$ such that $rx = 0 \in P$; hence $x \in P$. Thus $I \subseteq P$. ◻

Proposition 3.4. *Let I be a Q-ideal of a semiring R. Then I is a radical ideal of R if and only if R/I is a reduced semiring.*

Proof. Assume that $I = \sqrt{I}$ and let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I. Let $q + I \in R/I$ be such that $(q + I)^n = q_0 + I$ for some n. Then we must have $q^n + I \subseteq q_0 + I$. Since I is a k-ideal and $q_0 \in I$ by [6, Lemma 2.3 (i)], we must have $q \in I$; hence $q + I = q_0 + I$. Conversely, assume that R/I is a reduced semiring. It suffices to show that $\sqrt{I} \subseteq I$. Let $x = q_1 + a \in \sqrt{I}$ for some $q_1 \in Q$ and $a \in I$. Then there is a positive integer m such that $x^m = q_1^m + b \in I$ for some $b \in I$. There exists the unique element $q_2 \in Q$ such that $q_1^m + I \subseteq q_2 + I$, so $q_1^m + b = q_2 + c \in I$ for some $c \in I$; hence $q_2 \in I$. It then follows from [6, Lemma 2.3 (ii)] that $q_2 = q_0$. Therefore, $(q_1 + I)^n = q_0 + I$; hence $q_1 + I = q_0 + I$ since R/I is reduced; thus $x = q_1 + a \in I$, as required. ◻

Lemma 3.5. *Let R be a semiring, I a Q-ideal of R and P a k-ideal of R with $I \subseteq P$. Then $P/I \in \mathrm{Min}_k(R/I)$ if and only if $P \in \mathrm{Min}_k(I)$.*

Proof. If $P/I \in \mathrm{Min}_k(R/I)$, then P is a prime k-ideal of R by [5, Theorem 2.5]. Suppose that $I \subsetneqq J \subseteq P$, where J is a prime k-ideal. Then J/I is a prime k-ideal of R/I by [5, Theorem 2.5 and Proposition 2.2]; hence $J/I = P/I$ by minimality of P/I. It follows from [5, Lemma 2.13] that $P = J$. The other implication is similar. □

Theorem 3.6. *Let I be a radical Q-ideal of a semiring R. Then $V(\Gamma_I(R)) \cup I = \bigcup_{P \in \mathrm{Min}_k(I)} P$.*

Proof. By Lemma 2.1, it suffices to show that $p(I) = \bigcup_{P \in \mathrm{Min}_k(I)} P$. Let $x = q_1 + a \in p(I)$, where $q_1 \in Q$ and $a \in I$. We may assume that $x \notin I$ (so $q_1 \notin I$ since I is a k-ideal), so $x \in \Gamma_I(R)$. Then $xy \in I$ for some $y = q_2 + b \in R - I$, where $q_2 \in Q - I$ and $b \in I$. Since I is a k-ideal and $xy \in I$, we must have $q_1 q_2 \in I$. Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I. Let $(q_1 + I) \odot (q_2 + I) = q_3 + I$, where $q_3 \in Q$ is the unique element such that $q_1 q_2 + I \subseteq q_3 + I$, so $q_1 q_2 + c = q_3 + d$ for some $c, d \in I$, so $q_3 \in Q \cap I$; hence $q_3 + I = q_0 + I$ by [6, Lemma 2.3]. Therefore, $(0 :_{R/I} q_1 + I) \neq 0$; hence $q_1 + I \in L$ for some minimal prime k-ideal L of R/I by Proposition 3.3. Then by [5, Theorem 2.3 and Theorem 2.5], $L = P/I$ with $q_1 \in Q \cap P$ for some prime k-ideal P of R; hence $x \in P \in \mathrm{Min}_k(I)$ by Lemma 3.5. Thus, $p(I) \subseteq \bigcup_{P \in \mathrm{Min}_k(I)} P$. For the other containment, assume that $x \in P$ for some minimal prime k-ideal of I. If $x \in I$, then $x \in p(I)$ by Lemma 2.1. So we may assume that $x \notin I$. By Proposition 3.2, there exist $y \notin P$ and a positive integer n such that $yx^n \in I$, but $yx^{n-1} \notin I$. This implies that $x \in \Gamma_I(R)$, so we have equality. □

Let I be an ideal of a semiring R. An ideal J of R is called prime to I if $(I :_R J) = I$.

Proposition 3.7. *Let I be an ideal of a semiring R. Then the following hold:*

(i) If there are nonadjacent elements $a, b \in \Gamma_I(R)$ such that the ideal $J = \langle a, b \rangle$ is prime to I, then $\mathrm{diam}(\Gamma_I(R)) = 3$.

(ii) I is not primal if and only if there are elements a and b of $\Gamma_I(R)$ such that the ideal $J = \langle a, b \rangle$ is prime to I.

Proof. (i) Since a and b are nonadjacent, we must have $\mathrm{d}(a, b) \neq 1$. If $\mathrm{d}(a, b) = 2$, then there is an element $c \in R - I$ such that $ac, cb \in I$, so $c \in (I : J) = I$, which is a contradiction. Thus $\mathrm{d}(a, b) \neq 2$. Now the assertion follows from [6, Lemma 2.1].

(ii) Suppose that I is not primal. Then by Lemma 2.1, $V(\Gamma_I(R)) \cup I = p(I)$ is not an ideal of R, so there exist $a, b \in p(I)$ with $a + b \notin p(I)$. If $a, b \in I$, then $a + b \in I \subseteq p(I)$ by Lemma 2.1, which is a contradiction. So suppose that $a \in I$ but $b \notin I$. Then $b \in \Gamma_I(R)$ and $bc \in I$ for some $c \in R - I$, so

$(a+b)c \in I$; hence $a+b$ is not prime I, which is a contradiction. Similarly, for $a \notin I$ and $b \in I$, we get a contradiction. Thus, we must have $a, b \in R-I$, so $a, b \in \Gamma_I(R)$. Since the inclusion $I \subseteq (I : J)$ is clear, we will prove the reverse inclusion. If $r \in (I : J)$, then $r(a+b) \in I$, so $r \in I$ since $a+b$ is prime to I. Thus J is prime to I. The other implication is clear. □

We shall require the following theorem, and its proof is a slight modification of those in [11, Theorem 2.1 and Lemma 2.3].

THEOREM 3.8. *(i) Let R be a reduced semiring. If $|\mathrm{Min}_k(0)| \geq 3$ and there are non-zero elements $a, b \in Z(R)$ such that $(0 : \langle a, b\rangle) = 0$, Then* $\mathrm{diam}(\Gamma(R)) = 3$.

(ii) Let I be a k-ideal of a non-reduced semiring R. If $\mathrm{Ann}(I) \neq 0$ and q is a nilpotent element of R, then $(0 : qR + I) \neq 0$. In particular, if $a \in Z(R)$ and q is nilpotent, then $a + q \in Z(R)$ and $(0 : \langle a, q\rangle) \neq 0$.

THEOREM 3.9. *Let I be a radical Q-ideal of a semiring R and suppose that I is not a primal ideal of R and $|\mathrm{Min}_k(I)| \geq 3$. Then* $\mathrm{diam}(\Gamma_I(R)) = 3$.

PROOF. By Proposition 3.7 (ii), there exist $x = q_1 + a, y = q_2 + b \in \Gamma_I(R)$ such that $(I : \langle x, y\rangle) = I$, where $q_1, q_2 \in Q - I$ and $a, b \in I$. Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I. Let $q_3 + I \in (0 : \langle q_1 + I, q_2 + I\rangle)$. Then $(q_1 + I) \odot (q_3 + I) = q_0 + I$, where $q_1 q_3 + I \subseteq q_0 + I$, so $q_1 q_3 + a = q_0 + e$ for some $e \in I$. Since $q_0 \in I$ and I is a k-ideal, we must have $q_1 q_3 \in I$. Similarly, $q_2 q_3 \in I$. Therefore, $q_3 x, q_3 y \in I$, so $q_3 \in I = (I : \langle x, y\rangle)$; hence $q_3 = q_0$ by [6, Lemma 2. 3]. Then the ideal $\langle q_1 + I, q_2 + I\rangle$ of R/I has no non-zero annihilator. As R/I is a reduced semiring by Proposition 3.4 and $|\mathrm{Min}_k(R/I)| \geq 3$ by Lemma 3.5, it follows from Theorem 3.8 (i) that $\mathrm{diam}(\Gamma(R/I)) = 3$. Now the assertion follows from Theorem 2.12. □

THEOREM 3.10. *Assume that I is a radical Q-ideal of a semiring R and I is not a primal ideal of R. Then $\mathrm{diam}(\Gamma_I(R)) \leq 2$ if and only if $|\mathrm{Min}_k(I)| = 2$.*

PROOF. First, assume that $\mathrm{diam}(\Gamma_I(R)) \leq 2$. By Theorem 2.3, $p(I)$ is not an ideal of R, so there exist $a, b \in p(I)$ such that $a + b \notin p(I)$. If there is an element r of $(I : \langle a, b\rangle)$ with $r \notin I$, then $a + b \in \Gamma_I(R) \subseteq p(I)$, which is a contradiction, so we must have $\langle a, b\rangle$ is prime to I. Therefore, by Proposition 3.4 and Theorem 3.6, I has at least two minimal prime k-ideals. If I has more than two minimal prime k-ideals, then $\mathrm{diam}(\Gamma_I(R)) = 3$ by Theorem 3.9; hence I must have exactly two minimal prime k-ideals. Next, assume that $|\mathrm{Min}_k(I)| = 2$. If P_1 and P_2 are the only minimal prime k-ideals of I, then $p(I) = P_1 \cup P_2$ by Theorem 3.6 and we may assume $a \in P_1 - P_2$ and $b \in P_2 - P_1$. Clearly, $ab \in P_1 \cap P_2 = I$. Consider two distinct vertices x and y in $\Gamma_I(R)$. If $xy \in I$, then $\mathrm{d}(x, y) = 1$. On the other hand, if $xy \notin I$, then either $\langle x, y\rangle \subseteq P_1$ or $\langle x, y\rangle \subseteq P_2$ since $p(I) = P_1 \cup P_2$. If $\langle x, y\rangle \subseteq P_1$, then

$x - b - y$ is a path in $\Gamma_I(R)$; hence $\mathrm{d}(x, y) = 2$. A similar argument shows that if $\langle x, y\rangle \subseteq P_2$, then $\mathrm{d}(x, y) = 2$. It follows that $\mathrm{diam}(\Gamma_I(R)) \leq 2$. □

PROPOSITION 3.11. *Let I be a Q-ideal of a semiring R which is not a radical ideal, and let J be a k-ideal of R which is not prime to I. If $x \in \sqrt{I}$, then the ideal $Rx + J$ is not prime to I.*

PROOF. Since every Q-ideal is k-ideal, we must have $I + J$ is a k-ideal of R and $(I + J)/I = \{q + I : q \in Q \cap (I + J)\}$ is a k-ideal of R/I by [5, Lemma 2.12 and Proposition 2.2]. By assumption, there exist $r = q_1 + a \in (I : J)$ such that $r \notin I$, where $q_1 \in Q$ and $a \in I$ (so $q_1 \notin I$). Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I. Let $q' + I \in (I + J)/I$, where $q' = c + d \in (I + J) \cap Q$ for some $c \in I$ and $d \in J$. There exists the unique element $q_2 \in Q$ such that $q_1q' + I \subseteq q_2 + I$, so $q_1q' + e = q_2 + f$ for some $e, f \in I$. Since $rJ \subseteq I$, we must have $q_1J \subseteq I$, so $q_1q' \in I$; hence $q_2 \in I$. Therefore, $q_2 = q_0$ by [6, Lemma 2.3]. Thus $(q_1 + I) \odot (q' + I) = q_0 + I$. It follows that $(0 :_{R/I} (I + J)/I) \neq 0$. By Proposition 3.4, $R' = R/I$ is a non-reduced semiring. If $x = q_3 + t \in \sqrt{I}$ (where $q_3 \in Q$, $t \in I$), then $(q_3 + t)^n = q_3^n + u \in I$ for some $u \in I$; hence $q_3^n \in I$ since I is a k-ideal. Therefore, $(q_3 + I)^n = q_0 + I$, so $q_3 + I$ is a nilpotent element of R'. It follows from Theorem 3.8 (ii) that there exists a non-zero element $q + I$ (so $q \notin I$) in the annihilator of the ideal $R'(x + I) + (I + J)/I$ of R'. It suffices to show that $q \in (I : Rx + J)$. Let $rx + b \in Rx + J$, where $r \in R$ and $b \in J$. There are elements $q_4, q_5 \in Q$ and $s_1, s_2 \in I$ such that $r = q_4 + s_1$, $b = q_5 + s_2$, so we have

$$q(rx + b) = q(q_4 + s_1)(q_3 + t) + q(q_5 + s_2) = qq_4q_3 + qq_5 + w, \tag{3.1}$$

where $w \in I$. Since $(q+I)\odot[(q_3+I)\odot(q_4+I)\oplus(q_5+I)] = q_0+I$, there exist the unique elements $q_6, q_7 \in Q$ such that $q_3q_4+I \subseteq q_6+I$, $q_6+q_5+I \subseteq q_7+I$ and $qq_7 + I \subseteq q_0 + I$. An inspection will show that $qq_3q_4 + qq_5 \in I$. Now the assertion follows from (3.1). □

PROPOSITION 3.12. *Assume that I is not a radical Q-ideal of a semiring R. If $x \in \sqrt{I}$ and $a \in \Gamma_I(R)$, then $a + x \in \Gamma_I(R)$ and the ideal $\langle a, x\rangle$ is not prime to I.*

PROOF. It is easy to see that the k-closure $\mathrm{cl}(Ra)$ of Ra is a k-ideal of R. Then by Proposition 3.11, there is an element $u \in (I : Rx + \mathrm{cl}(Ra))$ with $u \notin I$. As $(I : Rx+\mathrm{cl}(Ra)) \subseteq (I : Rx+Ra)$, we must have $\langle a, x\rangle$ is not prime to I. □

THEOREM 3.13. *Let I be a Q-ideal of a semiring R which is not a radical ideal and suppose that I is not a primal ideal of R. Then* $\mathrm{diam}(\Gamma_I(R)) = 3$.

PROOF. By Proposition 3.7 (ii), there are elements $a, b \in \Gamma_I(R)$ such that the ideal $\langle a, b\rangle$ is prime to I, so $\mathrm{d}(a, b) \neq 2$. By Proposition 3.12, neither a

nor b can be elements of $\sqrt{I}$. If $ab \notin I$, then $\mathrm{d}(a,b) \neq 1$, so $\mathrm{d}(a,b) = 3$; hence $\mathrm{diam}(\Gamma_I(R)) = 3$. So we can assume that $ab \in I$. Then $(I : \langle a^2, b^2\rangle) =$

$$(I : \langle a^2, ab, b^2\rangle) = (I : \langle a, b\rangle^2) = (I : \langle a, b\rangle) = I.$$

Therefore, there is an element $z \in \sqrt{I}$ such that $z \notin (I : \langle a^2, b^2\rangle)$. Without loss of generality we may assume that $zb^2 \notin I$. By assumption and Proposition 3.12, we must have $a + bz \in \Gamma_I(R)$. Since $(I : \langle a + bz, b\rangle) = (I : \langle a, b\rangle) = I$, we get $\mathrm{d}(a + bz, b) \neq 2$. But $(a + bz)b = ab + b^2z \notin I$, so $\mathrm{d}(a + bz, b) \neq 1$. Thus $\mathrm{d}(a + bz, b) = 3$ and $\mathrm{diam}(\Gamma_I(R)) = 3$. □

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References

[1] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative rings*, J. Algebra **217** (1999), 434-447.

[2] P. Allen, Ideal theory in semirings, Dissertation, Texas Christian University, 1967.

[3] L. Dancheng and W. Tongsuo, *On ideal-based zero-divisor graphs*, preprint, 2006.

[4] S. Ebrahimi Atani, *On primal and weakly primal ideals over commutative semirings*, Glas. Mat. Ser. III **43(63)** (2008), 13-23.

[5] S. Ebrahimi Atani, *The ideal theory in quotients of commutative semirings*, Glas. Mat. Ser. III **42(62)** (2007), 301-308.

[6] S. Ebrahimi Atani, *Zero-divisor graph with respect to ideals of a commutative semirings*, Glas. Mat. Ser. III **43(63)** (2008), 309–320.

[7] S. Ebrahimi Atani and A. Yousefian Darani, *Zero-divisor graphs with respect to primal and weakly primal ideals*, J. Korean Math. Soc., **46** (2009), 313–325.

[8] L. Fuchs, *On primal ideals*, Proc. Amer. Math. Soc. **1** (1950), 1-6.

[9] V. Gupta and J. N. Chaudhari, *Right π-regular semirings*, Sarajevo J. Math. **2** (2006), 3–9.

[10] J. A. Huckaba, Commutative rings with zero divisors, Marcel Dekker, Inc., New York, 1988.

[11] T. G. Lucas, *The diameter of a zero-divisor graph*, J. Algebra **301** (2006), 174–193.

[12] J. R. Mosher, *Generalized quotients of hemirings*, Compositio Math. **22** (1970), 275–281.

[13] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), 4425–4443.

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