Knowledge Accumulation and Chaotic Business Cycles

RESEARCH PAPER

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Abstract

Within two-sector growth models, with physical goods and human capital produced under distinct technologies, a process of knowledge obsolescence/depreciation that is similar to the depreciation process of physical goods is generally considered. As a consequence, the long-term rate of per capita growth of the main economic aggregates is constant over time. This rate can be endogenously determined (in endogenous growth models in which production is subject to constant returns), or it can be the result of exogenous forces, like technological progress or population dynamics (in neoclassical growth theory in which decreasing marginal returns prevail).

In this paper, we introduce a new assumption about the generation of knowledge, which involves the notion of saturation, i.e., introducing additional knowledge to generate more knowledge becomes counterproductive after a given point. This new assumption is explored in scenarios of neoclassical and endogenous growth, and it is able to justify endogenous fluctuations. Saturation in the creation of knowledge implies that human capital does not grow steadily over time. Instead, cycles of various periodicities are observable for different degrees of saturation. Complete a-periodicity (chaos) is also found for particular values of the saturation parameter. This behavior of the human capital

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variable spreads to the whole economy given that this input is used in the production of final goods and, thus, main economic aggregates time paths (i.e., the time paths of physical capital, consumption, and output) also evolve following a cyclical pattern. With this argument, we intend to give support to the view of endogenous business cycles in the growth process, which is an alternative to the two mainstream views on business cycles: the RBC theory and the Keynesian interpretation.

Keywords: growth theory, endogenous business cycles, non-linear dynamics and chaos, knowledge saturation
JEL classification: O41, E32, C61

1 Introduction[1]

Does the creation of knowledge involve positive returns independently of the already accumulated amount of this input? In this paper, we discuss the implications of a growth setting in which knowledge is subject to saturation and loss of quality after a given threshold. The main argument is that one cannot accumulate knowledge endlessly without incurring relevant informational problems. For instance, organizational behavior theory suggests that groups make better decisions than individuals. However, in large groups, communication mismatches may emerge and productivity might be lost. Too much knowledge can generate a scenario where the difficulty to discern relevant from less relevant knowledge becomes harmful from the point of view of the accumulation of input.

Our concept of knowledge saturation is intrinsically related to problems arising from the transmission of information, i.e., knowledge accumulates as ideas are transmitted from one individual to another. Larger information sets imply that the decoding of messages becomes more demanding, and the likelihood of arising difficulties associated to noise or entropy increase. To develop the notion of knowledge saturation, we relate it to the concept of entropy in information theory. The original definition of entropy, proposed by Shannon (1948), refers to the measure of uncertainty within some sort of system. Given various possible outcomes, a higher level of entropy implies that the probabilities assigned to each outcome become closer to each other; and in the case of maximum entropy, a complete impossibility of making rational choices arises (all outcomes are equally probable; and, thus, the choice is simply random). The original work of Shannon (1948) and Shannon and Weaver (1949) was developed in the context of the theory of information; and, thus, it became a fundamental tool in communication studies, in which entropy is thought of as the loss of information that occurs with the

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transmission of some message.[2] In a more current use of the term, entropy may be understood as the opposite of synergy, that is, the loss of productivity that occurs when people work together rather than working by themselves (as discussed above, this is an event that becomes likely once oversized working groups are formed).

Within an economic interpretation of this idea, one can associate entropy/saturation with the presence of negative marginal returns in the accumulation of a given input. In the present analysis, we associate this comprising notion of entropy with the production of knowledge. The argument is that for a specific type of knowledge, there are positive returns until a given level of this input is accumulated; but after a given point, additional knowledge is synonymous with a net loss, as the dissemination of such knowledge suffers from decreasing quality. In the same way that the excess of information implies a loss of quality in the transmission of a message, the excess of accumulated knowledge implies that the dissemination of this factor loses quality and part of the input is simply lost.

While our definition of saturation is somehow too inclusive and departs from Shannon's initial notion of entropy, the mathematical concept used here relies on the aforementioned formulation. As we shall see in the next section, the accumulation of the knowledge input is the result of a production process (a trivial production function is assumed), but the following saturation term is associated to the rule of knowledge evolution through time: $E_t = -\tilde{h}_t \cdot \ln \tilde{h}_t$, with $\tilde{h}_t$ as the knowledge variable.[3] This term implies that an accumulation rule characterized by decreasing or constant marginal returns becomes a hump-shaped function which replaces the conventional concave or linear shape.[4] Therefore, we

[2] *See, e.g., Heath and Bryant (2000).*

[3] *In the proposed expression, we consider the natural logarithm instead of the base 2 logarithm of Shannon's original analysis. As referred in Sato, Akiyama and Crutchfield (2004), this does not change the interpretation of the notion of entropy; it just changes the measure unit (entropy is evaluated in nats (natural digits) instead of bits (binary digits)).*

[4] *Later, in Section 2, we present a phase diagram (Figure 8) describing this dynamic rule, which effectively reveals the hump-shaped form.*

introduce a new way of understanding the creation of knowledge. In our view, the accumulation of knowledge is not only subject to depreciation and obsolescence; it is subject to saturation, since large-scale 'common knowledge' can destroy partially the true meaning of the originally generated ideas.

A hump-shaped accumulation rule implies the possibility of 'strange' dynamic behavior. It is known[5] that this type of function is able to produce more than just the simple, long-term results of fixed-point stability and instability. Namely, periodic and a-periodic motion, that is, cycles of different orders and chaos are observable for some combinations of parameter values. Under this reasoning, our work is close to the path breaking idea of Day (1982), who explained endogenous cycles within an economic growth framework by considering the pollution effect which implied that after a given level of accumulated physical capital, the stock of this input began to be destroyed. This is, indeed, similar to our argument, which states that too much knowledge introduces saturation in the dissemination of knowledge, provoking the destruction of part of the existing stock of this input.

Any hump-shaped function possesses the fundamental properties to potentially generate endogenous cycles. In growth analysis, this requires that the process of accumulation of a given input must be subject to an initial phase of positive returns and, then, to a second phase of negative returns. The contribution that this paper intends to offer relates to explaining why a specific production factor is eventually subject to such kind of behavior.

The knowledge input is introduced in neoclassical and endogenous growth models in order to explain endogenous business cycles. Optimal growth models are able to explain the long-term growth, but they fail to address the issue of economic fluctuations, unless we consider some departure from the Walrasian competitive market structure. In fact, by

[5] *See, e.g., May (1976).*

allowing that knowledge is subject to saturation, we are introducing a kind of inefficiency that is able to support the existence of cycles.

The literature on endogenous cycles was introduced by Medio (1979), Stutzer (1980), Benhabib and Day (1981), Day (1982), and Grandmont (1985) not only in the context of intertemporal optimal control models but also under overlapping generations frameworks. This work has gained some strength with the discovery that under increasing returns or a strong externality effect in production, the standard one-sector growth model, with the consideration of the labor-leisure trade-off, is able to generate endogenous fluctuations.[6] Despite the relevance and intuitive appeal of the literature on endogenous business cycles, it continues to be somehow marginal relative to the two main strands of thought about business cycles: the Real Business Cycles theory and the Keynesian view of incomplete markets and nominal rigidities.

On the other hand, it is possible to think about the proposed approach as a way to bridge the gap between the two main competing business cycles theories. The Keynesian feature of the model of development relates to market inefficiency: agents are unable to process information and to communicate in a way that the accumulation of inputs can always benefit growth. From the RBC models, we recover the dynamic general equilibrium approach. In contrast with these models, our approach does not require any external shock to trigger business fluctuations. This is an intrinsic feature of the framework involving knowledge saturation. In RBC theory, it is an exogenous stochastic shock (technology, economic policy, or energy prices) that generates the change in output, which is spread over subsequent time periods as the result of the way the labor market works.[7]

[6] *See, e.g., Christiano and Harrison (1999), Schmitt-Grohé (2000), Guo and Lansing (2002), and Coury and Wen (2005).*

[7] *See King and Rebelo (1999) and Rebelo (2005) for a thorough discussion on the implications of RBC theory.*

In this paper, we intend to contribute to the literature on endogenous cycles by introducing a new source of fluctuations, which is, as stated, the presence of saturation in the creation of knowledge. The new assumption is worked out in neoclassical and endogenous growth scenarios. These maintain their essential features in terms of the characteristics of the growth process but for selected parameter values, the trends of growth are replaced by more realistic fluctuations around those trends.

The remainder of the paper is organized as follows. Section 2 studies the dynamics of the accumulation of knowledge under saturation. Section 3 introduces the knowledge rule in a neoclassical growth framework, and Section 4 considers the same rule in an endogenous growth setup. Finally, Section 5 presents a short conclusion.

2 Knowledge Saturation

We consider $\widetilde{h}_t$ to be a non-rival knowledge variable necessary to produce human capital. This variable is integrated into growth setups within the next two sections. In this section, we define and study a rule that characterizes knowledge movement over time. A production function for knowledge is assumed to be $f(\widetilde{h}_t) = \widetilde{B}\widetilde{h}_t^{\eta}$, where $\widetilde{B} > 0$ and $\eta>0$. We leave open the possibility of decreasing, constant, or increasing marginal returns in the generation of knowledge. If one considers that this type of knowledge is subject to a conventional process of depreciation/ obsolescence, then $\eta<1$ implies that $\widetilde{h}_t$ converges to a constant value (zero growth); $\eta=1$ indicates a positive constant growth for the long-term outcome, and $\eta>1$ is associated with an unstable outcome.

Instead of a simple depreciation/obsolescence process, we assume that the knowledge variable is subject to saturation. Therefore, we assume the following rule for the accumulation of knowledge over time,

$$\widetilde{h}_{t+1} - \widetilde{h}_t = \widetilde{B}\widetilde{h}_t^{\eta} - \widetilde{\delta}\widetilde{h}_t \ln \widetilde{h}_t, \quad \widetilde{h}_0 \text{ given} \tag{1}$$

where $\widetilde{\delta}$ is the saturation parameter. Through saturation, we have introduced a nonlinear component in the process of accumulation of knowledge, which will have a significant impact on the long-term behavior of the endogenous variable. The dynamics of Equation (1) can be studied locally or globally. We begin with a note about local dynamics.

Consider first the case where $\eta=1$. The constant returns case is highlighted separately because it is the only one that allows for a straightforward computation of the steady-state value and for an explicit stability result. In the presence of constant returns, the absence of saturation ($\widetilde{\delta}=0$) would mean that $\widetilde{h}_t$ would not assume a constant steady-state value. Instead, the variable would grow at a constant positive rate $\widetilde{B}$. When one introduces saturation in the process of knowledge creation, one will observe that the saturation term implies an inefficiency that may transform, for some parameter values, the constant growth process into a process of zero growth (and, thus, a constant equilibrium value for $\widetilde{h}_t$) or even into a process of periodic or a-periodic motion.[8] Nevertheless, this last possible outcome is not captured by a local analysis. Concerning local stability, Proposition 1 synthesizes the dynamic nature of Equation (1).

Proposition 1. *The knowledge accumulation difference equation with saturation and constant marginal returns has a unique steady-state point. This point is stable for* $\widetilde{\delta}<2$*; instability prevails for* $\widetilde{\delta}>2$*; and* $\widetilde{\delta}=2$ *corresponds to a bifurcation point.*

[8] *Saturation has the fundamental implication for a constant long-term level of knowledge (or for a cyclical process where knowledge is bounded around a constant mean). Otherwise, this variable can accumulate endlessly at a constant rate. This essentially signifies that the saturation hypothesis removes knowledge from the list of possible candidates for explaining persistent long-term growth (which is a fundamental feature of both neoclassical and endogenous growth models. In neoclassical growth models, the sources of growth are exogenous but they, nevertheless, exist - namely, the factors that can be included in the broad notion of total factor productivity). If knowledge does not play such role, one needs to associate persistent growth to other entities. This will be done in the next sections. For now, the main argument is that knowledge saturation can be the source of fluctuations but not the source of long-term sustained growth.*

Proof: Let $G(\widetilde{h}_t)=(1+\widetilde{B})\cdot\widetilde{h}_t-\widetilde{\delta}\widetilde{h}_t\ln\widetilde{h}_t$. The steady-state value of $\widetilde{h}_t$ is found by solving $G(\widetilde{h}_t)=\widetilde{h}_t$. In a straightforward way, one finds $\widetilde{h}^*=e^{\widetilde{B}/\widetilde{\delta}}$ to be the unique steady-state point. To inquire into what kind of stability underlies $\widetilde{h}^*$, one computes the derivative $\partial G(\widetilde{h}_t)/\partial\widetilde{h}_t\big|_{\widetilde{h}_t=\widetilde{h}^*}=1-\widetilde{\delta}$. Considering $\widetilde{\delta}>0$ (that is, that positive saturation exists) we just have to impose $1-\widetilde{\delta}>-1$ to guarantee that the derivative lies inside the unit circle. As a consequence, stability requires $\widetilde{\delta}<2$. Conversely, $\widetilde{\delta}>2$ implies instability. The value $\widetilde{\delta}=2$ indicates a point of transition between stable and unstable areas, and, thus, corresponds to a point of bifurcation. ■

Allowing for a parameter η different from 1, we cannot find an explicit steady-state value for $\widetilde{h}_t$; the solution for $\widetilde{h}^*:\widetilde{h}_t^{1-\eta}\ln\widetilde{h}_t=\widetilde{B}/\widetilde{\delta}$. The stability result is given in Proposition 2.

Proposition 2. *The knowledge accumulation difference equation with saturation is stable for* $\widetilde{h}^*<e^{\frac{2-\widetilde{\delta}}{(1-\eta)\cdot\widetilde{\delta}}}$*, unstable for* $\widetilde{h}^*>e^{\frac{2-\widetilde{\delta}}{(1-\eta)\cdot\widetilde{\delta}}}$*, and* $\widetilde{h}^*=e^{\frac{2-\widetilde{\delta}}{(1-\eta)\cdot\widetilde{\delta}}}$ *corresponds to a bifurcation point. These results apply in two circumstances:* ($\eta<1$; $\widetilde{\delta}<2$) *or* ($\eta>1$; $\widetilde{\delta}>2$).

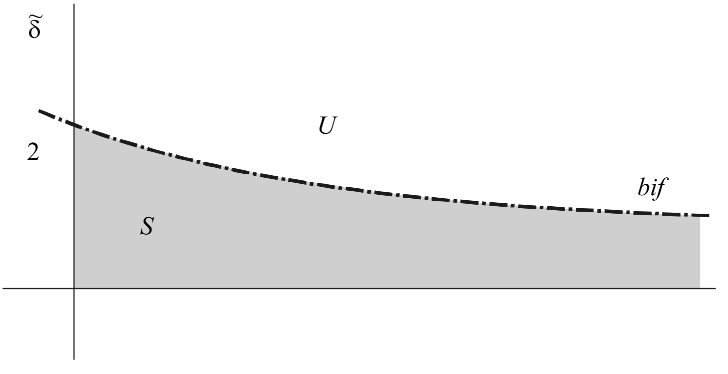
Proof: Consider a generic G function, including the possibilities of decreasing, constant, and increasing returns: $G(\widetilde{h}_t)=\widetilde{B}\widetilde{h}_t^{\eta}+\widetilde{h}_t-\widetilde{\delta}\widetilde{h}_t\ln\widetilde{h}_t$. Computing the derivative, $\partial G(\widetilde{h}_t)/\partial\widetilde{h}_t\big|_{\widetilde{h}_t=\widetilde{h}^*}=1-\widetilde{\delta}-(1-\eta)\cdot\widetilde{\delta}\ln\widetilde{h}^*$. Note that in the present case, we have to consider $\widetilde{h}^*>1$ in order to guarantee that $\ln\widetilde{h}^*>0$, a condition that is required for a reasonable steady-state result (which corresponds to a positive value of the knowledge variable). Note also that the case studied in Proposition 1 is just a particular case of this equation here and, thus, assuming $\eta=1$, the above derivative reduces to the one in the precedent analysis.

Stability requires $1-\widetilde{\delta}-(1-\eta)\cdot\widetilde{\delta}\ln\widetilde{h}^*>-1$, which implies $\widetilde{h}^*<e^{\frac{2-\widetilde{\delta}}{(1-\eta)\cdot\widetilde{\delta}}}$. Instability is to be given by the symmetric condition, $\widetilde{h}^*>e^{\frac{2-\widetilde{\delta}}{(1-\eta)\cdot\widetilde{\delta}}}$; and the bifurcation result corresponds to the border case $\widetilde{h}^*=e^{\frac{2-\widetilde{\delta}}{(1-\eta)\cdot\widetilde{\delta}}}$.

Finally, consider an important point: since we have established that $\widetilde{h}^*$ must be larger than 1, then expression $\frac{2-\widetilde{\delta}}{(1-\eta)\cdot\widetilde{\delta}}$ must be positive. Thus, only the two cases mentioned in the proposition, (i) $\eta<1$; $\widetilde{\delta}<2$ and (ii) $\eta>1$; $\widetilde{\delta}>2$, are effectively relevant in stating the presented result. ■

One can further explore the local properties of the model by studying in more detail the nature of the bifurcation point. This can be defined as the point in which the following combination of parameters holds, $\widetilde{B}=e^{\frac{2-\widetilde{\delta}}{\widetilde{\delta}}}\cdot\frac{2-\widetilde{\delta}}{1-\eta}$. Assuming a constant value for one of the parameters, one can draw in the space of the other parameters a bifurcation line that divides the areas of stability and instability. Consider, as an example, that $\eta=0.5$. Figure 1 displays the bifurcation line and the regions of stability/instability in the space ($\widetilde{B},\widetilde{\delta}$).

Figure 1 **Areas of Stability/Instability** *(η=0.5)*



In Figure 1, the region of stability (S) is located below the bifurcation line (*bif*), while the region of instability (U) is located above this line. To generalize, one presents the same graph as in Figure 1: now highlighting different bifurcation lines for different possibilities regarding the value of η (Figure 2).

Figure 2 **Bifurcation Lines for Different Values of** η

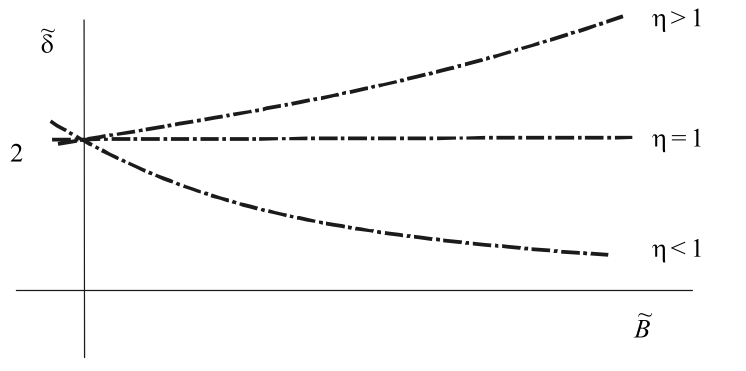


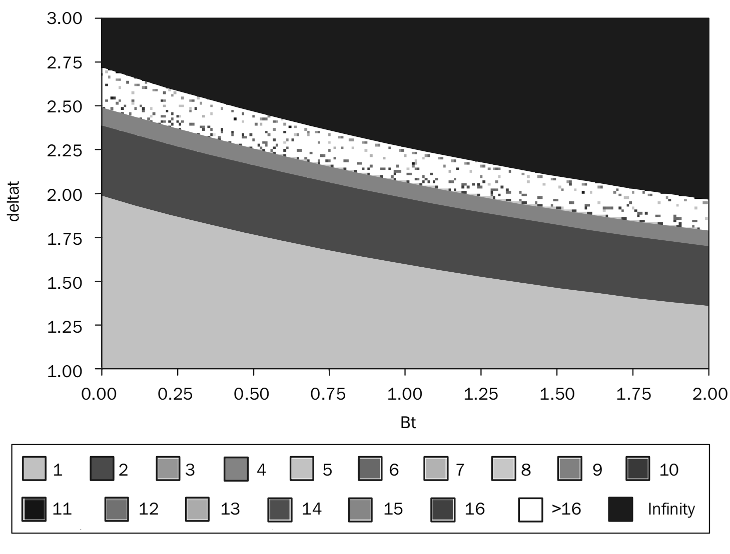
Figure 2 is illustrative of the nature of the bifurcation. For constant returns in the accumulation of knowledge, the bifurcation is not dependent on $\tilde{B}$ (only on $\tilde{\delta}$). Decreasing and increasing returns mean different slopes of the bifurcation line. In the first case, this is a negatively sloped curve in the space ($\tilde{B},\tilde{\delta}$), and it becomes positively sloped for $\eta>1$. In either case, the region of stability is located below the line.

The local stability analysis of Equation (1) has allowed us to understand how saturation changes the dynamics of a simple accumulation process. Without saturation, a stability result ($\eta<1$) or an instability outcome ($\eta<1$) would characterize the dynamics of (1) independently of the value of other parameters (in particular, $\tilde{B}$). With saturation, independently of assuming decreasing, constant, or increasing returns, a bifurcation that separates regions of stability and instability is always identified. The bifurcation does not depend on the productivity parameter $\tilde{B}$ only in the special case of constant marginal returns. Therefore, except for $\eta=1$, the value of $\tilde{B}$ is decisive for the stability result that is obtained, alongside with the value of the saturation parameter. When $\eta=1$, this is, indeed, a special case because stability is determined only by the fact that $\tilde{\delta}$ is above or below 2.

Local analysis gives important guidance about the stability properties of the model. Nevertheless, this analysis cannot capture some fundamental features of the dynamics of (1). To be precise, one has to engage in a global analysis, which would give evidence that the unstable region does not imply necessarily a divergence towards infinity. Global dynamics can only be understood through numerical examples and are better revealed if analyzed graphically. The following set of figures allows for a thorough understanding of the dynamics of Equation (1).[9]

We begin by presenting the areas of stability and instability identified in the local analysis, from the point of view of a global analysis. Three figures are drawn for different values of η (Figure 3 for η=0.5, Figure 4 for η=1, and Figure 5 for η=1.5). Concerning the stability area, the local analysis results are confirmed, but the unstable region does not translate immediately into a divergence towards infinity. After the bifurcation, the system undergoes a phase of periodic cycles of doubling order and chaos before arriving to the divergence result.

Figure 3 **Global Dynamics in the Parameters Space** *(η=0.5)*



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[9] *The various figures relating to global analysis are drawn using IDMC software (Interactive Dynamical Model Calculator). This is a free software program available at http://www.dss.uniud.it/nonlinear, and it is the copyright of Marji Lines and Alfredo Medio.*

Figure 4 **Global Dynamics in the Parameters Space** *(η=1)*

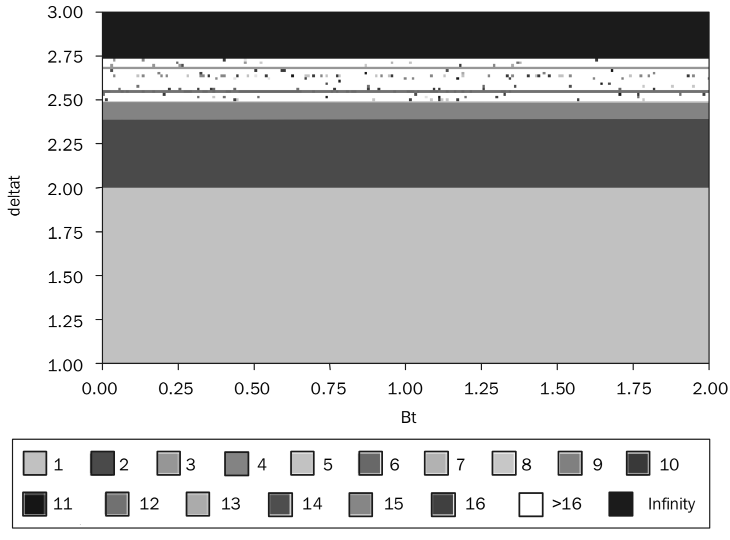
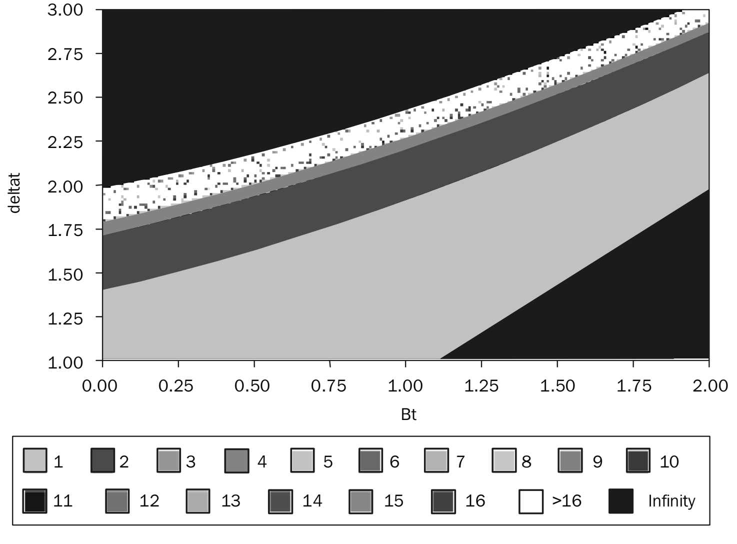


Figure 5 **Global Dynamics in the Parameters Space** *(η=1.5)*



With the global analysis, we expand the possibility of long-term outcomes for the time evolution of the knowledge variable. The knowledge variable can be subject to endogenous fluctuations as a result of introducing saturation (recall once again that without saturation we would have stability under decreasing returns and instability otherwise).

To better understand the properties of (1), mainly in the regions where cycles of various orders arise, a set of other graphical representations are shown. Figures 6 and 7 present bifurcation diagrams. In both figures, relating to different parameters, it is clear the period doubling route to chaos, which occurs in the areas identified in previous figures. Figures 8 and 9 use the same set of parameter values to characterize a situation of chaos. In Figure 8, we draw a phase diagram, where chaotic motion is clearly identified as a result of the hump-shaped form of the relation between the knowledge variable in two consecutive time moments. Figure 9 displays the time series of the knowledge variable in the long-term (after 1,000 observations).

Figure 6 **Bifurcation Diagram** $(\widetilde{h}_t, \widetilde{\delta})$ $(\widetilde{B} = 1; \eta = 0.5)$

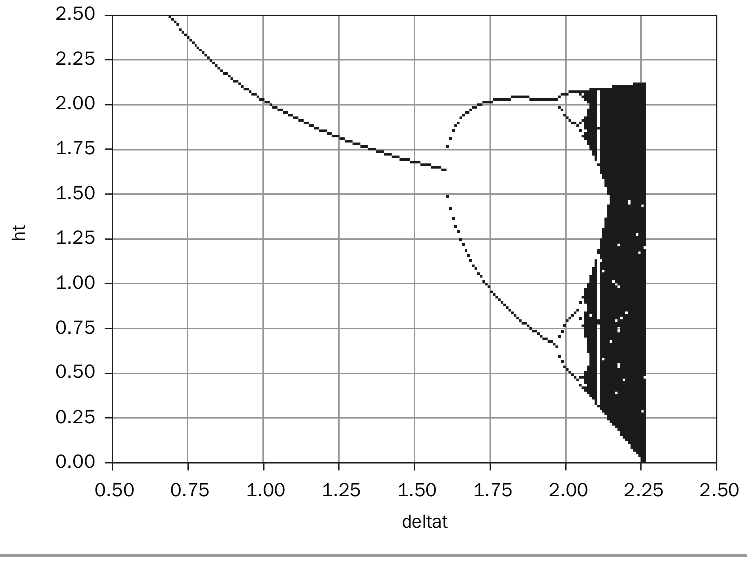


Figure 7 **Bifurcation Diagram** $(\widetilde{h}_t, \widetilde{B})$ $(\widetilde{\delta} = 2.7; \eta = 1.25)$

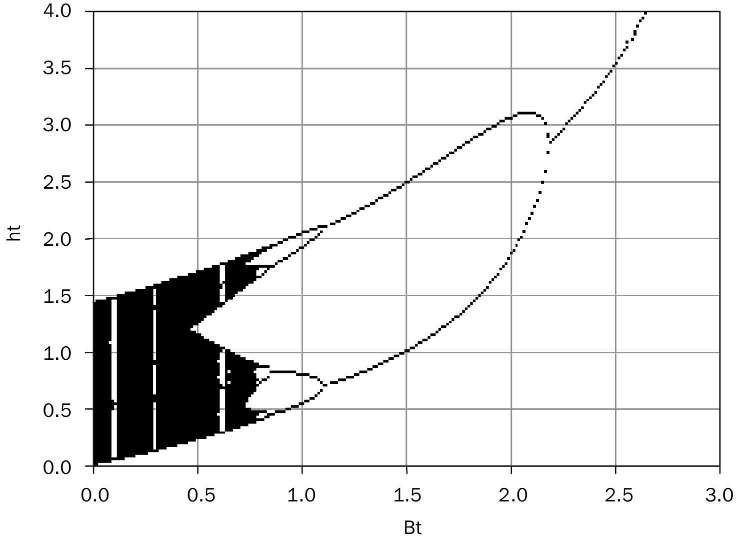


Figure 8 **Phase Diagram** $(\widetilde{B} = 0.5; \widetilde{\delta} = 2.7; \eta = 1.25)$

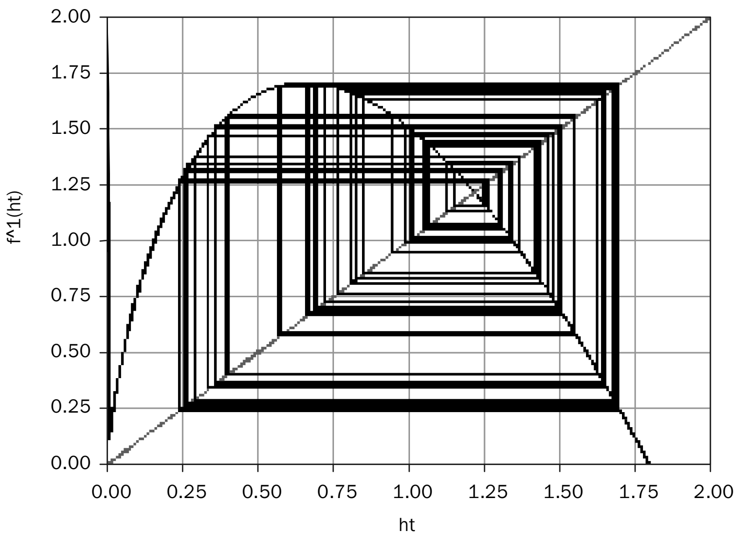
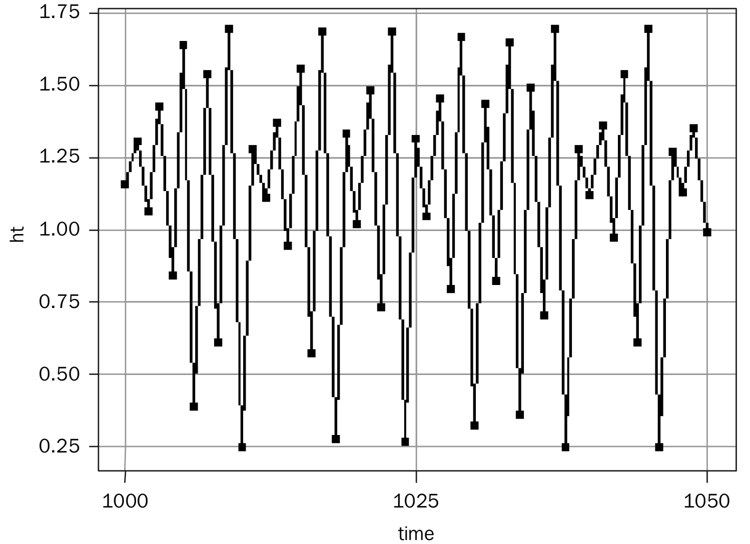
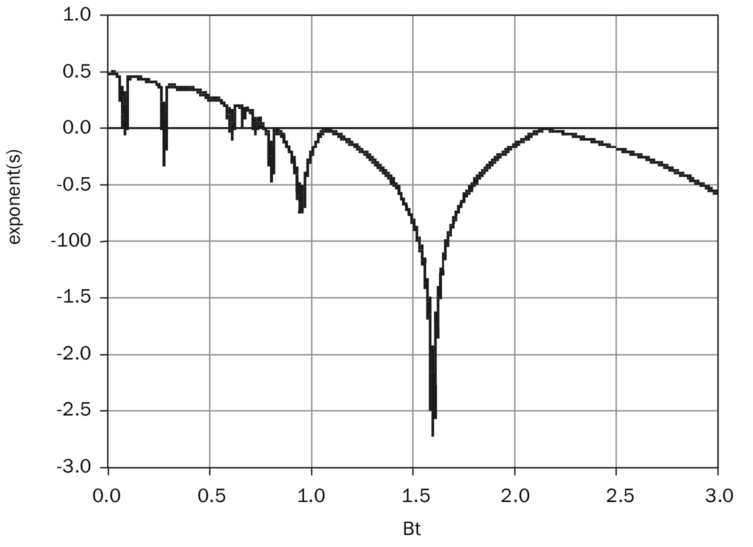


Figure 9 **Long-term Time Path** $(\widetilde{B}=0.5;\widetilde{\delta}=2.7;\eta=1.25)$



Finally, Figure 10 computes a measure for the evaluation of the existence of chaos. Lyapunov characteristic exponents (LCEs) measure the exponential divergence of nearby orbits: a positive LCE is synonymous with the divergence of nearby orbits or sensitive dependence on initial conditions, a phenomenon that is generally identified with chaotic motion. In this figure, we find a positive Lyapunov exponent for the same values of parameter $\widetilde{B}$ for which we have identified before the presence of no fixed-point or any kind of periodic cycles (this figure can be compared with Figure 7).

Figure 10 **Lyapunov Characteristic Exponents** $(\widetilde{h}_t, \widetilde{B})$ $(\widetilde{\delta} = 2.7; \eta = 1.25)$



3 Neoclassical Growth

Assume that knowledge variable $\widetilde{h}_t$ is an input into the production of human capital.[10] The human capital per capita, h_t, evolves over time according to accumulation rule (2),

$$h_{t+1} - h_t = B \cdot \left[(1-v) \cdot h_t\right]^{\theta} \cdot \widetilde{h}_t^{\zeta} - \delta h_t, \quad h_0 \text{ given} \tag{2}$$

In (2), $B>0$ is a productivity index; v is the share of human capital used in the production of physical goods (and, thus, 1-v represents the share of human capital associated with the production of this form of capital); and $\delta>0$ is a depreciation rate. Two particular cases of Equation (2) are studied. In this section, we concentrate on the absence of long-term

[10] *Here, we make the distinction between disembodied knowledge* ($\widetilde{h}_t$) *and a skills variable directly associated with labour productivity. Skills improve through a learning process involving already accumulated skills and the use of the common knowledge input which is subject to saturation.*

positive endogenous growth (the growth process has neoclassical features), while in Section 4 an endogenous growth setup is assumed. The difference in analysis is determined by the elasticity parameters θ and ζ.

In this section, we take constant returns to scale in the production of human capital ($0<\theta<1$ and $\zeta=1-\theta$). For the circumstances described in Section 2 in which $\widetilde{h}_t$ had a long-term stability solution (fixed-point or periodic or a-periodic motion around the steady-state point), the model displays neoclassical features (in the sense that there is not a process of sustained positive growth that is endogenously determined).[11] Endogenous variables tend to long-term constant values or they converge to a long-term position where endogenous fluctuations around a constant mean persist over time.

The endogenous growth model of the next section considers the knowledge variable as an externality over a constant returns equation of human capital accumulation ($\theta=1$ and $\zeta>0$). In this case, the growth problem exhibits a long-term constant growth rate (for $\widetilde{h}_t$ converging to a fixed point), or a long-term scenario with growth cycles; i.e., economic aggregates grow at an average constant rate, but endogenous fluctuations characterize the motion of the growth rate (and not only the motion of the capital and consumption aggregates themselves).

Let us concentrate for now on the case in which $\theta\in(0,1)$ and $\zeta=1-\theta$. Consider a standard utility maximization intertemporal problem under an infinite horizon and a discount factor $\beta<1$,

$$\underset{c}{Max}\sum_{t=0}^{+\infty}U(c_t)\cdot\beta^t \tag{3}$$

[11] Sustained growth can only be justified under the growth of some other, exogenous, component of the model (e.g., the productivity indexes associated to the production of human capital or physical capital).

The utility function is assumed under a simple concave form, $U(c_t) = \ln c_t$, where c_t stands for per capita consumption, and problem (3) is subject to three constraints: these are the knowledge equation in (1), the human capital equation in (2), and the third is a physical capital accumulation constraint, with a Cobb-Douglas production function that exhibits constant returns to scale,

$$k_{t+1} - k_t = Ak_t^{\alpha} \cdot (vh_t)^{1-\alpha} - c_t - \delta k_t, \quad k_0 \text{ given} \tag{4}$$

The physical capital variable, k_t, is a per capita variable; parameter $A>0$ is the productivity index in the final goods sector; $\alpha \in (0,1)$ represents the output – physical capital elasticity; and $\delta>0$ is the depreciation rate (that, for the sake of simplicity, is considered the same as in the human capital constraint).

Solving the optimal control problem (3) subject to (1), (2), and (4), we find, after the computation of optimality conditions, the following equation,

$$c_t = \beta c_{t-1} \cdot \left[1 - \delta + \alpha A \cdot \left(\frac{vh_t}{k_t} \right)^{1-\alpha} \right] \tag{5}$$

In order to simplify the dynamic analysis, we make the following assumption: the initial level of consumption chosen by the representative agent is already the steady state level, $c_0 = c^*$. Under this assumption, one can establish, through (5), a linear relation between the capital variables, which is,

$$k_t = v \cdot \left[\frac{\alpha A}{1/\beta - (1-\delta)} \right]^{1/(1-\alpha)} \cdot h_t \tag{6}$$

In Section 2, one has observed that an explicit equilibrium value of $\tilde{h}_t$ is attainable only for $\eta=1$. With this parameter value, the computation of steady-state values for our various variables is straightforward. The following results are obtained:

$$h^* = \left[\frac{B}{\delta} \cdot (1-v)^{\theta} \cdot e^{\left[(1-\theta)\cdot \widetilde{B}/\widetilde{\delta}\right]} \right]^{1/(1-\theta)} \tag{7}$$

$$k^* = v \cdot \left[\frac{\alpha A}{1/\beta - (1-\delta)} \right]^{1/(1-\alpha)} \cdot \left[\frac{B}{\delta} \cdot (1-v)^{\theta} \cdot e^{\left[(1-\theta)\cdot \widetilde{B}/\widetilde{\delta}\right]} \right]^{1/(1-\theta)} \tag{8}$$

$$\begin{aligned} c^* = A^{1/(1-\alpha)} \cdot & \left\{ \left[\frac{\alpha}{1/\beta - (1-\delta)} \right]^{\alpha/(1-\alpha)} - \delta \cdot \left[\frac{\alpha}{1/\beta - (1-\delta)} \right]^{1/(1-\alpha)} \right\} \\ & \cdot \left[\frac{B}{\delta} \cdot (1-v)^{\theta} \cdot e^{\left[(1-\theta)\cdot \widetilde{B}/\widetilde{\delta}\right]} \right]^{1/(1-\theta)} \end{aligned} \tag{9}$$

In expressions (7) to (9), some meaningful results are easy to identify. For instance, technology (A and B) contributes to higher, steady-state accumulated quantities of both forms of capital, while larger depreciation implies a fall in accumulated capital and in consumption.

The long-term outcomes of capital (physical and human) are determined by the behavior of the knowledge variable, $\widetilde{h}_t$. We have seen, in Section 2, that such behavior is directly influenced by the values of $\widetilde{\delta}, \widetilde{B}$ and η. Therefore, steady-state results (7) and (8) are not accomplished in every case. To illustrate a few possible equilibrium results, we draw the evolution of output in the long-term, under different combinations of parameter values (Figures 11 and 12). We just mention the values of the parameters in the knowledge equation, given that the others (A, B, θ, v, δ, α and β) are not relevant from a qualitative point of view. We take $\widetilde{B} = 1, \widetilde{\delta} = 2.25$ and $\eta = 0.5$ (Figure 11) and $\widetilde{B} = 0.5, \widetilde{\delta} = 2.7$ and $\eta = 1.25$ (Figure 12). The output corresponds to the income generated by the final goods production function, that is,

$$y_t = Ak_t^{\alpha} \cdot (vh_t)^{1-\alpha} = A^{1/(1-\alpha)} \cdot v \cdot \left[\frac{\alpha}{1/\beta - (1-\delta)} \right]^{\alpha/(1-\alpha)} \cdot h_t \cdot$$

Figure 11 **Long-term Time Path of the Output Variable in the Neoclassical Model** $(\tilde{B}=1;\tilde{\delta}=2.25;\eta=0.5)$

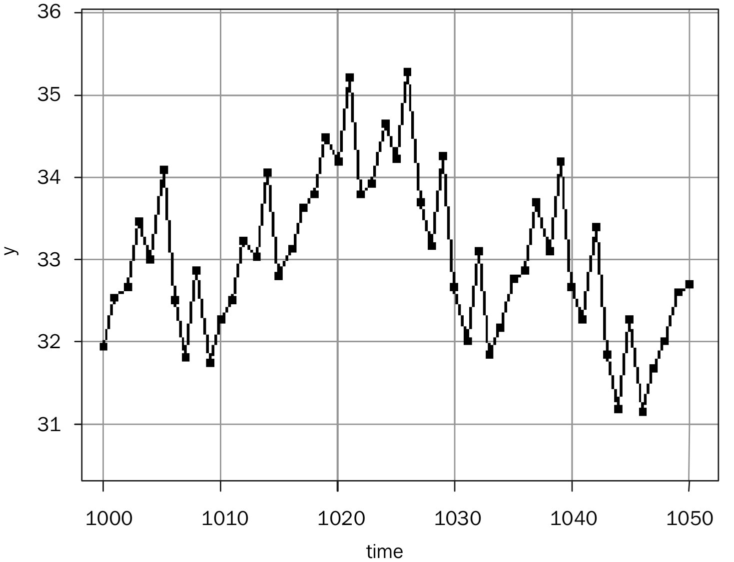
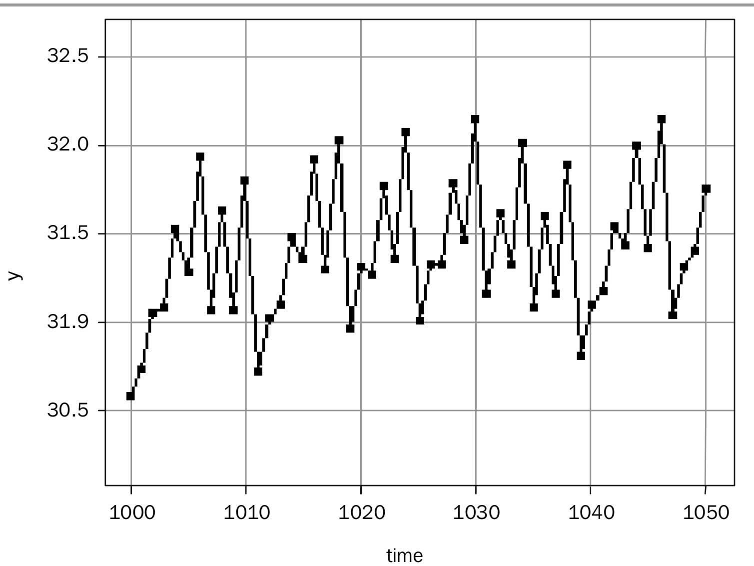


Figure 12 **Long-term Time Path of the Output Variable in the Neoclassical Model** $(\tilde{B}=0.5;\tilde{\delta}=2.7;\eta=1.25)$



We conclude that in a neoclassical growth model with saturation in the creation of knowledge, endogenous business cycles characterizing the time evolution of output, emerge under some circumstances which define the process of knowledge accumulation.

4 Endogenous Growth

The model in Section 3 presented neoclassical features in the sense that economic aggregates displayed zero average growth as a long-term solution. Now, considering $\theta=1$ and $\zeta>0$, the knowledge variable is introduced in the growth model as a positive externality over the accumulation of human capital, which is subject to constant marginal returns technology. Therefore, the model has endogenous growth features, meaning that capital and output grow at a positive (constant on average) growth rate. With the new assumption, the inclusion of the knowledge variable implies endogenous fluctuations in the growth rates.

It is the specific form of the final goods production function (Cobb-Douglas) and the type of considered capital accumulation constraint that allow for the automatic establishment of a correspondence between the growth rates of human capital, physical capital, and output.

Consider the same problem as in Section 3, so that Equation (5) is once again result of first order conditions. Here, we define variables that do not grow in the long-term. These are, following conventional endogenous growth analysis (Barro and Sala-i-Martin, 1995), the consumption – physical capital ratio, $\psi_t \equiv c_t / k_t$, and the physical capital – human capital ratio, $\omega_t \equiv k_t / h_t$. From (2), (4), and (5), we obtain,

$$\psi_{t+1} = \frac{\beta \cdot \left[1-\delta+\alpha A \cdot \left(v/\omega_{t+1}\right)^{1-\alpha}\right]}{A \cdot \left(v/\omega_t\right)^{1-\alpha} - \psi_t + (1-\delta)} \cdot \psi_t \tag{10}$$

$$\omega_{t+1} = \frac{A \cdot \left(v/\omega_t\right)^{1-\alpha} - \psi_t + (1-\delta)}{B \cdot (1-v) \cdot \widetilde{h}_t^{\zeta} + (1-\delta)} \cdot \omega_t \tag{11}$$

Steady-state values ψ^* and ω^*, which can be determined from (10) and (11), are constant values for the constant equilibrium value of knowledge, $\widetilde{h}^*$. Once again, to obtain long-term time trajectories that are, on average, constant, we use a simplified assumption regarding consumption, which in this case is $\psi_0 = \psi^*$. To understand the dynamics of the capital ratio, we present Figure 13, for $\widetilde{B} = 0.5, \widetilde{\delta} = 2.7$ and $\eta = 1.25$. For these values, we know that chaotic motion is present.

Figure 13 **Long-term Time Path of the Capital Ratio in the Endogenous Growth Model** $(\widetilde{B} = 0.5; \widetilde{\delta} = 2.7; \eta = 1.25)$

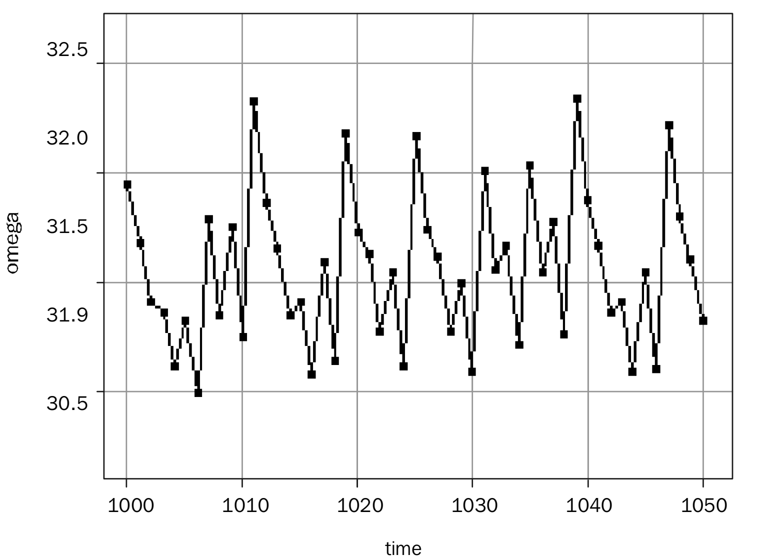


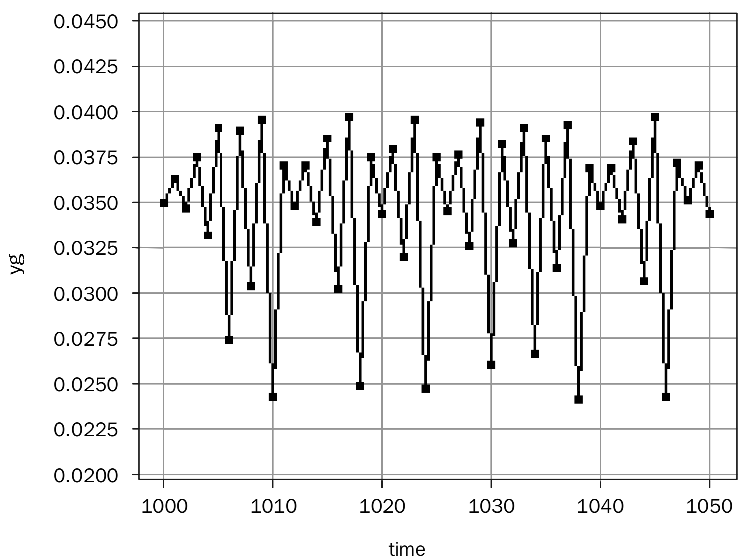
Figure 13 is drawn for a capital ratio. Each one of the capital variables, and also the per capita output, grow at a positive rate (that on average is constant). Figure 14 illustrates precisely the endogenous growth character of the model by representing, for the same set of parameter values, the growth rate of the income variable,

$$\frac{y_{t+1}-y_t}{y_t}=\left(\frac{\omega_{t+1}}{\omega_t}\right)^{\alpha}\cdot\frac{h_{t+1}}{h_t}-1=$$

$$=\left[A\cdot(v/\omega_t)^{1-\alpha}-\psi^{*}+(1-\delta)\right]^{\alpha}\cdot\left[B\cdot(1-v)\cdot\widetilde{h}_t^{\zeta}+(1-\delta)\right]^{1-\alpha}-1$$

Looking at Figure 14, we understand the relevance of the eventual presence of saturation in the generation of knowledge. This might lead to endogenous growth cycles that, nevertheless, do not disturb the positive growth trend (thus, making the endogenous growth paradigm more realistic).

Figure 14 **Long-term Time Path of the Output Growth Rate in the Endogenous Growth Model** $(\widetilde{B}=0.5;\widetilde{\delta}=2.7;\eta=1.25)$



The analysis, concerning both neoclassical and endogenous growth, has focused essentially on long-term results. Underlying these results is a stable transitional dynamics process (convergence to the steady state) associated to capital accumulation. The single force capable of disturbing the fixed-point stability result is attached to the knowledge accumulation equation. If this implies convergence to cyclical motion, this will spread to the behavior of the other macro variables.

5 Final Remarks

We have assumed the existence of a knowledge variable with special features. Knowledge is generated through an accumulation process, but it is also subject to saturation. Larger quantities of this input imply, after some point, that the accumulated amount of knowledge begins to decline. The impact of this process of knowledge accumulation on the growth of the main economic aggregates depends on the way this variable is linked with the generation of human capital.

If the knowledge variable is included in a human capital production function with constant returns to scale, the growth model can be interpreted as a neoclassical growth setup. Capital and output grow at a long-term endogenous zero rate (on average), that is, one can present long-term time series for the economic variables that have a constant mean. These time series are not necessarily constant over time. For some parameter values, periodic and chaotic cycles are obtained. Under this setup, endogenous cycles can coexist with neoclassical growth.

If the knowledge variable emerges as an externality in the production of human capital, the endogenous growth attributes of the original growth model are maintained (in the sense that capital and output grow at positive rates in the long-term). These rates are constant over time for some parameter values, but for others they fluctuate around a constant value. Thus, in the case of endogenous growth, saturation in knowledge creation implies endogenous cycles which characterize the growth rates of capital and output.

In synthesis, knowledge saturation can be understood as a source of endogenous business cycles, and it was introduced in growth models without changing the fundamental properties of the growth process, which remain, respectively, neoclassical or endogenous.

Literature

Barro, Robert J. and Xavier Sala-i-Martin, 1995, *Economic Growth*, New York, NY: McGraw-Hill.

Benhabib, Jess and Richard H. Day, 1981, “Rational Choice and Erratic Behaviour”, *Review of Economic Studies*, 48(3), pp. 459-471.

Christiano, Lawrence and Sharon G. Harrison, 1999, “Chaos, Sunspots and Automatic Stabilizers”, *Journal of Monetary Economics*, 44, pp. 3-31.

Coury, Tarek and Yi Wen, 2005, “Global Indeterminacy and Chaos in Standard RBC Models”, University of Oxford and Cornell University Working Paper.

Day, Richard H., 1982, “Irregular Growth Cycles”, *American Economic Review*, 72, pp. 406-414.

Grandmont, Jean-Michel, 1985, “On Endogenous Competitive Business Cycles”, *Econometrica*, 53(5), pp. 995-1045.

Guo, Jang-Ting and Kevin J. Lansing, 2002, “Fiscal Policy, Increasing Returns and Endogenous Fluctuations”, *Macroeconomic Dynamics*, 6, pp. 633-664.

Heath, Robert L. and Jennings Bryant, 2000, *Human Communication Theory and Research: Concepts, Contexts and Challenges*, New Jersey; NJ: Lawrence Erlbaum Associates.

King, Robert G. and Sergio Rebelo, 1999, “Resuscitating Real Business Cycles” in John B. Taylor and Michael Woodford, eds., *Handbook of Macroeconomics*, vol. 1B, pp. 928-1002.

May, Robert M., 1976, “Simple Mathematical Models with Very Complicated Dynamics”, *Nature*, 261, pp. 459-467.

Medio, Alfredo, 1979, *Teoria Nonlineare del Ciclo Economico*, Bologna: Il Mulino.

Rebelo, Sergio, 2005, "Real Business Cycle Models: Past, Present and Future", *Scandinavian Journal of Economics*, 107, pp. 217-238.

Sato, Yuzuru, Eizo Akiyama and James P. Crutchfield, 2004, "Stability and Diversity in Collective Adaptation", Santa Fe Institute Working Paper, No. 04-08-025.

Schmitt-Grohé, Stephanie, 2000, "Endogenous Business Cycles and the Dynamics of Output, Hours, and Consumption", *American Economic Review*, 90, pp. 1136-1159.

Shannon, Claude E., 1948, "A Mathematical Theory of Communication", *The Bell System Technical Journal*, 27, pp. 379-423 and pp. 623-656.

Shannon, Claude E. and Warren Weaver, 1949, *The Mathematical Theory of Communication,* Urbana, IL: University of Illinois Press.

Stutzer, Michael J., 1980, "Chaotic Dynamics and Bifurcations in a Macro-Model", *Journal of Economic Dynamics and Control*, 2, pp. 353-376.