GLASNIK MATEMATIČKI
Vol. 44(64)(2009), 167 – 175

FINITE p-GROUPS IN WHICH SOME SUBGROUPS ARE GENERATED BY ELEMENTS OF ORDER p

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Abstract. We prove that if a p-group G of exponent $p^e > p$ has no subgroup H such that $|\Omega_1(H)| = p^p$ and $H/\Omega_1(H)$ is cyclic of order $p^{e-1} \geq p$ and H is regular provided $e = 2$, then G is either absolutely regular or of maximal class. This result supplements the fundamental theorem of Blackburn on p-groups without normal subgroups of order p^p and exponent p. For $p > 2$, we deduce even stronger result than (respective result for $p = 2$ is unknown) a theorem of Božikov and Janko.

The 2-groups all of whose nonmetacyclic subgroups are generated by involutions, are classified in [BozJ]. Corollary 5 contains a stronger version of main theorem from [BozJ] for $p > 2$.

In what follows, G is a p-group, where p is a prime. By $\mathrm{cl}(G)$ we denote the class of G. Set $\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1\rangle$, $\mho_n(G) = \langle x^{p^n} \mid x \in G\rangle$, $\mho_n^*(G) = \langle x \in G \mid o(x) = p^n\rangle$, ($n$ is a positive integer). By $G', \mathrm{Z}(G), \Phi(G)$ we denote the derived subgroup, the center and the Frattini subgroup of G, respectively. A p-group G is said to be *absolutely regular* if $|G/\mho_1(G)| < p^p$ [Bla1]. A p-group G is said to be *regular* if for any $x, y \in G$ there exists $z \in \langle x, y\rangle'$ such that $(xy)^p = x^p y^p z^p$ [Hal]. By $\mathrm{K}_n(G)$ we denote the n-th member of the lower central series of G. A p-group G is said to be of *maximal class* if $|G| = p^m$, $m > 2$, and $\mathrm{cl}(G) = m - 1$ (so p-groups of maximal class are nonabelian). Next, Γ_1 is the set of all maximal subgroups of G.

Definition 1. *A p-group G is said to be an L_n-group, if $\Omega_1(G)$ is of order p^n and exponent p and $G/\Omega_1(G)$ is cyclic of order greater than p.*

In what follows we consider L_n-groups only for $n = p$.

2000 *Mathematics Subject Classification.* 20D15.
Key words and phrases. p-groups of maximal class, regular and absolutely regular p-groups, metacyclic p-groups, L_p-groups.

Let G be an L_p-group of exponent p^e; then $e > 2$ and $|G| = p^{p+e-1}$. Let us prove that $\mho_1(G)$ is cyclic of order p^{e-1}. We have

$$p^e = \exp(G) \le \exp(\mho_1(G)) \exp(G/\mho_1(G)) = p \cdot \exp(\mho_1(G))$$

so $|\mho_1(G)| \ge \exp(\mho_1(G)) \ge p^{e-1}$. Therefore, it suffices to show that $|\mho_1(G)| = p^{e-1}$. If G is regular, then $|\mho_1(G)| = |G/\Omega_1(G)| = p^{e-1}$. If G is irregular, then, by Hall's first regularity criterion [Ber2, Theorem 9.8(a)], we have $|G/\mho_1(G)| \ge p^p$ so $|\mho_1(G)| \le p^{-p}|G| = p^{e-1}$, and we conclude that $|\mho_1(G)| = p^{e-1}$, as required. It follows that $\exp(\Omega_{e-1}(G)) = p^{e-1}$ so we get $\Omega_e^*(G) = \langle G - \Omega_{e-1}(G)\rangle = G$.

Our main result is the following

Theorem 2. *Suppose that a p-group G of exponent $p^e > p$ is neither absolutely regular nor of maximal class. Then G contains a subgroup H of order p^{p+e-1} such that $|\Omega_1(H)| = p^p$, $H/\Omega_1(H)$ is cyclic of order p^{e-1} and $\Omega_2(H)$ is regular (so, if $e > 2$, then H is an L_p-group).*

Our proof of Theorem 2 is based on Blackburn's theory of p-groups of maximal class [Bla2] and some its consequences (see also [Ber1, §7] or [Hup, §III.13]). We also use some theorems from [Ber2, §13] (proofs of some of them we reproduce below; see Lemmas J(d), 6 and 7).

If a p-group G is either absolutely regular or of maximal class, then it has no subgroup H such as in conclusion of Theorem 2 (in the second case this follows from Lemma J(h)) so absolutely regular p-groups and p-groups of maximal class are excluded from hypothesis of Theorem 2.

Corollary 3. *Suppose that a p-group G of exponent greater than p is not absolutely regular. If all proper not absolutely regular subgroups of G are generated by elements of order p, then one and only one of the following holds:*

(a) $|G| = p^{p+1}$ *and* $|\Omega_1(G)| > p^{p-1}$ *so, if G is regular, then* $|\Omega_1(G)| = p^p$.
(b) $|G| = p^{p+1}$, $\mathrm{cl}(G) = p$, $|\Omega_1(G)| = p^{p-1}$ *(in that case, all proper subgroups of G are absolutely regular).*
(c) *G is of maximal class, $|G| > p^{p+1}$ and every irregular member of the set Γ_1 has two distinct subgroups of order p^p and exponent p (so, if $p = 2$, then G is dihedral).*

Corollary 4. *Let G be an irregular p-group, $p > 2$. Suppose that, whenever $H < G$ is neither absolutely regular nor of maximal class, then $\Omega_1(H) = H$. Then G is of maximal class.*

Corollary 5. *Let $p > 2$ and let M be a maximal metacyclic subgroup of a nonmetacyclic p-group G, where $|M| > p^2$. Suppose that, whenever $M < N \le G$ and $|N : M| = p$, then $\Omega_1(N) = N$. Then $p = 3$ and G is of maximal class.*

In particular, if $p > 2$ and a p-group G of exponent greater than p is neither metacyclic nor minimal nonmetacyclic and such that all proper nonmetacyclic subgroups of G are generated by elements of order p, then one and only one of the following assertions holds:

(a) G is regular of order p^4 and $|\Omega_1(G)| = p^3$.
(b) G is of maximal class and order 3^4, $|\Omega_1(G)| \geq 3^3$.
(c) $p = 3$, G is of maximal class, $|G| > 3^4$ and every irregular member of the set Γ_1 has two distinct (nonabelian) subgroups of order 3^3 and exponent 3.

Note that if a 3-group of maximal class has elementary abelian subgroup of order 3^3, then G is isomorphic to a Sylow 3-subgroup of the symmetric group of degree 3^2 ([Ber1, Theorem 5.2]). Therefore, a group of Corollary 5(c) has no abelian subgroups of order 3^3 and exponent 3.

In Lemma J we collected known results which are used in what follows.

LEMMA J. *Let $G > \{1\}$ be a p-group, $p > 2$.*

(a) *(P. Hall* [Hal]*) If G is regular, then* $\exp(\Omega_1(G)) = p$ *and* $|\Omega_1(G)| = |G/\mho_1(G)|$. *Absolutely regular p-groups, groups of exponent p and p-groups of class $< p$ are regular.*
(b) [Ber2, Exercise 9.1(b)] *If G is of maximal class and order p^m, then it has exactly one normal subgroup of order p^i for all $i < m - 1$.*
(c) *(Suzuki; see* [Ber2, Proposition 1.8]*) If $H < G$ is of order p^2 and $|C_G(H)| = p^2$, then G is of maximal class.*
(d) [Ber2, Exercise 13.10(a)] *If $A < G$ and all subgroups of G of order $p|A|$ and containing A are of maximal class (so that $|A| > p$), then G is also of maximal class.*
(e) [Ber1, Theorem 7.4] *Suppose that $L = L_1 \in \Gamma_1$ is irregular of maximal class. If G is not of maximal class, then $G/K_p(G)$ is of order p^{p+1} and exponent p and $\Gamma_1 = \{L = L_1, \dots, L_{p^2}, T_1, \dots, T_{p+1}\}$, where all L_i's are of maximal class and all T_j's are not of maximal class (T_j are also not absolutely regular since $|G/\mho_1(G)| \geq p^{p+1}$), $\eta(G) = \bigcap_{i=1}^{p+1} T_i$ has index p^2 in G and $G/\eta(G)$ is abelian of type (p,p) so $\bigcup_{i=1}^{p+1} T_i = G$ so that* $\exp(T_i) = \exp(G)$ *for some i.*
(f) *(*[Bla2]*; see also* [Ber1, Theorem 7.5]*) If G is neither absolutely regular nor of maximal class and $H \in \Gamma_1$ is absolutely regular, then $G = H\Omega_1(G)$, where $\Omega_1(G)$ is of order p^p and exponent p (in particular, $|\Omega_1(H)| = p^{p-1}$).*
(g) [Ber1, Remark 7.2] *and* [Bla3]. *If G is neither absolutely regular nor of maximal class, then the number of subgroups of order p^p and exponent p in G is $\equiv 1 \pmod p$.*
(h) *(Blackburn* [Bla2]*) If G is of maximal class and order greater than p^{p+1}, then $\Gamma_1 = \{G_1, G_2, \dots, G_{p+1}\}$, where G_1 is absolutely regular with $|\Omega_1(G_1)| = p^{p-1}$ (G_1 is called the fundamental subgroup of G),*

and $G_2, \dots, G_{p+1}$ are irregular of maximal class. All p-groups of maximal class and order p^{p+1} are irregular.

A p-group G of maximal class and order greater than p^{p+1} has no normal subgroup of order p^p and exponent p. Assume that this is false, and let $R \lhd G$ be of order p^p and exponent p. Then, by Lemma J(b), $R \leq \Phi(G) < G_1$, a contradiction since the fundamental subgroup G_1 is absolutely regular.

If a p-group G satisfies $\exp(\Omega_1(G)) > p$, then it is irregular (Lemma J(a)).

To facilitate the proof of Theorem 2, we prove the following three assertions.

Lemma 6. *If $H < G$ is such that $N = \mathrm{N}_G(H)$ is of maximal class, then G is also of maximal class.*

Proof. We use induction on $|G|$. One may assume that $N < G$; then H is not characteristic in N (otherwise, $N = G$). In that case, by Lemma J(b), we have $|N : H| = p$. Since $|\mathrm{Z}(N)| = p$ and $\mathrm{Z}(G) < N$, we get $\mathrm{Z}(G) = \mathrm{Z}(N)$ so $|\mathrm{Z}(G)| = p$ and $\mathrm{Z}(G) \leq \Phi(N) < H$. Set $\bar{G} = G/\mathrm{Z}(G)$. If $|\bar{H}| = p$, then $\mathrm{C}_{\bar{G}}(\bar{H}) = \bar{N}$ is of order p^2 so $\bar{G}$ is of maximal class (Lemma J(c)). Now let $|\bar{H}| > p$; then $\bar{N}$ is of maximal class so $\bar{G}$ is also of maximal class, by induction, and we are done since $|\mathrm{Z}(G)| = p$. □

Proof of Lemma J(d). In view of Lemma 6, it suffices to show that $N = \mathrm{N}_G(A)$ is of maximal class, so one may assume that $|N : A| > p$ and $N = G$; then $A \lhd G$ and $|G : A| > p$. Let $D < A$ be G-invariant of index p^2 and set $C = \mathrm{C}_G(A/D)$; then $|G : C| \leq p$ so $A < C$ since $|G : A| > p$. Let $B/A \leq C/A$ be of order p; then $|B| = p|A|$. Since B/D is abelian of order p^3, B is not of maximal class, a contradiction. □

Lemma 7. *Let G be a p-group. Suppose that $A \in \Gamma_1$ is absolutely regular and $M < G$ is irregular of maximal class. Then G is of maximal class.*

Proof. Assume that G is not of maximal class. Then, by Lemma J(d), $M < H \leq G$, where $|H : M| = p$ and H is not of maximal class. In that case, by Lemma J(e), $H/\mathrm{K}_p(H)$ is of order p^{p+1} and exponent p. It follows that H has no absolutely regular maximal subgroups. However, $A \cap H$ is an absolutely regular maximal subgroup of H, and this is a contradiction. □

Lemma 8. *All proper subgroups of an L_p-group G are regular; in particular, $\Omega_2(G)$ is regular.*

Proof. It suffices to show that all maximal subgroups of G are regular. Take $M \in \Gamma_1$.

Suppose that $\Omega_1(G) \not\leq M$; then $\Omega_1(M) = M \cap \Omega_1(G)$ is of order p^{p-1} (and exponent p). Since M is not of maximal class (indeed, $M/\Omega_1(M) \cong G/\Omega_1(G)$ is cyclic of order $> p$), it follows that M is absolutely regular so regular (Lemma J(g,a)).

Now let $\Omega_1(G) < M$. Let $D < \Omega_1(G)$ be a G-invariant subgroup of index p^2. Set $C = \mathrm{C}_G(\Omega_1(G)/D)$; then $|G : C| \leq p$. Take in $C/\Omega_1(G)$ a subgroup $H/\Omega_1(G)$ which is maximal in $G/\Omega_1(G)$; then H/D is abelian so $\mathrm{cl}(H) < p$, and hence H is regular (Lemma J(a)). Since $G/\Omega_1(G)$ has only one maximal subgroup, we get $H = M$. ∎

Suppose that $p > 2$ and G is an irregular L_p-group of exponent p^e. Assume that there is in G a normal cyclic subgroup C of order p^e. Set $D = C \cap \Omega_1(G)$; then $|D| = p$ and $G/D = (C/D) \times (\Omega_1(G)/D))$ so $\mathrm{d}(G) > 2$. By Lemma 8, all proper subgroups of G are regular. It follows that $\mathrm{d}(G) = 2$, contrary to what has just been proved. Thus, all cyclic subgroups of order p^e are not normal in G.

Proof of Theorem 2. Let $p = 2$. Since G is not of maximal class, there is in G a normal abelian subgroup R of type $(2, 2)$ (Lemma J(g)). Put $C = \mathrm{C}_G(R)$; then $|G : C| \leq 2$ so $\exp(C) \geq 2^{e-1}$. Suppose that $\exp(C) = 2^e$; then there is in $C - R$ an element x of order 2^e. In that case, $A = \langle x, R\rangle$ is abelian of type $(2^e, 2)$ or $(2^e, 2, 2)$. In any case, A contains an abelian subgroup H of type $(2^e, 2)$, and H is the desired subgroup. Now let $\exp(C) = 2^{e-1}$. Take $y \in G - C$ of order 2^e. Suppose that $U = \mathrm{C}_G(y)$ is cyclic; then $\mathrm{C}_G(U) = U$ and $\mathrm{Z}(G)$ is cyclic. If $e = 2$, then G is of maximal class (Lemma J(c)), contrary to the hypothesis. Thus, $e > 2$. In that case, $U \cap R$ is of order 2 so $H = RU$ is an L_2-subgroup of exponent 2^e. If $\mathrm{C}_G(y)$ is noncyclic, then there is an involution $x \in \mathrm{C}_G(y) - \langle y\rangle$; in that case, $H = \langle x, y\rangle = \langle x\rangle \times \langle y\rangle$ is abelian of type $(2^e, 2)$ so H is the desired subgroup.

In what follows we assume that $p > 2$. We use induction on $|G|$. Take an element $x \in G$ of order p^e. Let $x \in L < G$, where L is either absolutely regular or irregular of maximal class such that if $L < M \leq G$, then M is neither absolutely regular nor of maximal class (L exists, by hypothesis). Let $L < F \leq G$ and $|F : L| = p$; then F is neither absolutely regular nor of maximal class, by the choice of L. Therefore, if $F < G$, then F contains the desired subgroup H. Next assume that $F = G$; then $|G : L| = p$.

(i) Let L be absolutely regular. Then, by Lemma J(f), $G = L\Omega_1(G)$, where $\Omega_1(G)(< G)$ is of order p^p and exponent p. Set $H = \langle x, \Omega_1(G)\rangle$, where $x \in L$ has order p^e; then $\Omega_1(H) = \Omega_1(G)$ is of order p^p and $|H| = p^{p+e-1}$, by the product formula. If $e > 2$, then H is an L_p-group. It remains to show that if $e = 2$, then H is regular. This is true provided $H = G$ since, by hypothesis, G is not of maximal class, so $\mathrm{cl}(G) < p$ (Lemma J(a)). Now let $H < G$ and assume, by the way of contradiction, that H is irregular. Then H is of maximal class since $|H| = p^{p+1}$ (Lemma J(a) again) and, by assumption, L is absolutely regular of index p in G (indeed, $L \cap H = \Omega_1(L)$ is of order p^{p-1} so $|G| = p|L|$, by the product formula). It follows from Lemma 7 that G is of maximal class, contrary to the hypothesis. Thus, H is the desired subgroup.

(ii) Now let L be irregular of maximal class (indeed, since $\exp(L) = p^e > p$ and L is not absolutely regular, we get $|L| \geq p^{p+1}$ so it is irregular, by Lemma J(h), since it is of maximal class). Then, by Lemma J(e), we have $\Gamma_1 = \{L = L_1, L_2, \dots, L_{p^2}, T_1, \dots, T_{p+1}\}$, where $L_1, \dots, L_{p^2}$ are of maximal class and $T_1, \dots, T_{p+1}$ are neither absolutely regular nor of maximal class and $G = \bigcup_{i=1}^{p+1} T_i$. It follows that one of T_i's, say T_1, has exponent p^e. Therefore, by induction, there is $H \leq T_1$ of exponent p^e such that $\Omega_1(H)$ is of order p^p and exponent p, $H/\Omega_1(H)$ is cyclic of order p^{e-1}, and H is regular if $e = 2$. □

Lemma 9. *Suppose that A is a proper absolutely regular subgroup of a p-group G,* $\exp(A) > p$ *and, whenever $A < B \leq G$ and $|B : A| = p$, then $\Omega_1(B) = B$. Then G is of maximal class.*

Proof. By Lemma J(a), B is irregular so G is also irregular. Assume that G is not of maximal class. Let $|G : A| = p$; then $\Omega_1(G) = G$, by hypothesis. However, by Lemma J(f), $G = A\Omega_1(G)$, where $|\Omega_1(G)| = p^p < |G| = |\Omega_1(G)|$, a contradiction. Now let $|G : A| > p$. If $A < B < G$, where $|B : A| = p$, then by what has just been proved, B is of maximal class so G is also of maximal class, by Lemma J(d). □

Proof of Corollary 3. Suppose that G is regular and set $L = \Omega_1(G)$; then $|L| \geq p^p$ since G is not absolutely regular. By Lemma J(a), $|G : L| = p$ so G is as in (a) (indeed, $|L| = p^p$ since every maximal subgroup of G not containing L is absolutely regular because of it is not generated by elements of order p).

Next we assume that G is irregular. Since G has a proper absolutely regular subgroup of composite exponent, it follows from Lemma 9 that G is of maximal class. If all maximal subgroups of G are absolutely regular, then G is as in (b), by Lemma J(h). Obviously, every group of maximal class and order p^{p+1} satisfies the hypothesis. Now let $|G| > p^{p+1}$. If $M \in \Gamma_1$ is irregular, then $\Omega_1(M) = M$, by hypothesis. If $L = \Omega_1(\Phi(G))$, then L is of order p^{p-1} and exponent p (Lemma J(h,a,b)). Take an element $x \in M - L$ of order p and set $U_1 = \langle x, L\rangle$. Take an element $y \in M - U_1$ of order p and set $U_2 = \langle y, L\rangle$. Then U_1 and U_2 are distinct of order p^p and exponent p (Lemma J(a)), and the proof is complete since $U_1, U_2 < M$. □

Proof of Corollary 4. Assume that G is not of maximal class. Then there is in G a subgroup H such as in Theorem 2. Since H is neither absolutely regular nor of maximal class and $\Omega_1(H) \neq H$, we get a contradiction. □

Proof of Corollary 5. Let M be as in the statement of the corollary. Then M is absolutely regular since $p > 2$ so, by Lemma 9, G is of maximal class. Let $M < N \leq G$, where $|N : M| = p$. Then N is irregular of maximal class (Lemma 9). Since $p^{p-1} = |\Omega_1(M)| \leq p^2$ and $p > 2$, we get $p = 3$. □

Remark 10. Here we offer another proof of Theorem 2 in the case $\exp(G) = p^e > p^2$. We have to prove that G contains an L_p-subgroup of order p^{p+e-1}. We use induction on $|G|$. By Lemma J(g), there is $M \triangleleft G$ of order p^p and exponent p. Take a cyclic $X < G$ of order p^e and set $F = MX$; then F of exponent $p^e > p^2$ is neither absolutely regular nor of maximal class. Therefore, if $F < G$, the result follows by induction. Now let $F = G$. Suppose that $X \cap M > \{1\}$; then $|G| = p^{p+e-1}$. We claim that G is an L_p-group. It suffices to show that $\Omega_1(G) = M$. Assume that this is false. Since G/M is cyclic, we conclude that $|\Omega_1(G)| \le p^{p+1}$. Assume that $|\Omega_1(G)| = p^{p+1}$. Then $X \cap \Omega_1(G)$ is cyclic of order p^2 so $\Omega_1(G)$ is irregular and we conclude that it is of maximal class (Lemma J(a)). Since G is not of maximal class, the number $\mathrm{e}_p(G)$ of subgroups of order p^p and exponent p in G is congruent with 1 modulo p (Lemma J(g)), and all these subgroups lie in $\Omega_1(G)$. Since $\mathrm{e}_p(G) > 1$ and $\mathrm{d}(\Omega_1(G)) = 2$, it follows that all maximal subgroups of $\Omega_1(G)$ have exponent p, so $\exp(\Omega_1(G)) = p$ and $\Omega_1(G)$ is regular (Lemma J(a)), a contradiction. Thus, in the case under consideration, G is an L_p-group. Now let $X \cap M = \{1\}$. Let $R < M$ be a G-invariant subgroup of index p. Set $H = RX$. Let us show that H is an L_p-group. Indeed, H is not absolutely regular since $\Omega_1(X)R$ is of order p^p and exponent p (Lemma J(a)). Next, H/R is cyclic of order $p^e > p^2$ so H is not of maximal class. If $K/R < H/R$ is of order p, then $\Omega_1(H) \le K \le \Omega_1(H)$ so $\Omega_1(H) = \Omega_1(X)R = K$ is of order p^p and exponent p whence H is an L_p-subgroup.

Theorem 11. *Let H be a normal absolutely regular subgroup of a p-group G, $|H| > p^{p-1}$ and $\Omega_1(G) \not\le H$.*

(a) *If for every $z \in G - H$ of order p, the subgroup $V = \langle z, H\rangle$ is of maximal class, then G is also of maximal class.*

(b) *If, in addition, $|H| > p^p$ and, for every $z \in G - H$ of order p, we have $\Omega_1(\langle z, H\rangle) = \langle z, H\rangle$, then G is of maximal class.*

Proof. By hypothesis, $\exp(H) > p$. Assume that (a) and (b) are not true. Let $z \in G - H$ be of order p and $V = \langle z, H\rangle$. Then, by Lemma J(a), V is irregular. In (a), if $V = G$, then, by hypothesis, G is of maximal class. In (b), if $V = G$, then, by Lemma J(f), G is of maximal class. Assume that this is false. Then there is $R \triangleleft G$ of order p^p and exponent p. Let $R_0 \le R$ be G-invariant and minimal such that $R_0 \not\le H$. Then $F = R_0H = \langle x, H\rangle$ for any $x \in R_0 - H$. By hypothesis, $\Omega_1(F) = F$ so F is irregular of maximal class, contrary to Lemma J(b) (indeed, $|R_0| < p^p$ since F is of maximal class and order $> p^{p+1}$, and so $\Omega_1(H)$ and R_0 are F-invariant of indices greater that p and non-incident). Therefore, in both cases, one may assume that $V < G$ so $|G : H| > p$.

(a) Let $H \le H_0 < G$, where H_0 is absolutely regular such that $|H_0|$ is as large as possible. By Lemma J(d), $H_0 < B \le G$, where $|B : H_0| = p$ and B is not of maximal class. Then, by Lemma J(f), $B = H_0\Omega_1(B)$, where $\Omega_1(B)$ is of order p^p and exponent p so there exists an element $x \in \Omega_1(B) - H_0$ of order p. Set $U = \langle H, x \rangle$; then, by hypothesis and Lemma J(a,h), U is of maximal class and order $\geq p^{p+1}$ so irregular. We have $H_0, U < B$, H_0 is absolutely regular of index p in B and U is irregular of maximal class. Therefore, by Lemma 7, B is of maximal class, contrary to its choice.

(b) Set $\mathrm{N}_G(\Omega_1(H)) = N$. If the set $N - H$ has no elements of order p, then $\Omega_1(N) = \Omega_1(H)$ is characteristic in N so $N = G$, a contradiction since $\Omega_1(G) \not\leq H$, by hypothesis. Therefore, there is an element $y \in N - H$ of order p. By Lemma 6, N is not of maximal class. If N is absolutely regular, there is in $\Omega_1(N)/\Omega_1(H)$ an N-invariant subgroup $R/\Omega_1(H)$ of order p. In that case, HR is absolutely regular of order $p|H|$ and $\Omega_1(HR) \neq HR$, contrary to the hypothesis since $HR = \langle z, H \rangle$ for every $z \in R - H$. Otherwise, there is $S \lhd N$ of order p^p and exponent p (Lemma J(g)). Let S_0 be an N-invariant subgroup of S such that $S_0 \not\leq H$ and $|S_0|$ is as small as possible. Then $|S_0H : H| = p$ and $S_0H = \langle y, H \rangle$ for every $y \in S_0 - H$. Therefore, by hypothesis, $\Omega_1(S_0H) = S_0H$ so, by Lemma J(f), S_0H is of maximal class. Then, by Lemma J(b), $S_0 \leq \Phi(S_0H) < H$, contrary to the choice of S_0. □

Problems

Below G is a nonabelian p-group.

1. Study the 2-groups G containing a proper metacyclic subgroup M of order > 4 such that, whenever $M < N \leq G$ and $|N : M| = 2$, then $\Omega_1(N) = N$ ($\mathrm{d}(N) = 2$).
2. Classify the 2-groups all of whose proper nonabelian subgroups that are nonmetacyclic, are generated by elements of order p (compare with [BozJ]).
3. Study the p-groups G containing an abelian subgroup M of exponent greater than p such that, whenever $M < N \leq G$ and $|N : M| = p$, then $\Omega_1(N) = N$.
4. Study the irregular p-groups G, $p > 2$, such that $|\Omega_2^*(G)| = p^{p+1}$.
5. Study the p-groups G containing a proper minimal nonabelian subgroup M of order greater than p^3 and such that, whenever $M < N \leq G$ and $|N : M| = p$, then (i) $\Omega_1(N) = N$ (in that case, $p \leq 3$, by Lemma J(g,h)), (ii) $\mathrm{d}(N) = 2$.
6. Let A be a subgroup of index $> p^k > p$ in a p-group G. Suppose that all subgroups of G containing A as a subgroup of index p^k, are of maximal class. Is it true that G is also of maximal class (compare with Lemma J(d))?
7. Study the p-groups G containing a subgroup M of maximal class and order $> p^p$ such that, whenever $M < N \leq G$ and $|N : M| = p$, then $\Omega_1(N) = N$.

8. Study the p-groups G of order greater than p^{p+2} containing a maximal regular subgroup A of order p^{p+1}.
9. Suppose that G is an L_p-group. Is it true that G is regular if and only if $G/\mho_2(G)$ is regular?
10. Study the irregular p-groups G containing an L_p-subgroup of index p.
11. Does there exist a positive integer n such that a p-group G, $p > 2$, is regular if and only if $G/\mho_n(G)$ is regular?
12. Classify the 2-groups G such that $\Omega_1(G)$ is nonabelian of order 2^4.

Acknowledgements.
I thank the referee for a number of constructive remarks and suggestions.

References

[Ber1] Y. Berkovich, *On subgroups and epimorphic images of finite p-groups*, J. Algebra **248** (2002), 472–553.

[Ber2] Y. Berkovich, Groups of Prime Power Order, volume 1, Walter de Gruyter, 2008.

[Ber3] Y. Berkovich, *A generalization of theorems of Ph. Hall and N. Blackburn and an application to non-regular p-groups*, Math. USSR Izv. **35** (1971), 815–844.

[BJ] Y. Berkovich and Z. Janko, *Structure of finite p-groups with given subgroups*, Contemp. Math. **402** (2006), 13–93.

[Bla1] N. Blackburn, *Generalization of certain elementary theorems on p-groups*, Proc. London Math. Soc. (3) **11** (1961), 1–22.

[Bla2] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.

[Bla3] N. Blackburn, *Note on a paper of Berkovich*, J. Algebra **24** (1973), 323–334.

[BozJ] Z. Božikov and Z. Janko, *Finite 2-groups all of whose nonmetacyclic subgroups are generated by involutions*, Arch. Math. **90** (2008), 14–17.

[Hal] P. Hall, *A contribution to the theory of groups of prime power order*, Proc. London Math. Soc. **2**, No. 36 (1933), 29-95.

[Hup] B. Huppert, Endliche Gruppen. I., Springer, Berlin, 1967.

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Received: 30.4.2008.
Revised: 6.9.2008. & 1.10.2008.