MATHEMATICAL COMMUNICATIONS 1
Math. Commun., Vol. **14**, No. 1, pp. 1-6 (2009)

Generalization of absolute Cesàro summability factors

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Received March 24, 2008; accepted August 1, 2008

Abstract. In the present paper, a general theorem concerning $\varphi - |C,1|_k$ summability factors of infinite series under weaker conditions, has been proved.

AMS subject classifications: 40F05, 40D15, 40D25

Key words: absolute Cesàro summability, infinite series, summability factors

1. Introduction

Let (φ_n) be a sequence of positive real numbers and let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). By (t_n) we denote the n-th $(C,1)$ means of the sequence (na_n). The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty, \tag{1}$$

and it is said to be summable $\varphi - |C,1|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n|^k < \infty. \tag{2}$$

If we take $\varphi = n$, then $\varphi - |C,1|_k$ summability reduces to $|C,1|_k$ summability.

Mazhar [3] proved the following theorem for $|C,1|_k$ summability.

Theorem 1. *If*

$$\lambda_m = O(1), \ \textit{as } m \to \infty, \tag{3}$$

$$\sum_{n=1}^{m} n \log n \left|\Delta^2 \lambda_n\right| = O(1), \tag{4}$$

$$\sum_{\nu=1}^{m} \frac{|t_\nu|^k}{\nu} = O(\log m), \ \textit{as } m \to \infty, \tag{5}$$

then the series $\sum a_n \lambda_n$ *is summable* $|C,1|_k$, $k \geq 1$.

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Quite recently Özarslan [5] generalized the above Theorem 1 for $\varphi - |C,1|_k$ summability in the following form.

Theorem 2. *Let* (φ_n) *be a sequence of positive real numbers and conditions* (3) *and* (4) *of Theorem 1 are satisfied. If*

$$\sum_{\nu=1}^{m} \frac{\varphi_\nu^{k-1}}{\nu^k} |t_\nu|^k = O(\log m), \text{ as } m \to \infty, \tag{6}$$

$$\sum_{n=\nu}^{m} \frac{\varphi_n^{k-1}}{n^{k+1}} = O\left(\frac{\varphi_\nu^{k-1}}{\nu^k}\right), \tag{7}$$

then the series $\sum a_n \lambda_n$ *is summable* $\varphi - |C,1|_k$, $k \geq 1$.

It should be noted that if we take $\varphi_n = n$ in Theorem 2, then condition (6) reduces to condition (5) and condition (7) reduces to

$$\sum_{n=\nu}^{m} \frac{1}{n^2} = O\left(\frac{1}{\nu}\right), \tag{8}$$

which always holds.

2. The main result

The aim of this paper is to generalize Theorem 2 under weaker conditions. Now we shall prove the following theorem.

Theorem 3. *Let* (φ_n) *be a sequence of positive real numbers and condition* (3) *of Theorem 1 and condition* (7) *of Theorem 2 are satisfied. Let* (X_n) *be a positive non-decreasing sequence and* (λ_n) *a sequence such that*

$$|\lambda_n| X_n = O(1), \text{ as } n \to \infty, \tag{9}$$

$$\sum_{n=1}^{m} n \left|\Delta^2 \lambda_n\right| X_n = O(1), \tag{10}$$

$$\sum_{\nu=1}^{m} \frac{\varphi_\nu^{k-1}}{\nu^k} |t_\nu|^k = O(X_m \mu_m), \text{ as } m \to \infty, \tag{11}$$

where (μ_m) *is a positive non-decreasing sequence such that*

$$n X_n \mu_n \Delta\left(\frac{1}{\mu_n}\right) = O(1), \text{ as } n \to \infty, \tag{12}$$

then the series $\sum \frac{a_n \lambda_n}{\mu_n}$ *is summable* $\varphi - |C,1|_k$, $k \geq 1$.

If we take $X_n = \log n$ and $\mu_n = 1$, in Theorem 3, we get Theorem 2 and additionally if we also take $\varphi_n = n$, then we get Theorem 1.
We need the following lemma for the proof of our theorem.

Lemma 1 (see [1])**.** *Under the conditions on* (X_n) *and* (λ_n), *as taken in the statement on Theorem 3, the following conditions hold,*

$$nX_n\Delta\lambda_n = O(1)\,, \text{ as } n\to\infty, \tag{13}$$

$$\sum_{n=1}^{\infty} |\Delta\lambda_n|\, X_n < \infty. \tag{14}$$

3. Proof of Theorem 3

Let T_n be the n-th $(C,1)$ means of the sequence $\left(\frac{na_n\lambda_n}{\mu_n}\right)$, then by definition, we have

$$T_n = \frac{1}{n+1}\sum_{\nu=1}^{n}\frac{\nu a_\nu\lambda_\nu}{\mu_\nu}.$$

Applying Abel's transformation, we get

$$\begin{aligned}
T_n &= \frac{1}{n+1}\sum_{\nu=1}^{n-1}\Delta\left(\frac{\lambda_\nu}{\mu_\nu}\right)\sum_{r=0}^{\nu} ra_r + \frac{1}{n+1}\frac{\lambda_n}{\mu_n}\sum_{r=0}^{n} ra_r - \frac{a_0\lambda_1}{(n+1)\,\mu_1}\\
&= \frac{1}{n+1}\sum_{\nu=1}^{n-1}\Delta\left(\frac{\lambda_\nu}{\mu_\nu}\right)(\nu+1)\,t_\nu + \frac{\lambda_n t_n}{\mu_n} - \frac{a_0\lambda_1}{(n+1)\,\mu_1}\\
&= \frac{1}{n+1}\sum_{\nu=1}^{n-1}\left\{\Delta\lambda_\nu\left(\frac{1}{\mu_\nu}\right) + \lambda_{\nu+1}\Delta\left(\frac{1}{\mu_\nu}\right)\right\}(\nu+1)\,t_\nu + \frac{\lambda_n t_n}{\mu_n} - \frac{a_0\lambda_1}{(n+1)\,\mu_1}\\
&= \frac{1}{n+1}\sum_{\nu=1}^{n-1}\Delta\lambda_\nu\left(\frac{1}{\mu_\nu}\right)(\nu+1)\,t_\nu + \frac{1}{n+1}\sum_{\nu=1}^{n-1}\lambda_{\nu+1}\Delta\left(\frac{1}{\mu_\nu}\right)(\nu+1)\,t_\nu\\
&\quad + \frac{\lambda_n t_n}{\mu_n} - \frac{a_0\lambda_1}{(n+1)\,\mu_1}\\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}
\end{aligned}$$

Since $|T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}|^k \le 4^k\left(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k\right)$, to complete the proof of Theorem 3, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\,|T_{n,r}|^k < \infty, \text{ for } r = 1,2,3,4.$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\begin{aligned}
\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\,|T_{n,1}|^k &= \sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\frac{1}{n+1}\sum_{\nu=1}^{n-1}\Delta\lambda_\nu\left(\frac{1}{\mu_\nu}\right)(\nu+1)\,t_\nu\right|^k\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\left(\sum_{\nu=1}^{n-1}\frac{\nu\,|\Delta\lambda_\nu|}{\mu_\nu}\,|t_\nu|\right)^k
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\left(\sum_{\nu=1}^{n-1}\frac{\nu\left|\Delta\lambda_\nu\right|}{\mu_\nu}\left|t_\nu\right|^k\right)\left(\sum_{\nu=1}^{n-1}\frac{\nu\left|\Delta\lambda_\nu\right|}{\mu_\nu}\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}}\left(\sum_{\nu=1}^{n-1}\frac{\nu\left|\Delta\lambda_\nu\right|}{\mu_\nu}\left|t_\nu\right|^k\right)\\
&= O(1)\sum_{\nu=1}^{m}\frac{\nu\left|\Delta\lambda_\nu\right|}{\mu_\nu}\left|t_\nu\right|^k\left(\sum_{n=\nu}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}}\right)\\
&= O(1)\sum_{\nu=1}^{m}\frac{\nu\left|\Delta\lambda_\nu\right|}{\mu_\nu}\left|t_\nu\right|^k\frac{\varphi_\nu^{k-1}}{\nu^k}\\
&= O(1)\sum_{\nu=1}^{m-1}\Delta\left(\frac{\nu\left|\Delta\lambda_\nu\right|}{\mu_\nu}\right)\sum_{r=1}^{\nu}\frac{\varphi_r^{k-1}}{r^k}\left|t_r\right|^k\\
&\quad + O(1)\frac{m\left|\lambda_m\right|}{\mu_m}\sum_{r=1}^{m}\frac{\varphi_r^{k-1}}{r^k}\left|t_r\right|^k\\
&= O(1)\sum_{\nu=1}^{m-1}\left\{\frac{\left|\Delta\lambda_\nu\right|}{\mu_\nu}+\frac{(\nu+1)\left|\Delta^2\lambda_\nu\right|}{\mu_\nu}\right.\\
&\quad \left. + (\nu+1)\Delta\lambda_{\nu+1}\Delta\left(\frac{1}{\mu_\nu}\right)\right\}X_\nu\mu_\nu + O(1)\frac{m\left|\Delta\lambda_m\right|}{\mu_m}X_m\mu_m\\
&= O(1)\sum_{\nu=1}^{m-1}\left|\Delta\lambda_\nu\right|X_\nu + O(1)\sum_{\nu=1}^{m-1}\nu\left|\Delta^2\lambda_\nu\right|X_\nu\\
&\quad + O(1)\sum_{\nu=1}^{m-1}\nu\left|\Delta\lambda_\nu\right|X_\nu\mu_\nu\Delta\left(\frac{1}{\mu_\nu}\right) + O(1)\,m\left|\Delta\lambda_m\right|X_m\\
&= O(1)\,,\ \text{as } m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3 and Lemma 1.

Again

$$
\begin{aligned}
\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|T_{n,2}\right|^k &= \sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\frac{1}{n+1}\sum_{\nu=1}^{n-1}\lambda_\nu\Delta\left(\frac{1}{\mu_\nu}\right)\nu t_\nu\right|^k\\
&= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}}{n^{2k}}\left\{\sum_{\nu=1}^{n-1}\nu\left|\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)\left|t_\nu\right|^k\right\}\left\{\sum_{\nu=1}^{n-1}\nu\left|\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)\right\}^{k-1}\\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^{k+1}}\sum_{\nu=1}^{n-1}\nu\left|\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)\left|t_\nu\right|^k\\
&= O(1)\sum_{\nu=1}^{m}\nu\left|\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)\left|t_n\right|^k\sum_{n=1}^{\nu}\frac{\varphi_n^{k-1}}{n^{k+1}}\\
&= O(1)\sum_{\nu=1}^{m}\nu\left|\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)\left|t_n\right|^k\frac{\varphi_\nu^{k-1}}{\nu^k}
\end{aligned}
$$

$$\begin{aligned}
&= O(1)\sum_{\nu=1}^{m}\Delta\left(\nu\left|\lambda_\nu\right|\right)\Delta\left(\frac{1}{\mu_\nu}\right)\sum_{n=1}^{\nu}\frac{\varphi_n^{k-1}}{n^k}\left|t_n\right|^k \\
&\quad + O(1)\, m\left|\lambda_m\right|\Delta\left(\frac{1}{\mu_m}\right)\sum_{r=1}^{m}\frac{\varphi_r^{k-1}}{r^k}\left|t_r\right|^k \\
&= O(1)\sum_{\nu=1}^{m-1}\left\{\left|\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)+(\nu+1)\left|\Delta\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)\right\}X_\nu\mu_\nu \\
&\quad + O(1)\sum_{\nu=1}^{m-1}\left\{(\nu+1)\left|\lambda_{\nu+1}\right|\Delta^2\left(\frac{1}{\mu_\nu}\right)\right\}X_\nu\mu_\nu \\
&\quad + O(1)\, m\left|\lambda_m\right|\Delta\left(\frac{1}{\mu_m}\right)X_m\mu_m \\
&= O(1)\sum_{\nu=1}^{m-1}\left|\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)X_\nu\mu_\nu + O(1)\sum_{\nu=1}^{m-1}\nu\left|\Delta\lambda_\nu\right|\Delta\left(\frac{1}{\mu_\nu}\right)X_\nu\mu_\nu \\
&\quad + O(1)\sum_{\nu=1}^{m-1}(\nu+1)\left|\lambda_{\nu+1}\right|\Delta^2\left(\frac{1}{\mu_\nu}\right)X_\nu\mu_\nu + O(1)\, m\left|\lambda_m\right|\Delta\left(\frac{1}{\mu_m}\right)X_m\mu_m \\
&= O(1)\sum_{\nu=1}^{m-1}\frac{\left|\lambda_\nu\right|}{\nu} + O(1)\sum_{\nu=1}^{m-1}\left|\Delta\lambda_\nu\right| + O(1)\sum_{\nu=1}^{m-1}\left|\lambda_\nu\right| \\
&\quad + O(1)\, m\left|\lambda_m\right|\Delta\left(\frac{1}{\mu_m}\right)X_m\mu_m \\
&= O(1)\,,\ \text{as } m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3 and Lemma 1.

Also

$$\begin{aligned}
\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|T_{n,3}\right|^k &= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\frac{\lambda_n t_n}{\mu_n}\right|^k \leq \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|t_n\right|^k\frac{\left|\lambda_n\right|^k}{\mu_n} \\
&= O(1)\sum_{n=1}^{m}\varphi_n^{k-1}\frac{\left|\lambda_n\right|}{\mu_n}\frac{\left|t_n\right|^k}{n^k} \\
&= O(1)\sum_{n=1}^{m-1}\Delta\left(\frac{\left|\lambda_n\right|}{\mu_n}\right)\sum_{r=1}^{n}\frac{\varphi_r^{k-1}}{r^k}\left|t_r\right|^k + O(1)\frac{\left|\lambda_m\right|}{\mu_m}\sum_{r=1}^{m}\frac{\varphi_r^{k-1}}{r^k}\left|t_r\right|^k \\
&= O(1)\sum_{n=1}^{m-1}\left(\frac{\left|\Delta\lambda_n\right|}{\mu_n}\right)X_n\mu_n + O(1)\sum_{n=1}^{m-1}\left|\lambda_n\right|\Delta\left(\frac{1}{\mu_n}\right)X_n\mu_n \\
&\quad + O(1)\frac{\left|\lambda_m\right|}{\mu_m}X_m\mu_m \\
&= O(1)\sum_{n=1}^{m-1}\left|\Delta\lambda_n\right|X_n + O(1)\sum_{n=1}^{m-1}\frac{\left|\lambda_n\right|}{n} + O(1)\left|\lambda_m\right|X_m \\
&= O(1)\,,\ \text{as } m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3 and Lemma 1.
Finally,

$$\sum \frac{\varphi_n^{k-1}}{n^k} \left| T_{n,4} \right|^k = \sum \frac{\varphi_n^{k-1}}{n^k} \left| \frac{a_0 \lambda_1}{(n+1)\,\mu_1} \right|^k \leq A \sum \frac{\varphi_n^{k-1}}{n^{2k}} < \infty$$

by virtue of the hypotheses of Theorem 3 and Lemma 1.
Therefore, we get

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left| T_{n,r} \right|^k = O\left(1\right), \text{as } \ m \to \infty, \text{for } \ r = 1, 2, 3, 4.$$

This completes the proof of Theorem 3.

Acknowledgement

I am very thankful to Dr. Rajiv Sinha for his kind encouragement to prepare this paper.

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