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CONSTRUCTING NEAR-EMBEDDINGS OF CODIMENSION ONE MANIFOLDS WITH COUNTABLE DENSE SINGULAR SETS

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Abstract. We present, for all $n \geq 3$, very simple examples of continuous maps $f : M^{n-1} \to M^n$ from closed $(n-1)$-manifolds M^{n-1} into closed n-manifolds M^n such that even though the singular set $S(f)$ of f is countable and dense, the map f can nevertheless be approximated by an embedding, i.e. f is a *near-embedding*. In dimension 3 one can get even a piecewise-linear approximation by an embedding.

1. Introduction

Denote the *singular* set of an arbitrary continuous mapping $f\colon X \to Y$ between topological spaces by $S(f) = \{x \in X \mid f^{-1}(f(x)) \neq x\}$. A manifold which is connected, compact and has no boundary is said to be *closed*. The following conjecture was proposed by the first author in the mid 1980's:

Conjecture 1.1. *Let* $f\colon M^{n-1} \to M^n$ *be any continuous (possibly surjective) map from a closed* $(n-1)$*-manifold* M^{n-1} *into a closed n-manifold* M^n*,* $n \geq 3$*, such that* $\dim S(f) = 0$*. Then* f *is a near-embedding, i.e., for every* $\varepsilon > 0$ *there exists an embedding* $g\colon M^{n-1} \to M^n$ *such that for every* $x \in M^{n-1}$*,* $d(f(x), g(x)) < \varepsilon$*.*

For $n = 3$ it has since been shown to be equivalent to the Bing Conjecture from the 1950's (cf. [3]) and it is closely related to the 3-dimensional Recognition Problem, one of the central problems of geometric topology (cf. [8]). In particular, it is closely related to the general position of 3-manifolds, called the

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light map separation property LMSP*, introduced by Daverman and Repovš (cf. [7, Conjecture 5.4]).

In the case when $n = 3$, a very special case of Conjecture 1.1 was verified by Anderson ([1]) in 1965. Then in 1992 Brahm ([4]) proved Conjecture 1.1 for the case when $n = 3$ and the closure of the singular set is 0-dimensional, $\dim(\mathrm{Cl}S(f)) = 0$. However, in general the second property needs not be satisfied – it was shown in [5] that for $n = 3$ it can happen that $\dim(\mathrm{Cl}S(f)) = n - 1$.

Since the construction in [5] is very technical, there has been for a long time an open question if there is an *elementary* example of a continuous map $f : M^2 \to M^3$ such that $\dim S(f) = 0$ whereas $0 < \dim\mathrm{Cl}(S(f)) \leq 3$. The purpose of this note is to present such an example – it has a very simple construction and the verification of all asserted properties is straightforward. Moreover, unlike [5], our methods evidently generalize in a direct manner to yield continuous maps $f\colon M^{n-1} \to M^n$ of closed codimension one manifolds into closed n-manifolds, with properties analogous to (i) and (ii) below, for every $n \geq 3$.

THEOREM 1.2. *For every $n \geq 3$, there exists a continuous map $f\colon S^{n-1} \to S^n$ such that:*

(i) *the singular set $S(f)$ of f is countable and dense (hence 0-dimensional and nonclosed); and*

(ii) *f is a near-embedding.*

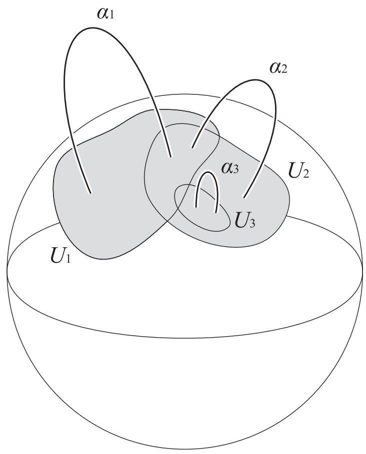
Remark. A question when a light map is a near-embedding is also interesting in view of the classical *Monotone-Light Factorization Theorem* (cf. e.g. [9]) which asserts that every continuous mapping $f : X \to Y$ from any compact space X to any space Y can be factorized as a product $f = l \circ m$ of a monotone map $m : X \to Z$ (i.e. each point inverse $m^{-1}(z)$ is connected) and a light map $l : Z \to Y$ (i.e. each point inverse $l^{-1}(y)$ is totally disconnected).

2. PROOF OF THEOREM 1.2

Let $n \geq 3$ and choose a countable basis $\{U_i\}_{i\in N}$ of open sets for S^{n-1} (which is considered as the standardly embedded $(n-1)$-sphere in S^n). We shall inductively construct a sequence of pairwise disjoint *tame* PL arcs α_i in S^n (i.e. for every $i \geq 1$ there is a homeomorphism $h_i : S^n \to S^n$ such that $h_i(\alpha_i) \subset S^1 \subset S^n$) with the property that:

1. for every i, $\alpha_i \cap S^{n-1} = \partial\alpha_i \subset U_i$;
2. for every i, $\mathrm{diam}(\alpha_i) < \frac{1}{2^i}$.

Begin with a tame PL arc $\alpha_1 \subset S^n$ such that $\partial\alpha_1 \subset U_1$ and $\mathrm{diam}(\alpha_1) < 1/2$. Assume inductively, that we have already constructed pairwise disjoint tame PL arcs $\alpha_1, \ldots, \alpha_{n-1} \subset S^n$ with all required properties. We can then clearly find a tame PL arc $\alpha_n \subset S^n$ such that $\partial\alpha_n \subset U_n$ and α_n is disjoint with $\alpha_1 \cup \ldots \cup \alpha_{n-1}$.



By our construction, $\{\alpha_n\}_{n\in N}$ is a *null-sequence*, i.e. $\lim_{i\to\infty}\operatorname{diam}\alpha_i = 0$. The decomposition $G = \{\alpha_i\}_{n\in N}$ of S^n into points and arcs is clearly *cellular* (i.e., each element of the decomposition G is the intersection of a *nested* sequence of closed n-cells $\{B^n_k\}_{k\in N}$ in S^n) (i.e., for ever $k, B^n_{k+1} \subset \operatorname{Int}B^n_k$) and *upper semicontinuous* (i.e. the quotient map $\pi : S^n \to S^n/G$ is closed).

Therefore it follows by [6, Theorem 7, page 56] that the decomposition G is *shrinkable* (i.e., the map π is approximable by homeomorphisms). In particular, the quotient space S^n/G is homeomorphic to S^n. The desired mapping $f\colon S^{n-1} \to S^n$ is now defined as the compositum $f = \pi \circ i$ of the inclusion $i\colon S^{n-1} \to S^n$ and the decomposition quotient mapping $\pi\colon S^n \to S^n/G$.

It follows by construction that the singular set $S(f) = \bigcup_i \partial\alpha_i$ is countable and dense. It is also clear that this map can be approximated arbitrarily closely by embeddings of S^{n-1} into S^n (by not shrinking the arcs all the way but only as much as it is necessary to make them sufficiently small).

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