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# Four-dimensional matrix transformation and the double Gibbs' phenomenon

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**Abstract.** In 1976 Fridy presented a series of theorems that chacterize when matrices preserve the Gibbs' phenomenon. In this paper we present a multidimensional extension of the results of Fridy. In particular, we prove necessary and sufficient conditions for a positive RH-matrix to preserve the double Gibbs' phenomenon for positive double sequences. Other related results are also established.
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## 1. Introduction

In Section 2 we present a definition of the double Gibbs' phenomenon (Definition 5). In Section 3 we present a number of lemmas and theorems which lead to a characterization of those doubly infinite positive RH-matrices which preserve the Gibbs' phenomenon.

## 2. Definitions, notations, and preliminary results

**Definition 1** (see [6])**.** *A double sequence $x = [x_{k,l}]$ has a Pringsheim limit $L$ (denoted by $P\text{-}\lim x = L$) provided that given an $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an $x$ more briefly as "P-convergent".*

**Definition 2** (see [6])**.** *A double sequence $x$ is called definite divergent, if for every (arbitrarily large) $G > 0$ there exist two natural numbers $n_1$ and $n_2$ such that $|x_{n,k}| > G$ for $n \geq n_1, k \geq n_2$.*

The four dimensional matrix $A$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

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The assumption of boundedness was made because a double sequence which is P-convergent is not necessarily bounded. Using this definition Robison and Hamilton, independently, both presented the following Silverman-Toeplitz type characterization of RH-regularity.

**Theorem 1.** *The four dimensional matrix $A$ is RH-regular if and only if*

$$RH_1 : P - \lim_{m,n} a_{m,n,k,l} = 0 \text{ for each } k \text{ and } l;$$

$$RH_2 : P - \lim_{m,n} \sum_{k,l=0}^{\infty} a_{m,n,k,l} = 1;$$

$$RH_3 : P - \lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } l;$$

$$RH_4 : P - \lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } k;$$

$$RH_5 : \sum_{k,l=0}^{\infty} |a_{m,n,k,l}| \text{ is P-convergent};$$

$$RH_6 : \text{there exist positive numbers } A \text{ and } B \text{ such that } \sum_{k,l>B} |a_{m,n,k,l}| < A.$$

**Definition 3** (see [5]). *The double sequence $[y]$ is a double subsequence of the sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then $y$ is formed by*

$$\begin{matrix} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & -. \end{matrix}$$

**Definition 4** (see [5]). *A number $\beta$ is called a Pringsheim limit point of the double sequence $[x]$ provided that there exists a subsequence $[y]$ of $[x]$ that has Pringsheim limit $\beta$ : $P$-$\lim[y] = \beta$.*

Let $\{x_{k,l}\}$ be a double sequence of real numbers and, for each $n$, let $\alpha_n = \sup_n\{x_{k,l} : k, l \geq n\}$. The *Pringsheim limit superior* of $[x]$ is defined as follows:

1. if $\alpha = +\infty$ for each n, then P-$\limsup[x] := +\infty$;
2. if $\alpha < \infty$ for some $n$, then P-$\limsup[x] := \inf_n\{\alpha_n\}$.

Similarly, let $\beta_n = \inf_n\{x_{k,l} : k, l \geq n\}$. Then the *Pringsheim limit inferior* of $[x]$ is defined as follows:

1. if $\beta_n = -\infty$ for each n, then P-$\liminf[x] := -\infty$;

2. if $\beta_n > -\infty$ for some $n$, then P-$\liminf[x] := \sup_n\{\beta_n\}$.

We now present a definition of the double Gibbs phenomenon.

**Definition 5.** *Let $f$ be a double real-valued sequence $\{f_{k,l}\}$ which is P-convergent to a function $\phi$ at each point of a deleted neighborhood $D$ of the point $(x_0, y_0) = \alpha$. If there exist subsequences $\{f_{k_i,l_j}\}$ of $\{f_{k,l}\}$ and a double sequence $\{x_{k,l}\}$ with $P - \lim_{i,j} x_{i,j} = \alpha$ and either*

$$P - \lim_{i,j} f_{k_i,l_j}(x_{i,j}) > P - \limsup_{i,j} \phi(x_{i,j}) \tag{1}$$

*or*

$$P - \lim_{i,j} f_{k_i,l_j}(x_{i,j}) < P - \liminf_{i,j} \phi(x_{i,j}) \tag{2}$$

*then $f$ is said to possess the double Gibbs phenomenon at $\alpha$.*

## 3. Main results

This presentation shall examine real matrix summability methods that map double sequences into double sequences. The transformation $A$ that transforms $f$ into $Af(x,y)$ for each $(x,y) \in D$ where $(Af)(x,y) = \{(Af)_{m,n}(x,y)\}_{m,n=0}^{\infty}$ is defined by

$$(Af)_{m,n}(x,y) = \sum_{k,l=1}^{\infty} a_{m,n,k,l} f_{k,l}(x,y).$$

Observe that if the limit function $\phi$ has limit $\alpha$, then (1) and (2) imply that the Gibbs phenomenon is equivalent to nonuniform P-convergence. This observation yields the following lemma.

**Lemma 1.** *The double real-valued sequence $\{f_{k,l}\}$ is uniformly P-convergent on $D$ if and only if $f$ is uniformly P-convergent on every countable subset of $D$.*

Let $\{f_{k,l}\}$ be double sequence of real-valued functions and $x$ a double sequence in $D$. We can now consider the following four-dimensional matrix of functions:

$$F_{m,n,k,l} = f_{m,n}(x_{k,l}) - \phi(x_{k,l}).$$

Note that each pairwise column of $\{F_{m,n}\}$ P-converges to 0 and thus Lemma 3 can be used to describe the connection between uniformly P-convergence and the double Gibbs Phenomenon in the following sense.

**Lemma 2.** *The double real-valued sequence $\{f_{k,l}\}$ is uniformly P-convergent on $D$ if and only if for every double sequence $x$ in $D$ the corresponding four dimension matrix $F$ has the property that its pairwise column double sequences P-converges uniformly to 0.*

**Lemma 3.** *If $f$ P-converges (point-wise) in a deleted neighborhood $D$ of $\alpha$, then the following are equivalent:*

1. *$f$ displays the double Gibbs phenomenon at $\alpha$;*
2. *there is a double number sequence $x$ P-converging to $\alpha$ for which the corresponding functions matrix $F$ is such that*

$$\lim_{m,n,k,l} F_{m,n,k,l} \neq 0.$$

The notation $\lim_{k,l,i,j} F_{k,l,i,j} = \lambda$ shall mean

$$P - \lim_{m,n}\Big(\sup_{k,l>m,n;i,j>m,n} F_{k,l,i,j} - \lambda\Big) = 0.$$

Let $\mathcal{G}$ denote the collection of matrices $F$ which have 0 as a pairwise Pringsheim column limit and which violates (1) or (2).

**Theorem 2.** *If $A$ is GP-preserving and $\{k_i\}_{i=0}^{\infty}$ and $\{l_j\}_{j=0}^{\infty}$ is any infinite subset of pairwise column indices, then*

$$P - \lim_{m,n}\big(\sup_{i,j} |a_{m,n,k_i,l_j}|\big) > 0.$$

**Proof**. Suppose that $A$ has a pairwise column such that

$$P - \lim_{m,n}\big(\sup_{i,j} |a_{m,n,k_i,l_j}|\big) = 0,$$

and consider the following:

$$F_{k,l,i,j} := \begin{cases} 1, \ (k,l) = (k_i, l_j) \\ 0, \ (k,l) \neq (k_i, l_j). \end{cases}$$

Since $\{k_i\}$ and $\{l_j\}$ are divergent single subsequences, $F$ is in $\mathcal{G}$. Then

$$(AF)_{m,n,i,j} = a_{m,n,k_i,l_j} \text{ for all } (m,n) \text{ and } (i,j),$$

and $\lim_{m,n,i,j} (AF)_{m,n,i,j} = 0$. (i.e. $AF$ is not in $\mathcal{G}$). Hence $A$ is not GP-preserving. □

For RH-regular matrices both pairwise columns and rows are P-null double sequences. In this case Theorem 2 can be simplified as follows:

**Theorem 3.** *If $A$ is an RH-regular matrix and $(k_i, l_j)$ a doubly infinite set of pairwise column indices, then*

$$P - \lim_{m,n}\big(\sup_{i,j} |a_{m,n,k_i,l_j}|\big) > 0 \tag{3}$$

*holds if and only if there exists a positive number $\epsilon$ and infinitely many terms $a_{m,n,k_i,l_j}$ such that $|a_{m,n,k_i,l_j}| \geq \epsilon$.*

**Corollary 1.** *If $A$ is RH-regular and GP-preserving, then*

$$P - \lim_{m,n}\Big(\sup_{(k,l)>(m,n);(i,j)>(m,n)} |a_{k,l,i,j}|\Big) > 0.$$

This immediately follows from (3).

This result has an invariant Pringsheim core type flavor, which was suggested by both Fridy ([2, Theorem 3]) and [5, Theorem 3.2], with a proof that is similar to Theorem 2, and is therefore omitted.

**Theorem 4.** *If $A$ is GP-preserving and $(k_i, l_j)$ a doubly infinite set of pairwise column indices, then*

$$P - \limsup_{m,n} \left| \sum_{i,j=1}^{\infty} a_{m,n,k_i,l_j} \right| > 0. \tag{4}$$

Thus far all lemmas and theorems have provided only sufficient conditions to ensure that $A$ is GP-preserving. However, note that

$$A(x)_{m,n} = \frac{x_{2m,2n} + x_{2m+1,2n+1}}{2}$$

is an RH-regular transformation that satisfies the conditions that there exists a positive number $\epsilon$ and infinitely many terms $a_{m,n,k_i,l_j}$ such that $|a_{m,n,k_i,l_j}| \geq \epsilon$ and

$$P - \lim_{m,n} \Big( \sup_{(k,l)>(m,n);(i,j)>(m,n)} |a_{k,l,i,j}| \Big) > 0$$

hold.

Consider the following four dimensional matrix

$$F_{m,n,k,l} := \begin{cases} 1, & \text{if } \; k = 2i, \quad l = 2j \\ -1, & \text{if } \; k = 2i, \quad l = 2j \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $F \in \mathcal{G}$ and $(AF)_{m,n,k,l} = 0$, so that $A$ is not GP-preserving. In addition, let $\mathcal{G}^+$ denote the collection of nonnegative matrices $F$ which have 0 as a pairwise Pringsheim column limit and fail to satisfy (1) or (2).

To obtain necessary conditions we need to place suitable restriction on $A$ and $F$. The following is a partial converse of Theorem 3.

**Theorem 5.** *If $A$ is an RH-regular nonnegative four dimensional matrix, then $A$ maps $\mathcal{G}^+$ into $\mathcal{G}^+$ if and only if there exists a positive number $\epsilon$ and infinitely many terms $a_{m,n,k_i,l_j}$ such that $|a_{m,n,k_i,l_j}| \geq \epsilon$.*

**Proof**. If $F$ is in $\mathcal{G}^+$, then there exists a positive number $\delta$ and two double Pringsheim sequences $(m_i, n_j)$ and $(k_i, l_j)$ such that for each pair $(i, j)$, $F_{m_i,n_j,k_i,l_j} \geq \delta$. By (3) there are infinitely many Pringsheim order terms $a_{m,n,k_i,l_j}$ such that $a_{m,n,k_i,l_j} \geq \epsilon > 0$. For each $a_{m,n,k_i,l_j}$ we have

$$(AF)_{m,n,k_i,l_j} \geq a_{m,n,k_i,l_j} F_{m_i,n_j,k_i,l_j} \geq \epsilon\delta.$$

Thus $AF$ satisfies (3), and, since $AF$ is nonnegative, it is clearly in $\mathcal{G}^+$. Note that the converse clearly follows from Theorem 2. □

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